On periodic points for systems of weakly coupled 1-dim maps

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Abstract

In paper we present the topological method of proving the existence of periodic in multidimensional nonuniformly hyperbolic dynamical systems. We apply our method to the maps of the form

$$F(x_1,\dots,x_n) := (f(x_1), f(x_2), \dots, f(x_n)) + G : \mathbb{R}^n \to \mathbb{R}^n$$

where $f : \mathbb{R} \to \mathbb{R}$ is either logistic or tent map in the chaotic region and G is sufficiently small in C^0 norm. We prove that F have inifinite number periodic points with symbolic dynamics. We prove also the result for spatial distribution of periodic points for F in the invariant set.

Another considered example is Rössler walking stick diffeomorphism, for which we prove the existence of inifinite number of periodic points.

Keywords: fixed point index, chaos, nonuniformly hyperbolic systems

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1 Introduction

The aim of this paper is to develop the method of TS-maps (topological shifts) introduced by the author in [?] for proving existence of infinite number of periodic points in multidimensional dynamical systems.

We describe our results in a concrete example. Let us consider the map $F : \mathbb{R}^n \to \mathbb{R}^n$ given by

$$F(x_1,\dots,x_n) := (f(x_1), f(x_2), \dots, f(x_n)) + G(x_1,\dots,x_n) \qquad (1)$$

where $f(x) = 1 - ax^2$, where a is close to 2 and G is small perturbation. The map F is not hyperbolic. From f it inherit the coexistence of hyperbolic dynamics in portions of phase space, together with highly nonhyperbolic behavior in certain critical regions.

When $G = 0$ and $a = 2$ (or slightly less then 2) then dynamics of F is given as the cartesian product of n independent copies of 1-dimensional chaotic maps, which are relatively well understood. There are two kind of results concerning dynamics of $f(x) = 1 - ax^2$. The 'ergodic' results: concerning existence of absolutely continuous invariant measure for f plus positive of Lyapunov exponent for positive Lesbegue measure in parameters space ([?],[?],[?]) and 'topological' results: kneading sequences, symbolic dynamics, infinite number of periodic points, transivity (existence of dense orbit) (see for example [?]).

For typical nonzero perturbation G F is not a product of 1-dim maps and F has more than one expanding direction. The research concerning the ergodic properties of such nonuniformly hyperbolic maps is just starting, see paper [?] and references cited there.

Using our method we are able to show that the following topological properties hold for F if G is sufficiently small in C^0 norm: F has symbolic dynamics, the number of periodic points is infinite and there exists a region in space that every not too small ball in this region contains a periodic point for F, see the theorem ?? for a precise result. The estimate for $|G|$ is also given.

Section ?? contains topological theorems which form the base for the method. The main tool used to obtain periodic points is the fixed point index. The properties of fixed point index and the proof of main topological theorem are contained in the Appendix.

Section ?? contains a result concerning C^0 small perturbations of cartesian products of tent maps.

Section ?? contains the above described result for the C^0 small perturbations of cartesian products of logistic maps.

Section ?? contains application of the method to the Rössler 'walking stick' diffeomorphism introduced in [?]. This map has two topologically expanding direction and one contracting direction. For this map we are able to prove the existence of symbolic dynamics with infinite number of periodic points.

2 Topological theorems

First we introduce some notations. Let $p \in \mathbb{R}^n$ by $x_i(p)$ we will denote the i-th coordinate of the point p. We will use the max norm on $\mathbb{R}^n$, so

$$|(x_1, \ldots, x_n)| := \max_i |x_i| \tag{2}$$

Let $Z \subset \mathbb{R}^n$, $x \in \mathbb{R}^n$ then we will the following notations $\mathrm{dist}\,(x, Z) = \inf\{|x - y| | y \in Z\}$. $B(x, \epsilon) = \{y | |x - y| < \epsilon\}$, $B(Z, \epsilon) = \{x | \mathrm{dist}\,(x, Z) < \epsilon\}$, $\mathrm{diam}\, Z = \sup_{x,y \in Z} |x - y|$.

We by $\mathcal{C}$ we will denote a parallelograms in $\mathbb{R}^n$, so

$$\mathcal{C} := \{X \subset \mathbb{R}^n \mid X = \prod_{i=1}^{n}[x_{ai}, x_{bi}]\} \tag{3}$$

We define for $X \in \mathcal{C}$ *the internal diameter*

$$\operatorname{idiam}(X) = \min_i |x_{bi} - x_{ai}|, \quad \text{where } X = \textstyle\prod_{i=1}^{n}[x_{ai}, x_{bi}] \tag{4}$$

Definition 1 *Let $f : \mathbb{R} \to \mathbb{R}$ be continuous, $\delta \geq 0$, $X = [x_a, x_b]$ and $Y = [y_a, y_b]$. We say that X* f-covers Y (with a margin δ) *iff $[y_a - \delta, y_b + \delta]$ is contained either in $(f(x_a), f(x_a))$ or in $(f(x_b), f(x_a))$.*

Definition 2 *Let $X = \prod_{i=1}^{n}[x_{ai}, x_{bi}]$ For $i \in \{1, \ldots, n\}$ we define the i-the upper and down edge of X respectively by*

$$U_i(X) = \{p \in X \mid x_i(p) = x_{bi}\} \tag{5}$$

$$D_i(X) = \{p \in X \mid x_i(p) = x_{ai}\} \tag{6}$$

Definition 3 *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be continuous, $\delta \geq 0$, $X = \prod_{i=1}^{n}[x_{ai}, x_{bi}]$ and $Y = \prod_{i=1}^{n}[y_{ai}, y_{bi}]$. Let $1 \leq i \leq n$, we say that X* F-covers Y in i direction (with a margin δ) *one of the two following conditions hold*

$$[y_{ai} - \delta, y_{bi} + \delta] \subset (\max x_i(F(D_i(X))), \min x_i(F(U_i(X)))) \tag{7}$$

$$[y_{ai} - \delta, y_{bi} + \delta] \subset (\max x_i(F(U_i(X))), \min x_i(F(D_i(X)))) \tag{8}$$

If (??) is satisfied, then we will say that orientation is preserved, *if (??) is true then we will say that* orientation is reversed.

Definition 4 *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be continuous, $\delta \geq 0$, $X = \prod_{i=1}^{n}[x_{ai}, x_{bi}]$ and $Y = \prod_{i=1}^{n}[y_{ai}, y_{bi}]$. Let $1 \leq i_1 < i_2 < \ldots < i_k \leq n$ be a sequence of integers. We say that X* F-covers Y in $(i_1, i_2, \ldots, i_k)$-direction (with a margin δ) *if the following conditions hold*

- *for every $l = 1, \ldots, k$ X F-covers Y in the i_l direction with margin δ*
- *for every j not in the sequence $i_1, i_2, \ldots, i_k$ hold*

$$x_j(F(X)) \subset [y_{aj} + \delta, y_{bj} - \delta] \tag{9}$$

To illustrate the notions introduced above let us consider the following example, let $n = 3$ and $f_1, f_2, f_3 : \mathbb{R} \to \mathbb{R}$ and the segments X_i, Y_i for $i = 1, 2$ be such that X_i f_i-covers Y_i with margin $\delta < 1$ and be such that $f_3(x) = 0$ for $x \in \mathbb{R}$, we set $X_3 = Y_3 = [-1, 1]$. We set $X = X_1 \times X_2 \times X_3$, $Y = Y_1 \times Y_2 \times Y_3$. Consider the map $F : \mathbb{R}^3 \to \mathbb{R}^3$ given by $F(x_1, x_2, x_3) = (f_1(x_1), f_2(x_2), f_3(x_3))$. It is easy to see that the set X F-covers Y in $(1, 2)$-direction with margin δ. Consider now a perturbation $\tilde{F} : \mathbb{R}^n \to \mathbb{R}^n$ of F such that $|\tilde{F} - F|_{|X} \leq \delta$. Then it is easy to see that X G-covers Y in $(1, 2)$-direction.

Theorem 1 *Suppose we have a family of sets $N_j \in \mathcal{C}$, for $j = 0, \ldots, l-1$, continuous maps $F_j : N_j \to \mathbb{R}^n$ for $j = 0, 1, \ldots, l-1$ and a sequence $1 \leq i_1 < i_2 < \ldots < i_k \leq n$, such that for $j = 0, \ldots, l-1$ N_j F_j-covers $N_{(j+1) \bmod l}$ in $(i_1, \ldots, i_k)$-direction, then there exists $x \in \mathit{int}\, N_0$ such that*

$$F_j \circ F_{j-1} \circ \ldots F_1(x) \in \mathit{int}\, N_{j+1}, \quad \mathit{for}\ j = 0, 1, \ldots, l-2 \tag{10}$$

$$F_{l-1} \circ F_{l-2} \circ \ldots F_1(x) = x \tag{11}$$

The proof of this theorem is based on the fixed point index and is contained in the appendix.

The above theorem can be easily extended to maps on manifolds. To this end one need for every set N_j a local chart ϕ_j in which this set is parallelogram and the relation N_j F-covers N_{j+1} in $(i_1, \ldots, i_k)$-direction should be expressed using coordinates obtained by the charts ϕ_j and ϕ_{j+1}.

Also the requirement that F_j is defined on entire $\mathbb{R}^n$ is unnecessary, it is enough to assume that the domain of F_j contains N_j.

We will now describe two kinds of application of the theorem ??

2.1 Symbolic dynamics. Topological shifts

We present here a generalization of the notion of TS-map (topological shifts) introduced (as window chains) in [?], [?] and in [?], to the situation of more then one expanding direction.

Let be a continuous map $F : \mathbb{R}^n \to \mathbb{R}^n$. Let us fix a sequence $1 \leq i_1 < i_2 < \ldots < i_k \leq n$. Let $\{e_i\}$ be a canonical basis in $\mathbb{R}^n$. Then we will call the linear subspace spanned by vectors $\{e_{i_1}, \ldots, e_{i_k}\}$ *topologically expanding direction*. The reason for this name will be clear from theorems ??, ??.

Definition 5 *Let $X, Y \in \mathcal{C}$. We will say that there exists* a F-transition of length *m from X to Y iff there exists a sequence of sets $\{N_j\}_{j=0,1,\ldots,l} \subset \mathcal{C}$ and the sequence of integers $\{m_j\}_{j=0,\ldots,l-1}$, such that*

$$N_0 \subset X, \quad N_l = Y$$

$$N_j\ F^{m_j}\text{-covers}\ N_{j+1}\ \text{in}\ (i_1, \ldots, i_k)\text{-direction, for}\ j = 0, \ldots, l-1$$

$$m_0 + m_1 + \ldots m_{l-1} = m$$

We will use the following notation graphical notation $X \xrightarrow{F^m} Y$.

The following proposition is an easy consequence of the above definition and theorem ??.

Proposition 2 *Suppose we have a family of sets $N_j \in \mathcal{C}$, for $j = 0, \ldots, l-1$ and a sequence of positive integers m_j for $j = 0, \ldots, l$, such that*

$$N_0 \xrightarrow{F^{m_0}} N_1 \xrightarrow{F^{m_1}} \cdots \xrightarrow{F^{m_{l-1}}} N_l \xrightarrow{F^{m_l}} N_0$$

Then there exists $x \in int\ N_0$ such that

$$F^{m_j+m_{j-1}+\ldots+m_0}(x) \in int\ N_{j+1}, \quad for\ j = 1, \ldots, l-1 \tag{12}$$
$$F^{m_l+m_{l-1}+\ldots+m_0}(x) = x \tag{13}$$

Suppose now that we have a family of sets $N_j \in \mathcal{C}$ for $j = 0, 1, \ldots, l-1$ and an integer m such that

$$N_j \cap N_k = \emptyset, \quad \text{for } j, k = 0, \ldots, l-1,\ j \neq k \tag{14}$$
$$N_j \xrightarrow{F^m} N_k, \quad \text{for } j, k = 0, \ldots, l-1 \tag{15}$$

We set $N = \bigcup_{j=0}^{l-1} N_j$. An invariant part of the set N is defined by

$$\mathrm{Inv}(N, F) := \bigcap_{i \in \mathbb{Z}} F_{|N}^{-i}(N) \tag{16}$$

Let $\Sigma_l := \{0, 1, \ldots, l-1\}^{\mathbb{Z}}$, $\Sigma_l^+ := \{0, 1, \ldots, l-1\}^{\mathbb{N}}$. Σ_l, Σ_l^+ are topological spaces with the Tichonov topology. On Σ_l, Σ_l^+ we have the shift map σ given by

$$(\sigma(c))_i = c_{i+1}$$

For $i \in \mathbb{N}$ we define a map $\pi_i : \mathrm{Inv}(N, F^m) \to \{0, 1, \ldots, l-1\}$ given by $\pi_i(x) = j$ iff $F^{mi}(x) \in N_j$. Now we define a map $\pi : \mathrm{Inv}(N, F^m) \to \Sigma_l^+$ by $\pi(x) := (\pi_i(x))_{i \in \mathbb{N}}$. The map π assigns to the point x the indices of N_i-s its F^m-trajectory goes through. It is easy to see that

$$\pi \circ F^m = \sigma \circ \pi. \tag{17}$$

If F is also a homeomorphism, then the definition of π_i can be extended to all integers and the domain of π is Σ_l.

Obviously the semiconjugacy (??) alone is not a sign of complicated dynamics. The dynamics is complicated if the set $\mathrm{Inv}(N, F^m)$ is infinite. The following theorem gives is a generalization of results from [?]

Theorem 3 *Let the family of sets $\{N_j\}_{j=0,\ldots,l-1} \subset \mathcal{C}$ satisfies the conditions (??)(??).*
Then $\Sigma_l^+ = \pi(Inv(N, F^m))$.
The preimage of any periodic sequence from Σ_l^+ contains periodic points of F^m.
If we additionally suppose that F is a homeomorphism, then $\Sigma_l = \pi(Inv(N, F^m))$.

Proof Since periodic sequences are dense in Σ_l^+, then it is enough to show that every periodic sequence $c \in \Sigma_l^+$ is realized as the image of some periodic point for F^m.

Let $c \in \Sigma_l^+$ be periodic. Let $(c_0, c_1, \ldots, c_k)$ be finite subsequence of c representing its period. From our assumptions follows that

$$N_{c_0} \stackrel{F^m}{\longrightarrow} N_{c_1} \stackrel{F^m}{\longrightarrow} \cdots \stackrel{F^m}{\longrightarrow} N_{c_k} \stackrel{F^m}{\longrightarrow} N_{c_0} \tag{18}$$

From proposition ?? follows the existence of the point of $x \in \text{int } N_{c_0}$ such that $\pi(x) = c$.

□

2.2 Distribution of periodic points in the phase space, the existence of dense orbit

Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be continuous. Let $(i_1, \ldots, i_k) = (1, \ldots, n)$. This mean that all directions are topologically expanding.

Definition 6 *Let $X \in \mathcal{C}$, $\delta > 0$. We will say that F* has property TE (topological expansion) for X and δ, *if there exists a finite covering $\{M_j\} \subset \mathcal{C}$ of X, (i.e. $\bigcup M_j = X$), such that for every j diam $(M_j) \leq \delta$ and there exists an integer $l(M_j, \delta)$ such that*

$$M_j \stackrel{F^{l(M_j,\delta)}}{\longrightarrow} X \tag{19}$$

Theorem 4 *Assume that F has* TE property *for X and δ. Then there exists $x \in X$ such that*

- *x is periodic point for F*
- *for every closed ball $Z = cl\ B(z, 2\delta)$, $Z \subset X$ there exists m such that $F^m(x) \in Z$*

Proof Let $\{M_j\}_{j=0,\ldots,k}$ be finite covering of X , which follows from *TE property* for X and δ. We have

$$M_0 \stackrel{F^{l(M_0,\delta)}}{\longrightarrow} M_1 \stackrel{F^{l(M_1,\delta)}}{\longrightarrow} M_2 \ldots \stackrel{F^{l(M_{k-1},\delta)}}{\longrightarrow} M_k \stackrel{F^{l(M_k,\delta)}}{\longrightarrow} M_0 \tag{20}$$

From proposition ?? follows the existence of x with desired properties. □

Obviously in the above theorem to get something nontrivial we need that δ should be small when compared with the size of the set X. We will see that this is indeed the case in the examples considered in the next two sections.

Theorem 5 *Suppose that F has* TE property *for $X \in \mathcal{C}$ and every $\delta > 0$. Then*

- *the set of periodic points is dense in X*
- *there exists $x \in X$ such that positive orbit of this point is dense in X.*

Proof

The assertion concerning density of periodic points follows directly from previous theorem. It remains to prove the existence of dense orbit.

For every $l \in \mathbb{N}$ let $\mathcal{M}^l = \{M_i^l\}_{i=0,\ldots,i_l}$, $M_i^l \subset X$ be covering of X by closed balls of radius less or equal that $1/2^l$, whose existence follows from TE property for $\delta = 1/2^l$. We define new sequence $\{M_i\}$, $i \in \mathbb{N}$ which is union of sequences $\mathcal{M}^l$. From TE property it follows that for every $i \in \mathbb{N}$ there exists l_i such that

$$M_i \xrightarrow{F^{l_i}} X \tag{21}$$

Hence we obtain

$$M_i \xrightarrow{F^{l_i}} M_{i+1} \tag{22}$$

$$M_i \xrightarrow{F^{l_i}} M_0 \tag{23}$$

Let us set $L_i = \Sigma_{k=0}^{i-1} l_k$ Consider now finite subsequence $M_0, M_1, \ldots, M_i$. We have

$$M_0 \xrightarrow{F^{l_0}} M_1 \xrightarrow{F^{l_1}} \ldots \xrightarrow{F^{l_{i-1}}} M_i \xrightarrow{F^{l_i}} M_0 \tag{24}$$

From proposition ?? follows that there exists a periodic point $x_i \in M_0$ such that

$$F^{L_k}(x_i) \in M_k, \quad \text{for } k = 1, 2, \ldots, i \tag{25}$$

Let x be a limit of convergent subsequence of $\{x_i\}$. Passing to the limit in (??) we obtain

$$F^{L_k}(x) \in M_k, \quad \text{for } k \in \mathbb{N} \tag{26}$$

To finish the proof let us remark that for every ball $B(p, r) \subset N$ there exists k such that $M_k \subset B(p, r)$.

□

3 Infinite number of periodic points for perturbations of products of tent maps

In this section we consider perturbations of products of tent maps. We show that theorems ??, ?? can be applied to these maps. Another reason for investigation of these maps is the existence of the conjugacy between the tent map t_2 and the logistic map $1-2x^2$, so the results concerning existence of transitions (as defined in previous section) between some intervals for tent map are true also for the logistic map. We will exploit this conjugacy in the section ??.

In the sequel we will use the following convention maps f, t_μ, g which are denoted with small letters are one dimensional and maps denoted by capitals are multidimensional.

Let us define a tent map $t_\mu : \mathbb{R} \to \mathbb{R}$ by

$$t_\mu(x) := \begin{cases} 1+\mu x, & \text{for } x \leq 0 \\ 1-\mu x, & \text{for } x \geq 0 \end{cases} \tag{27}$$

Let us denote by

$$I_{\mu,\epsilon} := [t_\mu(1-\epsilon)+\epsilon, 1-\epsilon] = [1-\mu+\epsilon(1+\mu), 1-\epsilon]. \tag{28}$$

We will assume further that μ and ϵ are such that

$$1-\mu+\epsilon(1+\mu) < 0 < 1-\epsilon \tag{29}$$

$$|1-\mu+\epsilon(1+\mu)| < 1-\epsilon \tag{30}$$

The condition (??) means that $0 \in \operatorname{int} I_{\mu,\epsilon}$. The condition (??) means that the interval $I_{\mu,\epsilon} \cap \mathbb{R}_-$ is shorter than $I_{\mu,\epsilon} \cap \mathbb{R}_+$.

Conditions (??), (??) hold for example for μ close to 2 and ϵ small.

Let f be a continuous perturbation of t_μ given by

$$f(x) = t_\mu(x) + g(x), \quad \text{where } |g(x)| < \epsilon \tag{31}$$

Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be perturbation of cartesian product of n tents given by

$$F(x_1,\ldots,x_n) = (t_\mu(x_1), t_\mu(x_2), \ldots, t_\mu(x_n)) + G(x_1,\ldots,x_n) \tag{32}$$

where $|G([-1,1]^n)| < \epsilon$.

Let us remark that form (??) it follows

$$I_{\mu,\epsilon} \subset f(I_{\mu,\epsilon}) \tag{33}$$

moreover it is easy to see that the interval $I_{\mu,\epsilon}$ is a maximal set in $[-1,1]$ which has this property. Namely the following proposition holds

Proposition 6 *If $J \subset [-1,1]$ such that for every continuous map $g : [-1,1] \to \mathbb{R}$, such that $|g| < \epsilon$ holds*

$$J \subset (t_\mu + g)(J) \tag{34}$$

then $J \subset I_{\mu,\epsilon}$

Proof

Consider function $f = t_\mu + g$ such that

$$f(x) = \min\{t_\mu(x) + \epsilon, 1 - \epsilon\} \tag{35}$$

It is easy to see that

$$f([-1,1]) = [t_\mu(-1) + \epsilon, 1 - \epsilon] \tag{36}$$

Since we have

$$t_\mu(1-\epsilon) = t_\mu(-1+\epsilon), \text{ and } -1+\epsilon < t_\mu(-1) + \epsilon \tag{37}$$

we obtain

$$f^2([-1,1] = [t_\mu(1-\epsilon) + \epsilon, 1 - \epsilon] = I_{\mu,\epsilon} \tag{38}$$

By our assumption $J \subset f^2([-1,1])$, so the assertion follows.

□

In multidimensional case we obtain the following

Proposition 7 *If $J \subset [-1,1]^n$ such that for every continuous map $G : [-1,1]^n \to \mathbb{R}^n$, such that $|G| < \epsilon$ holds*

$$J \subset F(J) \tag{39}$$

then $J \subset I^n_{\mu,\epsilon}$

In this section we define the expanding directions as $(i_1, i_2, \ldots, i_k) = (1, \ldots, n)$.

The aim of this section is to prove the following theorem

Theorem 8 *Let F, G satisfy (??) and μ, ϵ be such that (??)(??) are satisfied. Then there exists $\delta = \delta(\mu, \epsilon)$ such that*

1. *there exists $x \in \prod_{i=1}^n I_{\mu,\epsilon}$ such that*

- *x is periodic point for F*
- *for every closed ball $Z = cl\, B(z, 2\delta)$, $Z \subset \prod_{i=1}^n I_{\mu,\epsilon}$ there exists m such that $F^m(x) \in Z$*

2. *for every finite family of sets $\{N^s\}_{s=0,\ldots,k-1}$, $N^s \subset \prod_{i=1}^n I_{\mu,\epsilon}$, of the form*

$$N^s = J^{s,1} \times J^{s,2} \times \ldots \times J^{s,n} \tag{40}$$

where $J^{s,i}$ are closed intervals of length greater than δ and $0 \notin int J^{s,i}$ and

$$N^s \cap N^{s'} = \emptyset, \qquad \text{for } s \neq s'. \tag{41}$$

There exists m and a continuous map $\pi : Inv(\bigcup_s N^s, F^m) \to \Sigma_k^+$, such that

$$\sigma \circ \pi = \pi \circ F^m \tag{42}$$

π is onto. The preimage of any periodic sequence in Σ_k^+ contains a periodic point for F^m.

The crucial step in the proof of the above theorem is the following lemma, which shows that F for sufficiently small ϵ and δ has TE property for $\prod_{i=1}^n I_{\mu,\epsilon}$. Namely

Lemma 9 *Let F, G satisfy (??) and μ, ϵ be such that (??)(??) are satisfied. Then there exists $\delta = \delta(\mu,\epsilon)$ such that for every set $M \subset \prod_{i=1}^n I_{\mu,\epsilon}$, of the form*

$$M = J^1 \times J^2 \times \ldots \times J^n \tag{43}$$

where J^i are closed intervals of length greater than δ and $0 \notin intJ^i$ there exists m such that

$$M \xrightarrow{F^m} \prod_{i=1}^n I_{\mu,\epsilon} \tag{44}$$

Before we prove this lemma we need the following

Lemma 10 *There exists $\delta = \delta(\mu,\epsilon)$, such that for every closed interval $J \subset I_{\mu,\epsilon}$, such that $0 \notin int\, J$ and length of J is greater than δ there exists s and a sequence of closed intervals $\{J_i\}_{i=0,\ldots,l}$ such that*

$$J_0 := J, \quad J_l = [0, 1-\epsilon] \tag{45}$$

$$0 \notin int\, J_i, \tag{46}$$

$$J_i\, t_\mu\text{-covers } J_{i+1} \text{ with margin } \epsilon, \text{ for } i=0,\ldots,l-1 \tag{47}$$

Proof

We will denote by $d([a,b]) = b-a$ the length of $[a,b]$.

Let us consider the following procedure for constructing $J_{i+1} = [a_{i+1}, b_{i+1}]$ from a given $J_i = [a_i, b_i]$.

Step 1. Since $0 \notin J_i$, so t_μ is strictly monotone on J_i and $t_\mu(J_i) = [a,b]$, where either $a := t_\mu(a_i)$ and $b := t_\mu(b_i)$ or $a := t_\mu(b_i)$ and $b := t_\mu(a_i)$.

Step 2. we set $a := a+\epsilon$, $b := b-\epsilon$. If not $a < b$ then we interrupt the procedure returning **fail**. if $a < 0 < b$ then go to step 3 otherwise go to step 4.

Step 3. - cut then if $b \geq |a|$ we set $a := 0$, in the opposite case we set $b := 0$. This means that we choose the longer one of two segments $(a,0)$ and $(0,b)$.

Step 4. We set $a_{i+1} := a$, $b_{i+1} := b$ and we return **success**.

It is clear from the construction of the above procedure that if *success* is returned then

$$0 \notin \mathrm{int}\, J_i, \tag{48}$$

$$J_i\, t_\mu\text{-covers } J_{i+1} \text{with margin } \epsilon \tag{49}$$

It is easy to see that the procedure return *fail* iff

$$|t_\mu(a_i) - t_\mu(b_i)| \leq 2\epsilon \tag{50}$$

and since t_μ is a linear function on J_i it is easy that this happens iff

$$\mu d(J_i) \leq 2\epsilon \tag{51}$$

Let us remark that it follows from (??-??) that a result of application of this procedure to a segment $[0, 1-\epsilon]$ is again $[0, 1-\epsilon]$, so this interval is a fixed point for this procedure.

Suppose that we start with J_0 and we apply recursively the above procedure to produce a sequence J_i until we obtain *fail* message or $J_i = [0, 1-\epsilon]$.

We will now show that if $d(J_0)$ is sufficiently big then we always get $J_i = [0, 1-\epsilon]$ for some i.

It is easy to see that if the above procedure returns *success* then we have

$$d(J_{i+1}) \geq \mu d(J_i)/2 - \epsilon, \quad \text{if } cut \text{ was executed} \tag{52}$$

$$d(J_{i+1}) = \mu d(J_i) - 2\epsilon, \quad \text{otherwise.} \tag{53}$$

From (??) it follows the necessary condition for constructing J_{i+1} is

$$d(J_i) > 2\epsilon/\mu \tag{54}$$

We show now that two consecutive cuts starting from J_{i-1} are possible only if $J_{i+1} = [0, 1-\epsilon]$. For the proof suppose that J_i is obtained after the cut. Then either $J_i = [0, x]$ or $J_i = [-x, 0]$ for some $x > 0$. But the image of $[-x, 0]$ is the same as image of the segment $[0, x]$, so we can assume that $J_i = [0, x]$, where $0 < x < 1 - \epsilon$. Consider now the next cut, which takes place when we construct J_{i+1}. After execution of the step 2 we have $a = t_\mu(x) + \epsilon$ and $b = 1 - \epsilon$. The condition for the cut is

$$t_\mu(x) + \epsilon < 0 < 1 - \epsilon \tag{55}$$

Let us observe that from $0 < x < 1 - \epsilon$ and since of t_μ is strictly decreasing on $\mathbb{R}_+$ we get

$$t_\mu(1-\epsilon) + \epsilon < t_\mu(x) + \epsilon < 0 \tag{56}$$

From this and (??) follows that

$$|t_\mu(x) + \epsilon| < 1 - \epsilon \tag{57}$$

So executing step 3 and 4 we obtain $J_{i+1} = [0, 1-\epsilon]$.

Let us fix $q > 1$, but close to 1. We set

$$\delta > \max\{\frac{2\epsilon}{\mu - q}, 2\epsilon(2/\mu^2 + 1/\mu), \epsilon\frac{\mu+2}{\mu^2/2 - q}\} \tag{58}$$

Suppose that $d(J_0) \geq \delta$, then we show that one of the following condition holds

$$J_1 = [0, 1-\epsilon] \tag{59}$$
$$J_1, J_2 \,\text{exist}, \quad J_2 = [0, 1-\epsilon] \tag{60}$$
$$J_1, J_2 \,\text{exist}, \quad d(J_2)/d(J_0)) > q \tag{61}$$

Suppose that (??),(??) do not hold, we have to prove (??). Since the case of two consecutive cuts was shown to lead to (??) we need to consider the following three cases.

1. there is no cut in construction of J_1 and J_2

2. there is a cut in construction of J_1

3. there is a cut in construction of J_2

ad 1. we have $d(J_1) = \mu d(J_0) - 2\epsilon$. Hence $d(J_1)/d(J_0) \geq \mu - 2\epsilon/\delta \geq q > 1$ if $\delta \geq \frac{2\epsilon}{\mu - q}$. Since length of J_1 is greater than J_0 construction of J_2 is possible and we obtain $d(J_2)/d(J_0) > q^2 > q$.

ad 2. $d(J_1) \geq \mu d(J_0)/2 - \epsilon$. From (??) it follows to construct J_2 must hold

$$d(J_1) > 2\epsilon/\mu \tag{62}$$

This forces the following condition for δ to satisfy

$$\delta > 2\epsilon(2/\mu^2 + 1/\mu). \tag{63}$$

For J_2 we get

$$d(J_2) \geq \mu(\mu d(J_0)/2 - \epsilon) - 2\epsilon = \mu^2 d(J_0)/2 - \epsilon(\mu + 2) \tag{64}$$
$$d(J_2)/d(J_0) \geq \mu^2/2 - \epsilon(\mu + 2)/\delta \tag{65}$$

and finally

$$d(J_2)/d(J_0) \geq q, \quad \text{if} \quad \delta \geq \epsilon \frac{\mu + 2}{\mu^2/2 - q} \tag{66}$$

ad 3.

$$d(J_1) \qquad = \mu d(J_0) - 2\epsilon \tag{67}$$
$$d(J_2) \qquad \geq \mu d(J_1)/2 - \epsilon \tag{68}$$
$$d(J_2) \quad \geq \mu^2 d(J_0)/2 - \epsilon(\mu + 1) \tag{69}$$

We obtain

$$d(J_2)/d(J_0) \geq \mu^2/2 - \frac{\epsilon}{\delta}(\mu + 1) \tag{70}$$

and to have $d(J_2)/d(J_0) \geq q$ we obtain for δ the condition

$$\delta \geq \epsilon \frac{\mu + 1}{\mu^2/2 - q} \tag{71}$$

Now it is easy to see that the sequence J_i will reach $[0, 1-\epsilon]$ if $d(J_0) \geq \delta$.

Letting $q \to 1$ we can set

$$\delta(\mu, \epsilon) = \max\{\frac{2\epsilon}{\mu - 1}, 2\epsilon(2/\mu^2 + 1/\mu), \epsilon\frac{\mu + 2}{\mu^2/2 - 1}\} \tag{72}$$

□

Proof of lemma ?? Let δ be that obtained in lemma ??. From lemma ?? follows that for $i = 1, \ldots, n$ there exists a sequence of segments J^i_j such that

$$J^i_0 = J^i \tag{73}$$
$$J^i_j \, t_\mu\text{-covers } \; J^i_{j+1} \; \text{ with margin } \epsilon, \text{for } j = 0, \ldots, l_{s,i} - 1 \tag{74}$$
$$J^i_{l_i} = [0, 1-\epsilon] \tag{75}$$

It follows from (??) that

$$[0, 1-\epsilon] \subset I_{\mu,\epsilon} \tag{76}$$
$$[0, 1-\epsilon] \, t_\mu\text{-covers } \; I_{\mu,\epsilon} \text{ with margin } \epsilon \tag{77}$$

We set $l = \max_i l_i + 1$. From conditions (??–??) follows that we can prolongate the sequences J^i to get

$$J^i_0 = J^i \tag{78}$$
$$J^i_j \, t_\mu\text{-covers } \; J^i_{j+1} \; \text{ with margin } \epsilon, \text{for } j = 0, \ldots, l - 1 \tag{79}$$
$$J^i_l = I_{\mu,\epsilon} \tag{80}$$

We define

$$M_j := J^1_j \times J^2_j \times \ldots \times J^n_j \tag{81}$$

Since by assumption $|G|_{[-1,1]} < \epsilon$, them from (??–??) and the definition of M_j follows

$$M_0 = M \tag{82}$$
$$M_j \; F\text{-covers } \; M_{j+1} \text{ for } j = 0, \ldots, l_s - 1 \tag{83}$$
$$M_l = I^n_{\mu,\epsilon} \tag{84}$$

□

Proof of theorem ??

Let δ be that obtained in lemma ??.

To obtain first assertion let us observe that by lemma ?? F has *TE property* for $\prod_{i=1}^n I_{\mu,\epsilon}$ and δ, so the assertion follows from theorem ??.

Now we prove the second assertion. From lemma ?? it follows that there exists a sequence of integers l_s, such that

$$N^s \xrightarrow{F^{l_s}} \prod_{i=1}^{n} I_{\mu,\epsilon} \tag{85}$$

It follows from (??) that

$$[0, 1-\epsilon]^n \subset \prod_{i=1}^{n} I_{\mu,\epsilon} \tag{86}$$

$$[0, 1-\epsilon]^n \ F\text{-covers } \ \prod_{i=1}^{n} I_{\mu,\epsilon} \text{ in } (1, \ldots, n\text{-direction} \tag{87}$$

We set $m = \max l_s$. From (??),(??),(??) it follows that

$$N^s \xrightarrow{F^{m}} \prod_{i=1}^{n} I_{\mu,\epsilon} \tag{88}$$

We can apply now theorem ?? to get desired statement.

□

Suppose now that $\mu = 2$. We are interested in the range of ϵ and δ for which assertion 2 of theorem ?? guarantee an existence of infinite number of periodic points. From the proof of lemma ?? (condition (??)) it follows that

$$\delta(2, \epsilon) = 4\epsilon \tag{89}$$

Form (??) follows that

$$\epsilon < \frac{1}{3} \tag{90}$$

It is easy to check that when (??) is true then (??) is satisfied.

Observe that $I_{2,\epsilon} = (-1 + 3\epsilon, 1 - \epsilon)$.

The the condition for the assertion 2 with $k = (2l)^n$ reads $-1 + 3\epsilon + l\delta(2, \epsilon) < 0$, since in both segments $(-1 + 3\epsilon, 0)$ and $(0, 1 - \epsilon)$ we can insert the same number l of segments of length slightly greater than $\delta(2, \epsilon)$. We obtain

$$\epsilon < \frac{1}{4l + 3} \tag{91}$$

4 Perturbations of products of map $1 - 2x^2$

The aim of this section is to extend results of section ?? to products of maps $x \mapsto 1 - 2x^2$.

Let $l : \mathbb{R} \to \mathbb{R}$ be given by

$$l(x) := 1 - 2x^2 \tag{92}$$

Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be continuous function such that

$$F(x_1, x_2, \ldots, x_n) = (l(x_1), l(x_2), \ldots, l(x_n)) + G(x_1, x_2, \ldots, x_n) \tag{93}$$

where $|G|_{[-1,1]^n} < \epsilon$. We intent to prove for such an F theorem analogous to the theorem ??.

First we need to introduce notations. Let us denote by

$$I_\epsilon := [l(1-\epsilon) + \epsilon, 1-\epsilon] = [-1 + 4\epsilon - 2\epsilon^2, 1-\epsilon]. \tag{94}$$

The conditions corresponding to (??-??) are

$$-1 + 4\epsilon - 2\epsilon^2 < 0 < 1 - \epsilon \tag{95}$$

$$|-1 + 4\epsilon - 2\epsilon^2| < 1 - \epsilon \tag{96}$$

These inequalities give

$$\epsilon < 1 - \frac{1}{\sqrt{2}} \tag{97}$$

Our task is to prove the following

Theorem 11 *Let F, G satisfy (??) and ϵ be such that (??)(??) are satisfied. Then there exists $\delta = \delta(\epsilon)$ such that*

1. *there exists $x \in \prod_{i=1}^n I_\epsilon$ such that*

- *x is periodic point for F*
- *for every closed ball $Z = cl\, B(z, 2\delta)$, $Z \subset \prod_{i=1}^n I_\epsilon$ there exists m such that $F^m(x) \in Z$*

2. *for every finite family of sets $\{N^s\}_{s=0,\ldots,k-1}$, $N^s \subset \prod_{i=1}^n I_\epsilon$, of the form*

$$N^s = J^{s,1} \times J^{s,2} \times \ldots \times J^{s,n} \tag{98}$$

where $J^{s,i}$ are closed intervals of length greater than δ and $0 \notin int J^{s,i}$ and

$$N^s \cap N^{s'} = \emptyset, \quad \text{for } s \neq s'. \tag{99}$$

There exists m and a continuous map $\pi : Inv(\bigcup_s N^s, F^m) \to \Sigma_k^+$, such that

$$\sigma \circ \pi = \pi \circ F^m \tag{100}$$

π is onto. The preimage of any periodic sequence in Σ_k^+ contains a periodic point for F^m.

We prove this theorem using the fact that the tent map t_2 conjugated to the map l on the interval $[-1,1]$. The coordinate transformation is given by

$$x = \phi(\theta) = \sin(\frac{\pi}{2}\theta) \tag{101}$$

In the sequel we will use the following convention the symbols with tilde denote objects expressed using θ coordinates. So according to this convention we have

$$\tilde{l}(\theta) = t_2(\theta). \tag{102}$$

To express ϵ and δ using θ coordinates we need the following

Lemma 12 *Let $x_1 = \phi(\theta_1)$, $x_2 = \phi(\theta_2)$. $x_1, x_2 \in [-1,1]$. Then*

$$\frac{\pi}{4}|\theta_1 - \theta_2|^2 \leq |x_1 - x_2| \leq \frac{\pi}{2}|\theta_1 - \theta_2| \tag{103}$$

Proof The right inequality follows directly from mean value theorem applied to ϕ.

Suppose $x_2 > x_1$. Let $\Delta x := x_2 - x_1$. It is easy to observe that

$$\arcsin(x_2) - \arcsin(x_2 - \Delta x) \leq \arcsin(1) - \arcsin(1 - \Delta x) \tag{104}$$

Hence

$$\theta_2 - \theta_1 \leq \frac{2}{\pi}(\arcsin(1) - \arcsin(1 - \Delta x)) = \theta(1) - \theta(1 - \Delta x) \tag{105}$$

We set $\Delta\theta := \theta(1) - \theta(1 - \Delta x)$. We have

$$1 - \Delta x = \sin(\frac{\pi}{2}(1 - \Delta\theta)) = \cos(\frac{\pi}{2}\Delta\theta) \tag{106}$$

We have the following estimate

$$\cos x \leq 1 - x^2/\pi, \text{ for } x \in [-\pi/2, \pi/2]. \tag{107}$$

From (??) and (??) we obtain

$$(\frac{\pi}{2}\Delta\theta)^2/\pi \leq \Delta x \tag{108}$$

and finally

$$\Delta\theta \leq \frac{2}{\sqrt{\pi}}\sqrt{\Delta x} \tag{109}$$

□

Proof of theorem ??

The strategy of proof is obvious. We have to go to θ variables, calculate $\tilde{\epsilon}$ and $\tilde{\delta}$ and then go back to x-variables.

The change of variables $(x_1,\ldots,x_n) = (\phi(\theta_1),\ldots,\phi(\theta_n))$ transforms the map F into a map $\tilde{F}$

$$\tilde{F}(\theta_1,\ldots,\theta_n) = (t_2(\theta_1),t_2(\theta_2),\ldots,t_2(\theta_n)) + \tilde{G}(\theta_1,\ldots,\theta_n) \tag{110}$$

such that $|\tilde{G}|_{|[-1,1]} < \tilde{\epsilon}$, where by lemma ??

$$\tilde{\epsilon} = 2\sqrt{\epsilon/\pi} \tag{111}$$

From (??) it follows that $\tilde{\epsilon} < 1/3$, so we have

$$\epsilon < \pi/36 \tag{112}$$

From (??) we have

$$\tilde{\delta} = 4\tilde{\epsilon} \tag{113}$$

From lemma ?? it follows that

$$\delta = \frac{2}{\pi}\tilde{\delta} = \frac{16}{\pi}\sqrt{\epsilon/\pi} = 16\sqrt{epsilon/\pi^3} > 2.9\sqrt{\epsilon} \tag{114}$$

□

To obtain point 2 of the assertion we have to check the following condition $k\delta < 1-\epsilon$, where $k > 1$. When we insert for δ the value $2.9\sqrt{\epsilon}$ we obtain the inequality to solve

$$2.9k\sqrt{\epsilon} < 1-\epsilon \tag{115}$$

As the solution we obtain

$$\epsilon < \left(-2.9k + \sqrt{2.9^2k^2+4}\right)^2/4 \tag{116}$$

For $k = 2$ we obtain $\epsilon < 2.8 \cdot 10^{-2}$

5 3-dim Rössler 'walking stick' map

Let us define

$$f_\mu(x) := \mu x(1-x) \tag{117}$$

We consider the following *walking stick map* introduced by Rössler in [?] $F(x,y,z) = (\bar{x},\bar{y},\bar{z})(x,y,z)$

$$\begin{aligned}
\bar{x}(x,y,z) &= f_{3.8}(x) - 0.05(y+0.35)(1-2z) && (118)\\
\bar{y}(x,y,z) &= 0.1[(y+0.35)(1-2z)-1](1-1.9x) && (119)\\
\bar{z}(x,y,z) &= f_{3.78}(z) + 0.2y && (120)
\end{aligned}$$

This map is a diffeomorphism and for the parameter values considered have two topologically expanding directions $(1,3)$. Our aim to apply theorem ?? to this map to obtain a symbolic dynamics.

Let us set $\epsilon = 0.022$, $\epsilon_1 = 0.01$, $\epsilon_2 = 0.02$.

Let us define

$$\begin{aligned}
x_{max} &:= \max_{x \in [0,1]} f_{3.8}(x) - \epsilon_1 = 3.8/4 - \epsilon_1 = 0.94 && (121)\\
x_{min} &:= f_{3.8}(x_{max}) = 0.21432 && (122)\\
y_{max} &:= 0.1095 && (123)\\
z_{max} &:= \max_{z \in [0,1]} f_{3.78}(z) - \epsilon = 3.78/4 - \epsilon = 0.923 && (124)\\
z_{min} &:= f_{3.8}(z_{max}) = 0.26864838 && (125)
\end{aligned}$$

Let $D := [x_{min}, x_{max}] \times [-y_{max}, y_{max}] \times [z_{min}, z_{max}]$.

We show that

$$|\bar{y}| < y_{max}, \quad \text{for } (x,y,z) \in D \tag{126}$$

$$\begin{gathered}
|\bar{y}| = 0.1|[(y+0.35)(1-2z)-1](1-1.9x)| \leq \\
0.1|(y_{max}+0.35)(1-2z_{max})-1||1-1.9x_{max}| < \\
0.1(0.46 \cdot 0.846 + 1)0.786 = 0.1 \cdot 1.38916 \cdot 0.786 < 0.1092
\end{gathered}$$

We show now

$$\begin{aligned}
|\bar{x}(x,y,z) - f_{3.8}(x)| &< \epsilon_2, \quad \text{for } (x,y,z) \in D && (127)\\
|\bar{z}(x,y,z) - f_{3.78}(z)| &< \epsilon, \quad \text{for } (x,y,z) \in D && (128)
\end{aligned}$$

The equation (??) follows immediately from (??). To get (??) we compute

$$\begin{gathered}
|\bar{x}(x,y,z) - f_{3.8}(x)| = |0.05(y+0.35)(1-2z)| \leq \\
0.05(y_{max}+0.35)|1-2z_{max}| < 0.05 \cdot 0.46 \cdot 0.846 < 0.0195
\end{gathered}$$

We set

$$Z_0 := [0.295, 0.5], \quad Z_1 := [0.809, 0.922]$$

It is easy to check that

$$\begin{aligned}
&Z_0 \ f_{3.78}\text{-covers } Z_1 \text{ with the margin } \epsilon && (129)\\
&Z_1 \ f_{3.78}\text{-covers } Z_0 \text{ with the margin } \epsilon && (130)
\end{aligned}$$

For $(x,y,z) \in [x_{min}, x_{max}] \times [-y_{max}, y_{max}] \times Z_0$ we have

$$\begin{gathered}
|\bar{x}(x,y,z) - f_{3.8}(x)| = |0.05(y+0.35)(1-2z)| \leq \\
0.05(y_{max}+0.35)|1-2 \cdot 0.295| < 0.05 \cdot 0.46 \cdot 0.41 < 0.0095
\end{gathered}$$

Hence

$$|\bar{x}(x,y,z) - f_{3.8}(x)| < \epsilon_1, \quad \text{for } (x,y,z) \in [x_{min}, x_{max}] \times [-y_{max}, y_{max}] \times Z_0 \tag{131}$$

We set

$$X_{00} = [0.2347, 0.5], \qquad X_{01} = [0.5, 0.7653] \tag{132}$$
$$X_1 = [0.6927, 0.94]. \tag{133}$$

Observe that X_{01} is an image of X_{00} under reflection $x \to 1-x$.

It is easy to check that

$$X_{00},\ X_{01} \text{ both } f_{3.8}\text{-cover } X_1 \text{ with a margin } \epsilon_1 \tag{134}$$
$$X_1\ f_{3.8}\text{-covers } X_{00} \cup X_{01} \text{ with a margin } \epsilon_2 \tag{135}$$

We set

$$N_{00} = X_{00} \times [-y_{max}, y_{max}] \times Z_0 \tag{136}$$
$$N_{01} = X_{00} \times [-y_{max}, y_{max}] \times Z_0 \tag{137}$$
$$N_1 = X_1 \times [-y_{max}, y_{max}] \times Z_1 \tag{138}$$

From (??), (??), (??),(??), (??),(??), (??) and (??) follows that N_{00} and N_{01} F-cover N_1 in $(1,3)$-direction, and N_1 F-covers in $(1,3)$-direction both sets N_{00} and N_{01}.

We want to apply the theorem ?? to F, $m = 2$ and the sets N_{00}, N_{01}, but $N_{00} \cap N_{01} \neq \emptyset$. We overcome this problem by observing that there exist a sets $\tilde{N}_{00} \subset \operatorname{int} N_{00}$ and $\tilde{N}_{01} \subset \operatorname{int} N_{01}$, such that $\tilde{N}_{00}$ and $\tilde{N}_{01}$ F-cover N_1 in $(1,3)$-direction. We have

$$\tilde{N}_{00} \cap \tilde{N}_{01} = \emptyset \tag{139}$$
$$\tilde{N}_{00} \xrightarrow{F^2} \tilde{N}_{00}, \tilde{N}_{01} \quad \tilde{N}_{01} \xrightarrow{F^2} \tilde{N}_{00}, \tilde{N}_{01} \tag{140}$$

and applying theorem ?? we get the following

Theorem 13 *There exists a continuous map $\pi : Inv(\tilde{N}_{00} \cup \tilde{N}_{01}, F^2) \to \Sigma_2$,*

$$\pi \circ F^2 = \sigma \circ \pi \tag{141}$$

such that $\Sigma_2 = \pi(Inv(\tilde{N}_{00} \cup \tilde{N}_{01}, F^2))$.
The preimage of any periodic sequence from Σ_2 contains periodic points of F^2.

6 Concluding remarks.

In paper we have presented a method for proving an existence of infinite number of periodic points for multidimensional nonuniformly hyperbolic dynamical systems. We applied our method to C^0-small perturbation of cartesian product of logistic maps. To get stronger results, like density of periodic points or existence dense orbit in some region of space, one have to assume that the perturbation is not only small in C^0-norm, but is also small in C^2-norm. We hope to treat this problem in another paper.

7 Appendix

7.1 Fixed point index

The tool used in proof of theorem ?? is *the fixed point index*. Below we list the basic properties of this notion. The proofs can be found find in the Dold's book [?, Ch. VII, sec. 5].

Let $D \subset \mathbb{R}^d$ be an open set and $f : D \to \mathbb{R}^d$ be continuos and compact. Suppose that

$$\mathrm{Fix}(f, D) = \{x \in D \mid f(x) = x\} \quad \text{is compact.} \tag{142}$$

Then *fixed point index for f in D* is defined. We denote it by $\mathrm{I}(f, D)$.

If $\overline{D} \subset \mathrm{dom}(f)$ then (??) follows from the condition

$$\mathrm{Fix}(f, \overline{D}) \cap \partial D = \emptyset \tag{143}$$

Let us summarize the properties of the fixed point index (see [?, Ch. VII, sec. 5]).

Index is an integer.

$$I(f, D) \in \mathbb{Z}. \tag{144}$$

Homotopy property. Let $H : [0, 1] \times D \to \mathbb{R}^d$ be continuous. Suppose that

$$\mathrm{Fix}(H, D) := \bigcup_{\lambda \in [0,1]} \mathrm{Fix}(H_\lambda, D) \quad \text{is compact} \tag{145}$$

Then

$$\forall \lambda \in [0, 1] \quad I(H_\lambda, D) = I(H_0, D) \tag{146}$$

If $[0, 1] \times \overline{D} \subset \mathrm{dom}(H)$ then (??) follows from the following condition

$$\mathrm{Fix}(H, \overline{D}) \cap \partial D = \emptyset \tag{147}$$

Excision property. Suppose that $C \subset D$, C is open and

$$\mathrm{Fix}(f,D) \subset C \tag{148}$$

Then

$$I(f,C) = I(f,D). \tag{149}$$

Fixed point property.

$$\text{if} \quad I(f,D) \neq 0, \quad \text{then} \quad \exists x \in D \quad f(x) = x \tag{150}$$

in particular

$$I(f,\emptyset) = 0 \tag{151}$$

Fixed point index for affine maps. Suppose that $f(x) = A(x-x_0)+x_0$, where A is a linear map and $x_0 \in \mathbb{R}^d$. If the equation $A(x) = x$ has no nontrivial solutions and $x_0 \in \mathrm{int}(D)$, then

$$I(f,D) = \mathrm{sgn}(\det(\mathrm{Id} - A)), \quad \text{where } Id \text{ denotes the identity map} \tag{152}$$

7.2 Continuation lemma

For an indexed family of continuous homotopies $F_j : [0,1] \times \mathbb{R}^n \to \mathbb{R}^n$, $j = 0,1,\ldots,l-1$. In the sequel we will us notation $F_\lambda(v) := F(\lambda, v)$. For an indexed family of set Z_j, $j = 0,1,\ldots,l-1$ we introduce notation

$$Z^\lambda := \{v \in Z_0 | F_{j-1,\lambda} \circ F_{j-2,\lambda} \circ \ldots \circ F_{1\lambda}(v) \in Z_j, \quad \text{for } j = 0,1,2,\ldots,l-2\}$$

The following lemma plays a crucial role in the proof of the theorem **??**

Lemma 14 *Suppose that for $j = 0,1,\ldots,l-1$ the set N_j $F_{j,\lambda}$ covers $N_{(j+1) \bmod l}$ in the $(i_1,i_2,\ldots,i_k)$-direction then then the fixed point index $I(F_{l-1,\lambda} \circ F_{l-2,\lambda} \circ \ldots \circ F_{1,\lambda}, int\, N^\lambda)$ is well defined and constant (i.e. does not depend on λ).*

Proof

Without loss of generality we can assume that $(i_1,\ldots,i_k) = (1,\ldots,k)$.

Since $N_j \in \mathcal{C}$, hence

$$N_j = \prod_{i=1}^{n} [v_{ji} - r_{ji}, v_{ji} + r_{ji}]. \tag{153}$$

For $\delta > 0$ we define

$$\begin{aligned} V(N_j,\delta) := \prod_{i=1}^k [v_{ji} - r_{ji} + \delta, \quad & v_{ji} + r_{ji} - \delta] \times \\ & \prod_{i=k+1}^n [v_{ji} - r_{ji}, v_{ji} + r_{ji}] \\ H(N_{ji},\delta) := \prod_{i=1}^k [v_{ji} - r_{ji}, \quad & v_{ji} + r_{ji}] \times \\ & \prod_{i=k+1}^n [v_{ji} - r_{ji} + \delta, v_{ji} + r_{ji} - \delta] \end{aligned}$$

For δ sufficiently small the sets $V(N_j,\delta)$, $H(N_j,\delta)$ are nonempty and

$$\partial N_j \subset V(N_j,\delta) \cup H(N_j,\delta) \tag{154}$$

Since N_j $F_{j,\lambda}$-covers $N_{(j+1) \bmod l}$ in $(1,\ldots,k)$, then we have for for $j = 0,1,\ldots,l-1$ and $\lambda \in [0,1]$ the following conditions

$$F_{j,\lambda}(V(N_j,0)) \cap N_{(j+1) \bmod l} = \emptyset \tag{155}$$
$$H(N_{(j+1) \bmod l}, 0) \cap F_{j,\lambda}(N_j) = \emptyset \tag{156}$$

From compactness of N_j we easily conclude that there exists $\delta > 0$, such that for $j = 0,1,\ldots,l-1$ and $\lambda \in [0,1]$ the following conditions hold

$$F_{j,\lambda}(V(N_j,2\delta)) \cap N_{(j+1) \bmod l} = \emptyset \tag{157}$$
$$H(N_{(j+1) \bmod l}, 2\delta) \cap F_{j,\lambda}(N_j) = \emptyset \tag{158}$$

We define

$$C_j := \prod_{i=1}^n (v_{ij} - r_{ij} + \delta, v_{ji} + r_{ji} - \delta) \tag{159}$$
$$D_j := \prod_{i=1}^n (v_{ji} - r_{ji} + 2\delta, v_{ji} + r_{ji} - 2\delta) \tag{160}$$

We have

$$B(D_j,\delta) \subset C_j, \qquad B(C_j,\delta) \subset \operatorname{Int} N_j \tag{161}$$

Obviously the sets D^λ, C^λ are both open and the set N^λ is closed for $\lambda \in [0,1]$.

We will show now that for $\lambda \in [0,1]$

$$F_{l-1,\lambda} \circ F_{l-2,\lambda} \circ \ldots F_{1,\lambda}(x) \neq x, \quad \text{provided} \quad x \in N^\lambda \setminus D^\lambda. \tag{162}$$

Suppose that (??) does not hold. Then there exists an $x \in N^\lambda \setminus D^\lambda$ such that

$$F_{l-1,\lambda} \circ F_{l-2,\lambda} \circ \ldots F_{1,\lambda}(x) = x \tag{163}$$

and at least one the following conditions holds

$$x \in N_0 \setminus D_0 \tag{164}$$

$$F_{j_0-1,\lambda} \circ F_{j_0-2,\lambda} \circ \dots F_{1,\lambda}(x) \in (N_{j_0} \setminus D_{j_0}), \quad \text{for some } 1 \leq j_0 \leq l-2 \tag{165}$$

Assume (??), proof for (??) is analogous. It follows from (??) and (??) that

$$F_{j_0-1,\lambda} \circ F_{j_0-2,\lambda} \circ \dots F_{1,\lambda}(x) \in V(N_{j_0}, 2\delta) \tag{166}$$

It follows from this and from (??)

$$F_{j_0,\lambda} \circ F_{j_0-1,\lambda} \circ \dots F_{1,\lambda}(x) \notin N_{(j_0+1) \bmod l} \tag{167}$$

So if $j_0 < l-1$ then $x \notin N^\lambda$ and if $j_0 = l-1$ then it follows from (??) that x can not be a fixed point for a map $F_{l-1,\lambda} \circ F_{l-2,\lambda} \circ \dots F_{1,\lambda}$. In both cases we get contradiction, so (??) holds.

Let $Z \subset \mathbb{R}^n$. We introduce the notation

$$\mathrm{Fix}_\lambda(Z) := \{x \in Z \quad | \quad F_{l-1,\lambda} \circ F_{l-2,\lambda} \circ \dots \circ F_{1,\lambda}(x) = x\} \tag{168}$$

From (??) it follows immediately that

$$\mathrm{Fix}_\lambda(D^\lambda) = \mathrm{Fix}_\lambda(C^\lambda) = \mathrm{Fix}_\lambda(\mathrm{int}N^\lambda) = \mathrm{Fix}_\lambda(N^\lambda). \tag{169}$$

Let $\lambda_0 \in [0,1]$. It follows from the uniform continuity of continuous maps on compact sets that there exists an interval Λ open in $[0,1]$, $\lambda_0 \in \overline{\Lambda}$, such that for every $\lambda_1, \lambda_2 \in \overline{\Lambda}$, $j = 0, 1, \dots, l-1$ and $x \in \bigcup N_j$ holds

$$|F_{j,\lambda_1} \circ F_{j-1,\lambda_1} \circ \dots \circ F_{1,\lambda_1}(x) - F_{j,\lambda_2} \circ F_{j-1,\lambda_2} \dots \circ F_{1,\lambda_2}(x)| \leq \delta. \tag{170}$$

We will show now the following inclusions for $\lambda \in \overline{\Lambda}$

$$D^\lambda \subset C^{\lambda_0} \subset \mathrm{int}(N^\lambda). \tag{171}$$

Let $x \in D^\lambda$. Then

$$F_{j,\lambda} \circ F_{j-1,\lambda} \circ \dots F_{1,\lambda}(x) \in D_{j+1}, \quad \text{for } j = 0, 1, \dots, l-2.$$

But it follows from (??), (??) that

$$F_{j,\lambda_0} \circ F_{j-1,\lambda_0} \circ \dots F_{1,\lambda_0}(x) \in C_{j+1}, \quad \text{for } j = 0, 1, \dots, l-2.$$

Thus $x \in C^{\lambda_0}$. Proof of the second inclusion is analogous.

It follows from (??)(??) for $\lambda \in \overline{\Lambda}$ that

$$\mathrm{Fix}_\lambda(C^{\lambda_0}) = \mathrm{Fix}_\lambda(N^\lambda) \tag{172}$$

Since $\mathrm{Fix}_\lambda(N_\alpha^{\lambda_0})$ is a compact set and from (??)(??) follows that the fixed point indices of the map $F_{l-1,\lambda} \circ F_{l-2,\lambda} \circ \ldots F_{1,\lambda}$ relatively to the sets $\mathrm{int} N^\lambda$, C^{λ_0}, D^λ are well defined and from the excision property of the fixed point index [?, Ch.VII Th. 5.4] we conclude

$$\begin{aligned} \mathrm{I}(F_{l-1,\lambda} \circ F_{l-2,\lambda} \circ \ldots F_{1,\lambda}, D^\lambda) = \\ \mathrm{I}(F_{l-1,\lambda} \circ F_{l-2,\lambda} \circ \ldots F_{1,\lambda}, C^{\lambda_0}) = \\ \mathrm{I}(F_{l-1,\lambda} \circ F_{l-2,\lambda} \circ \ldots F_{1,\lambda}, \mathrm{int} N^\lambda), \quad \forall \lambda \in \overline{\Lambda}. \end{aligned} \tag{173}$$

Substituting $\lambda := \lambda_0$ we derive

$$\mathrm{I}(F_{l-1,\lambda_0} \circ F_{l-2,\lambda_0} \circ \ldots F_{1,\lambda_0}, D^{\lambda_0}) = \mathrm{I}(F_{l-1,\lambda_0} \circ F_{l-2,\lambda_0} \circ \ldots F_{1,\lambda_0}, C^{\lambda_0}). \tag{174}$$

From (??)(??) it follows that

$$\mathrm{Fix}_{\overline{\Lambda}}(C^{\lambda_0}) := \bigcup_{\lambda \in \overline{\Lambda}} \mathrm{Fix}_\lambda(C^{\lambda_0}) \subset C^{\lambda_0} \tag{175}$$

Moreover, from (??)(??) it follows that

$$\mathrm{Fix}_{\overline{\Lambda}}(C^{\lambda_0}) = \bigcup_{\lambda \in \overline{\Lambda}} \mathrm{Fix}_\lambda(N^\lambda) \tag{176}$$

From the above condition and compactness of maps $F_{j,\lambda}$ we see that $\mathrm{Fix}_{\overline{\Lambda}}(C^{\lambda_0})$ is compact.

Now from the homotopy property of the fixed point index we obtain

$$\mathrm{I}(F_{l-1,\lambda} \circ F_{l-2,\lambda} \circ \ldots F_{1,\lambda}, C^{\lambda_0}) = \mathrm{I}(F_{l-1,\lambda_0} \circ F_{l-2,\lambda_0} \circ \ldots F_{1,\lambda_0}, C^{\lambda_0}) \quad \forall \lambda \in \overline{\Lambda}. \tag{177}$$

From (??),(??),(??) we conclude

$$\mathrm{I}(F_{l-1,\lambda} \circ F_{l-2,\lambda} \circ \ldots F_{1,\lambda}, D^\lambda) = \mathrm{I}(F_{l-1,\lambda_0} \circ F_{l-2,\lambda_0} \circ \ldots F_{1,\lambda_0}, D^{\lambda_0}) \quad \forall \lambda \in \overline{\Lambda}. \tag{178}$$

From the connectedness of $[0,1]$, (??) and (??) we get

$$\mathrm{I}(F_{l-1,\lambda} \circ F_{l-2,\lambda} \circ \ldots F_{1,\lambda}, \mathrm{Int} N^\lambda) = \mathrm{I}(F_{l-1,0} \circ F_{l-2,0} \circ \ldots F_{1,0}, \mathrm{Int} N^0) \qquad \forall \lambda \in [0,1]$$

This finishes the proof. □

7.3 Proof of topological theorem

Proof of theorem ??

As we can assume that that $(i_1, \ldots, i_k) = (1, \ldots, k)$. Let

$$N_j = \prod_{i=1}^{n} [v_{ji} - r_{ji}, v_{ji} + r_{ji}]. \tag{179}$$

We want to connect maps F_j by homotopy with affine maps M_j for which we can easily calculate the fixed point index of interest.

Let us choose $d < u \in \mathbb{R}$ such that for every j $N_j \subset [d,u]^n$.

We start now to define affine map $M_j : \mathbb{R}^n \to \mathbb{R}^n$ as a cartesian product of one-dimensional affine maps $M_j(x_1,\dots,x_n) = (m_{j1}(x_1),\dots,m_{jn}(x_n))$. First we define m_{ji} for $i > k$ (i is not an expanding dimension). In this case we set $m_{ji}(x_i) = v_{ji}$.

If $i \leq k$ (i is exapanding direction), then if N_j F_j-covers $N_{(j+1) \bmod l}$ in the i-direction preserving orientation then we define

$$m_{ji}(v_i - r_i) := d, \quad m_{ji}(v_i + r_i) := u \tag{180}$$

in the other case (orientation reversed) we set

$$m_{ji}(v_i - r_i) := u, \quad m_{ji}(v_i + r_i) := d \tag{181}$$

We now extend the maps m_{ji} to affine maps defined on $\mathbb{R}$.

We define the a family of homotopies $G_j : [0,1]\times\mathbb{R}^n \to \mathbb{R}^n$ as a convex combination of M_j and G_i, so

$$G_j(\lambda, x) := (1-\lambda)F_j(x) + \lambda M_j(x) \tag{182}$$

It follows from the construction of M_j and G_j easily that for every $\lambda \in [0,1]$

$$\begin{array}{ll} & \text{if } N_j \ F_j\text{-covers the set } N_{(j+1) \bmod l}. \text{ in } (1,\dots,k)\text{-direction} \\ \text{then} & N_j \ G_{j,\lambda}\text{-covers the set } N_{(j+1) \bmod l} \text{ in } (1,\dots,k)\text{-direction} \end{array} \tag{183}$$

So we can apply to G_j lemma **??** to obtain

$$I(F_{l-1} \circ F_{l-2} \circ \dots \circ F_1, \mathrm{int} N^0) = I(M_{l-1} \circ M_{l-2} \circ \dots \circ M_1, \mathrm{int} N^1) \tag{184}$$

So to finish the proof it remains to calculate the fixed point index of the superposition of M_j's.

Let us set for $p = 1,\dots,l-2$

$$Y_p := \{x \in N_0 \mid M_s \circ M_{s-1} \circ \dots \circ M_1(x) \in N_{s+1}, \quad \text{for } s = 1,\dots,p\} \tag{185}$$

For $p = 0$ we set $Y_0 = N_0$. Obviously $Y_{l-2} = N^1$.

It is easy to prove inductively for $p = 0,\dots,l-2$,

$$Y_p = \prod_{i=1}^{k} J_{p,i} \times \prod_{i=k+1}^{n} [v_{0,i} - r_{0,i}, v_{0,i} + r_{0,i}] \tag{186}$$

$$\begin{aligned} M_p \circ \ & M_{p-1} \circ \dots \circ M_1(Y_p) = \\ & \prod_{i=1}^{k} [v_{p+1,i} - r_{p+1,i}, v_{p+1,i} + r_{p+1,i}] \times \prod_{i=k+1}^{n} \{v_{p+1,i}\} \end{aligned} \tag{187}$$

where $J_{p,i} \subset [v_{0,i} - r_{0,i}, v_{0,i} + r_{0,i}]$ for $i = 1, \ldots, k$.

From the definition of M_j it follows that there exist a family of affine maps $a_i : \mathbb{R} \to \mathbb{R}$, $i = 1, \ldots, n$ such that

$$M_{l-1} \circ M_{l-2} \circ \ldots \circ M_1(x_1, \ldots, x_n) = (a_1(x_1), a_2(x_2), \ldots, a_n(x_n)) \tag{188}$$

Moreover, it follows from (??) and (??) for $\lambda = 1$ that

$$\begin{gathered} M_{l-1} \circ M_{l-2} \circ \ldots \circ M_1(Y_{l-2}) = \\ \prod_{i=1}^{k} [d, u] \times \prod_{i=k+1}^{n} \{v_{0,i}\} \end{gathered} \tag{189}$$

So we obtain

$$J_{l-2,i} \subset \operatorname{int} a_i(J_{l-2,i}), \quad \text{if } i \leq k \tag{190}$$

$$a_i(x_i) = v_{0i} \in \operatorname{int} J_{l-2,i}, \quad \text{otherwise} \tag{191}$$

From this and (??) it follows that

$$I(M_{l-1} \circ M_{l-2} \circ \ldots \circ M_1, \mathrm{int} N^1) = \pm 1 \tag{192}$$

□

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