COMPOSED ENSEMBLES OF RANDOM UNITARY MATRICES

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Composed ensembles of random unitary matrices are defined via products of matrices, each pertaining to a given canonical circular ensemble of Dyson. We investigate statistical properties of spectra of some composed ensembles and demonstrate their physical relevance. We discuss also the methods of generating random matrices distributed according to invariant Haar measure on the orthogonal and unitary group.

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I. INTRODUCTION

Random unitary matrices are often used to describe the process of chaotic scattering [1,2], conductance in mesoscopic systems [3] and statistics of quantum, periodically driven systems (see [4] and references therein). They may be defined by circular ensembles of unitary matrices, first considered by Dyson [5]. He defined circular orthogonal, unitary or symplectic ensembles (COE, CUE and CSE), which display different transformation properties [6]. Distribution of matrix elements and their correlations are known for these canonical ensembles [7–10].

Our investigations are motivated by many successful applications of the Random Matrix Theory to problems of quantum chaos, i.e. to the description of quantum properties of systems chaotic in the classical limit. Random matrices of three canonical circular ensembles appear to provide quantitatively verifiable predictions concerning statistical properties of quasi-energy spectra, transition amplitudes etc. for quantum chaotic systems [4]. For systems without generalized time–reversal symmetry one should use CUE, while COE consisting of unitary symmetric matrices corresponds to the time reversal invariant systems (with integer spin). The so-called circular Poissonian ensemble (CPE) of diagonal unitary matrices with independent unimodular eigenvalues has also found applications for certain classically integrable systems.

In this paper we shall study statistical properties of composed ensembles defined by products of unitary matrices, each drawn with a given probability distribution. Products of matrices come into attention in a natural way when we consider the evolution of kicked systems. Unitary propagators transporting wave functions of such systems over one period of the kicking perturbation are products of "free" evolution propagators and unitary transformations corresponding to instantaneous kicks. Moreover, products of two unitary matrices appear also in the theory of chaotic scattering [11,12].

The paper is organized as follows. In section 2 we briefly recall the necessary definitions and introduce notation. Section 3 contains results concerning spectral properties of composed ensembles of random unitary matrices. The paper is completed by concluding remarks. In appendices we review methods of generating random matrices according to the invariant Haar measures on the orthogonal and unitary group.

II. CANONICAL ENSEMBLES OF UNITARY MATRICES

Circular ensembles of matrices where defined by Dyson [5] as the subsets of the set of unitary matrices. Uniqueness of the ensembles is imposed by introducing measures invariant under appropriate groups of transformations [13]. Specifically the circular unitary ensemble (CUE) consists of all unitary matrices with the (normalized) Haar measure μ_U on the unitary group U_N. The circular orthogonal ensemble (COE) is defined on the set S_N of all symmetric unitary matrices $S = S^T = (S^\dagger)^{-1}$ by the property of being invariant under all transformations by an arbitrary unitary matrix W,

$$S \to W^T S W, \tag{2.1}$$

where ^{T} denotes the transposition. The normalized measure on COE will be denoted by μ_S.

Eigenvalues of an $N \times N$ unitary matrix lie on the unit circle, $\lambda_i = \exp(i\varphi_i)$; $0 \le \varphi_i \le 2\pi$, $i = 1, \ldots, N$. The joint probability distribution (JPD) of eigenvalues for each ensemble was given by Dyson [5]

$$P_\beta(\varphi_1, \ldots, \varphi_N) = C_\beta \prod_{i<j} |e^{i\varphi_i} - e^{i\varphi_j}|^\beta, \tag{2.2}$$

where C_β is a normalization constant and β equals to 1 and 2 for COE and CUE, respectively. This number is sometimes called repulsion parameter, since it determines the behaviour of levels spacings as $P(s) \sim s^\beta$ for small s [6].

The above formula with $\beta = 0$ describes spectra of circular Poissonian ensemble (CPE) of diagonal unitary matrices with N independent phases drawn with uniform distribution in $[0, 2\pi)$. The set of diagonal matrices will be denoted by D_N and the normalized measure on CPE, (which is simply the product measure of N measures on the unit circle,) by μ_D. For further consideration we will also need an ensemble of orthogonal matrices with the probability density μ_O defined by the (normalized) Haar measure on the orthogonal group in N dimension. We shall call this ensemble as Haar orthogonal ensemble (HOE). It is invariant with respect to all transformations $O_1 \to O_2 O_1 O_3$, where O_2 and O_3 denote arbitrary orthogonal matrices. The joint distribution of eigenvalues of this ensemble $P_{ort}(\varphi_1, \ldots, \varphi_N)$ can be found in the book of Girko [14] and is recalled in the appendix A. In this appendix we propose a method of generating such matrices numerically and study some properties of their spectra.

III. SPECTRA OF PRODUCTS OF MATRICES OF CIRCULAR ENSEMBLES

A. Notation

We are interested in the spectral properties of products of unitary matrices, each pertaining to a given ensemble. Let us introduce a following notation: D denotes a diagonal unitary matrix of CPE, S denotes a symmetric matrix of COE, U represents a unitary matrix of CUE and O an orthogonal matrix typical to HOE. As usual, the symbol SU represents a product of two concrete matrices S and U. On the other hand $S*U$ will denote the composed ensemble of unitary matrices defined as the image of the mapping

$$S_N \times U_N \ni (S,U) \mapsto SU \in U_N \tag{3.1}$$

with the measure induced by this mapping in its image by the product measure $\mu_S \times \mu_U$ on the Cartesian product $S_N \times U_N$. Indices can be added to any matrix, if needed. For example $S_1 S_2$ denotes a product of two symmetric matrices, while $S_1 * S_2$ represents the composed ensemble defined as the image of $S_N \times S_N$, which is different from the ensemble of squared symmetric matrices $S_1 * S_1$ obtained by the mapping

$$S_N \ni S \mapsto S^2 \in U_N. \tag{3.2}$$

B. Results

No	Composed ensemble	Measure	Spectrum	Remarks
A1	U	μ_U	P_2	CUE
A2	$U*S$	μ_U	P_2	a)
A3	$U*D$	μ_U	P_2	a)
A4	$U*O$	μ_U	P_2	a)
A5	U_1*U_2	μ_U	P_2	CUE
A6	X_1*U*X_2	μ_U	P_2	b)
B1	S	μ_S	P_1	COE
B2	U^T*U	μ_S	P_1	d)
B3	U^T*D*U	μ_S	P_1	d)
B4	$S*D$	μ_S	P_1	g)
B5	S_1*S_2	?	P_1	f)
B6	$S_1*S_2*S_1$	?	P_1	e)
B7	$S_1^\alpha*S_2*S_1^\alpha$	?	P_1	e)
B8	X^T*S*X	?	P_1	e)
C1	S_1*S_2*D	?	P_2	n_1)
C2	$S_1*S_2*S_3$	?	P_2	n_1)
C3	S_1*S_1	?	-	h)
C4	S_1*S_1*D	?	P_1	n_2)
C5	S_1*D*S_1	?	P_1	n_2)
D1	D_1*D_2	μ_D	P_0	c)
D2	O_1*O_2	μ_O	P_{ort}	c)
D3	$O*S$	?	P_2	n_3)
D4	$O*S*D$	?	P_2	n_3)
D5	O_1*D*O_2	?	P_2	n_4)
D6	$O*S_1*O^T*S_2$	?	P_1	e), f) (see B8,B5)
D7	$D_1*O*D_2*O^T$	?	P_1	n_5)
D8	$O*D_1*O^T*D_1$	?	P_1	n_6)
D9	$D_1*O_1*D_2*O_1^T*O_2*D_3*O_2^T$	?	P_2	n_7)
D10	$U*D_1*U^T*D_2$	?	P_1	d), g) (see B3,B4)
D11	$U*D_1*U^\dagger*D_2$	?	P_2	n_8)
D12	$S*D_1*S^\dagger*D_2$	?	P_2	n_9)

TABLE I. Composed ensembles, their measures (? represents an unknown measure), and their joint probability distribution of eigenvalues. Apart of ensembles defined in the text, symbol X represents an arbitrary ensemble of unitary matrices and α denotes an arbitrary positive real number.

Main results of this paper concerning the spectra of products of unitary matrices are collected in Table 1. For convenience we added also some previously known results. JPD P_β represents the formula (2.2), which depending on β describes properties of all canonical ensembles. Last column of the table gives a reference to the further text. Some items have not been proved rigorously yet, but are based on numerical results.

We shall start the discussion of above results with an important note. The fact that the joint probability distribution of eigenvalues characteristic to a given composed ensemble is same as, for example, for CUE, does not mean at all that the measures of both ensemble are the same. In other words, if probability measures of two ensembles are equal ($\mu_a = \mu_b$), then the corresponding JPD are the same ($P_a = P_b$). Reverse is not true, what explains why composition of ensembles is not transitive. For example JPD of S is the same as for $S_1 * S_2$ but differs from this for $S_1 * S_2 * S_3$.

C. Remarks and references

Detailed remarks and references to the table are collected below.

a) Let us consider an arbitrary subset X of the unitary group U_N with an arbitrary measure μ_X, and the mapping:

$$f : U_N \times X \ni (U, A) \mapsto UA \in U_N \tag{3.3}$$

The product measure $\mu_{U\times X} = \mu_U \times \mu_X$ in $U_N \times X$ induces a measure in the image of f i.e. in U_N. Since μ_U is left-invariant i.e. invariant with respect to the left multiplication by $V \in U_N$ the same is true for the product measure i.e. $\mu_U \times \mu_X$ is invariant under the transformation $(U, A) \mapsto (VU, A)$. In consequence also the measure induced on U_N is left-invariant. There is only one (normalized) left-invariant measure on U_N - the Haar measure, hence the resulting ensemble $U * A$ is CUE. The cases (A2-A5) from the table are particular examples.

b) Since the Haar measure on U_N is also right-invariant an analogous reasoning shows that $B * U$ gives the CUE ensemble for an arbitrary ensemble of unitary matrices X from which the matrices B are drawn. Further, since $U * A$ and $B * U$ are CUE so is $B * U * A$ for A and B from arbitrary ensembles X_1 and X_2 of unitary matrices (The case A6 from the table above).

c) Similar results are valid for diagonal (or orthogonal) matrices. We must only substitute in the previous reasoning, CUE by CPE (or the ensemble of orthogonal matrices) with measures μ_D (or μ_O) and X by an arbitrary subset of diagonal (or orthogonal) matrices. The cases D1 and D2 from the table correspond to this situation.

d) It is easy to prove [15] that the mapping

$$g : U_N \ni U \mapsto U^T U \in S_N \tag{3.4}$$

induces in its image (the full set of symmetric unitary matrices) the COE measure μ_S i.e. in our notation $U^T * U$ =COE. This corresponds to the B2 and B3 in the table. In the latter case let's observe that $U^T D U = V^T V$, where $V = D^{1/2} U$ and $D^{1/2}$ denotes an arbitrary diagonal unitary matrix such that $D^{1/2} D^{1/2} = D$. The mapping

$$U_N \times D_N \mapsto V = D^{1/2} U \in U_N \tag{3.5}$$

induces, according to b), μ_U in U_N which reduces B3 to B2 with U substituted by V.

e) Let, as previously, X denote an arbitrary subset of U_N with an arbitrary probabilistic measure μ_X. From a) it is now clear that the composite mapping $g \circ f$ where f and g are given by (3.3) and (3.4)

$$g \circ f : U_N \times X \ni (U, A) \mapsto A^T U^T U A \in S_N \tag{3.6}$$

induces COE measure μ_S in the image S_N. Consider now two following mappings

$$\begin{aligned} h &: U_N \times X \ni (U, A) \mapsto (U^T U, A) \in S_N \times X \\ k &: S_N \times X \ni (S, A) \mapsto A^T S A \in S_N . \end{aligned} \tag{3.7}$$

According to the above, h induces in its image the measure $\mu_S \times \mu_X$. Since $g \circ f = k \circ h$ they induce the same measure in their image S_N and, as a consequence, k induces μ_S in S_N i.e. in our notation $A^T * S * A$ =COE for A from an arbitrary ensemble X. This corresponds to the case B8 in the table and its special forms B6 and B7.

f) Until now we considered the situations where the ensemble obtained by multiplication of matrices coincided with CUE, COE or CPE. Our main interest consists, however, in examination of statistical properties of spectra of resulting matrices. This allows us to investigate more general situations in which products either do not have specific symmetry properties or the induced measure is not equal to μ_U, μ_O, μ_S or μ_D. As an example let us consider the mapping

$$s : S_N \times S_N \ni (S_1, S_2) \mapsto S_1 S_2 \in U_N . \tag{3.8}$$

Observe that the image of s is the whole set U_N. Indeed, it is enough to show that an arbitrary unitary matrix U is a product of two symmetric unitary matrices. To this end lets denote by W an arbitrary unitary matrix which diagonalizes U (such a matrix W exists since U is unitary) i.e.

$$U = WDW^\dagger, \quad WW^\dagger = W^* W^T = I \tag{3.9}$$

where D is diagonal and unitary. Now take $S_1 = WW^T$ and $S_2 = W^* D W^\dagger$. Both S_1 and S_2 are unitary and symmetric and $S_1 S_2 = WDW^\dagger = U$. Nevertheless the measure induced on U_N by the COE measures $\mu_S \times \mu_S$ on $S_N \times S_N$ is not equal to CUE measure μ_U. Indeed, for all S_1, S_2 the matrix $S_1 S_2$ is unitary similar to $S_1^{1/2} S_2 S_1^{1/2}$, where $S_1^{1/2}$ is an arbitrary unitary, symmetric matrix such that $S_1^{1/2} S_1^{1/2} = S_1$, (such a unitary, symmetric $S_1^{1/2}$ exists since S_1 is unitary and symmetric). It means that the spectra of $S_1 S_2$ and $S_1^{1/2} S_2 S_1^{1/2} = (S_1^{1/2})^T S_2 S_1$ coincide. But from e) above we know that the mapping

$$S_N \times X \ni (S_2, S_1^{1/2}) \mapsto (S_1^{1/2})^T S_2 S_1 \in S_N \tag{3.10}$$

induces COE measure μ_S in the image S_N for S_2 from COE and arbitrary X. It follows that the eigenvalues of $(S_1^{1/2})^T S_2 S_1 = S_1 S_2$ are distributed according to (2.2) with $\beta = 1$, which, on one side, proves that the the mapping does not give CUE and, on the other side, covers the case B5 from the table.

g) Similar reasoning proves the validity of B4. Indeed, observe that since $S = U^T U$ for some unitary U the matrix $SD = U^T U D$ is unitary similar to to UDU^T, but from already proven case B3 from the table we know that such multiplication produces COE.

h) A superposition of two spectra has JPD different from canonical P_β. The two level correlations can be expressed as combination of correlations of both initial spectra (with rescaled argument) [16], while level spacing distribution may be obtained as a special case of Berry–Robnik distribution [17] (for two equal chaotic layers).

n_i**)** Conjectures based on numerical results. Conjectures indexed by the same index are equivalent. Random orthogonal matrices where generated as described in Appendix A. A modified version of an algorithm for generation of random unitary matrices, first presented in Ref. [15], is given in the appendix B. We generated several realizations of discussed products, diagonalized them numerically and compared the level spacing distribution $P(s)$ and number variance $\Sigma^2(L)$ with known predictions of canonical ensembles [6]. Our numerical results are valid thus in the limit of large N (practically $N \approx 20$ and larger). We have performed additional cross-checking by repeating calculations (with similar results) using random matrices generated out of eigenvectors. In order to verify or reject hypothesis concerning properties of the spectra the long range correlations where found to be more informative than spacing distribution. In Fig. 1 we display number variance averaged over spectra of exemplary composed ensembles - ($O * S, S_1 * S_2 * D, U * D * U^\dagger * D$) typical of CUE, and other ($D_1 * O * D_1 * O^T$, $D_1 * O * D_2 * O^T$) typical of COE.

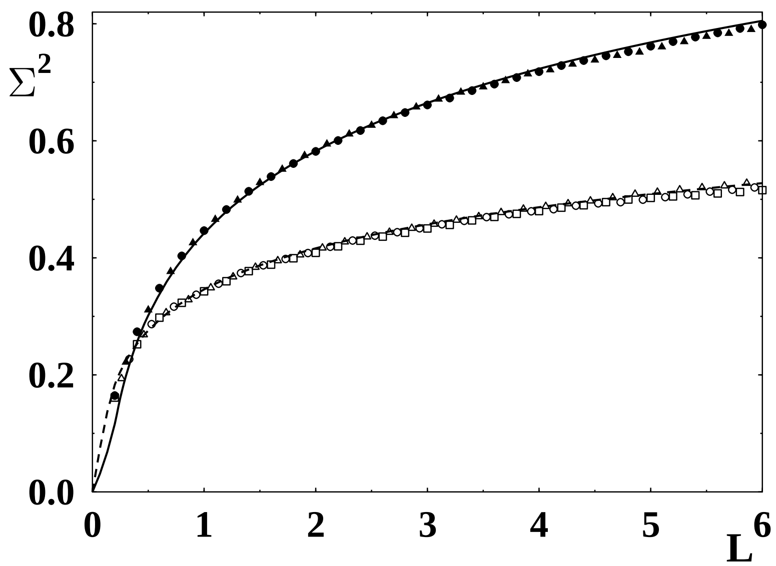

FIG. 1. Number variance $\Sigma^2(L)$ for three ensembles $D3(\square)$, $C1(\triangle)$, and $D11(\circ)$ with JPD P_2 typical of CUE, and two ensembles $D7$ and $D8$ (full symbols) with COE like JPD P_1. Solid and dashed lines stand for RMT results for COE and CUE.

m) Consider composed ensemble defined as a product of n matrices, each pertaining to a given ensemble. For large n we expect the product to be distributed uniformly with respect to the Haar measure, thus displaying the CUE-like spectral fluctuations. This remark obviously hold if at least one matrix belongs to CUE (see ensemble A6). On the other hand, it does not hold if all n matrices belong to the Poissonian ensemble, since their product displays the JPD P_0.

D. Intermediate ensembles

Observe that the JPD of the composed ensembles A5, B5 and D1 can be written as

$$P[U_\beta * U_\beta] = P[U_{\beta'}] \tag{3.11}$$

with $\beta = \beta'$ equal to $2, 1,$ and 0. The number β, characterizing the degree of the level repulsion, (2.2), for ensembles interpolating between CPE and CUE may take any real value in $[0, 2]$.

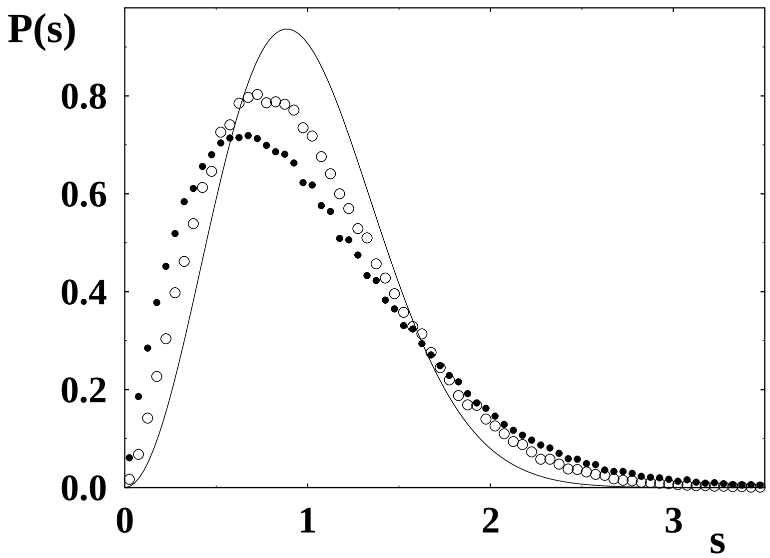

FIG. 2. Level spacing distribution $P(s)$ for an ensemble U_δ interpolating between Poisson–CUE ($\bullet$), and the composed ensemble $U_\delta * U_\delta$ ($\circ$) for the transition parameter $\delta = 0.5$. Solid line represents CUE distribution.

In order to investigate, to what extend the formula (3.11) may be generalized, we constructed numerically random unitary matrices pertaining to ensembles interpolating between CPE and CUE as described in appendix B. Fig. 2 presents level spacing distribution taken of 3000 matrices of size $N = 50$, while the value of the parameter δ, controlling the transition CPE-CUE, is set to 0.5. Level spacing distribution of the composed ensemble defined *via* product of such two independent matrices is represented by open symbols. It is closer to the CUE prediction and can be approximated by distribution typical of another ensemble with larger value of the control parameter δ. In other words, for this family of interpolating ensembles the relation (3.11) seems to hold with β' being an unknown function of β satisfying $\beta' \geq \beta$.

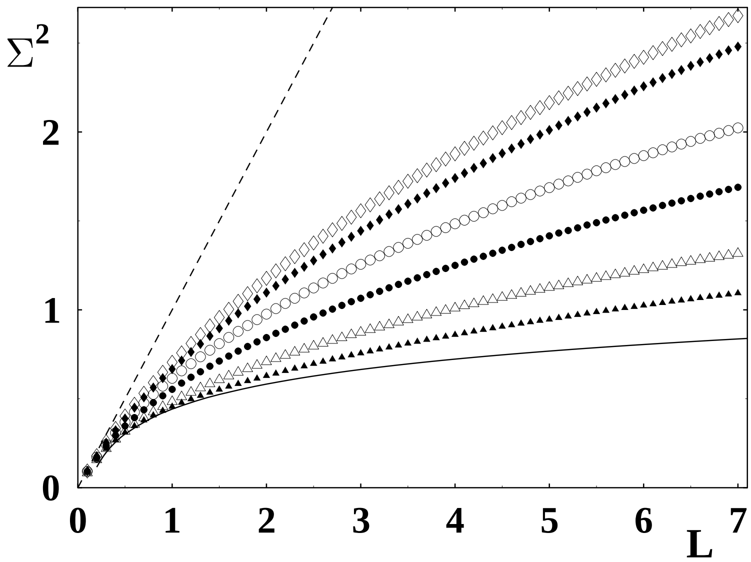

FIG. 3. Number variance $\Sigma^2(L)$ for interpolating ensembles U_δ (open symbols) and the corresponding composed ensembles $U_\delta * U_\delta$ (closed symbols) for $\delta = 0.1$ ($\diamond$), $\delta = 0.3$ ($\circ$) and $\delta = 0.7$ (∇). Dashed and solid lines stand for Poisson and CUE results, respectively.

Further tests of the long range correlations of the spectra allowed us to support this conjecture. Figure 3 shows the number variance $\Sigma^2(L)$ [6] for simple and composed interpolating ensembles for three values of the control parameter. In every case the spectra of products of two matrices (full symbols) are less rigid that the spectra of the simple interpolating ensemble (open symbols). This property can be understood realizing that such interpolating unitary matrices enjoy band structure, as demonstrated in Fig. 4 for an exemplary matrix of size $N = 35$. Vaguely speaking, a product of two band matrices possess a band of a double width, and the spectral properties of composed ensembles are thus closer to these typical of CUE.

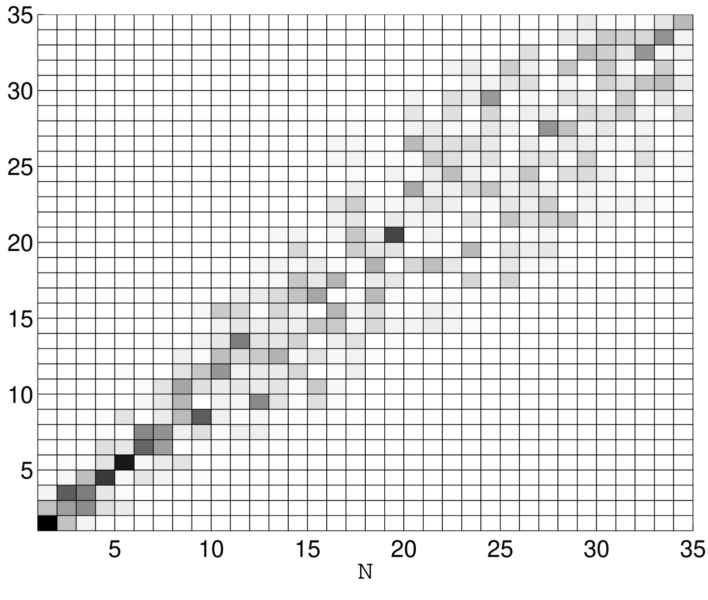

FIG. 4. Squared moduli of elements of a random matrix $|U_{kl}|^2$ taken from an interpolating ensemble U_δ with $\delta = 0.5$. Observe a band structure of the unitary matrix U.

E. Physical applications

Describing quantized physical systems one encounters often a structure of one of the above mentioned composed ensembles. Analyzing a concrete physical system we deal with deterministic matrices, so the assumptions concerning randomness of each matrix forming the composed ensemble can not be rigorously fulfilled. It seems however, that the assumptions concerning randomness are too strong: we provide below examples of quantum systems which are characterized by JPD found for an appropriate composed ensemble, although some composing matrices do not display required properties of presupposed canonical ensembles. To show this one may study the statistical properties of a *semiclassical ensemble*, i.e. the properties of several quantum realizations of the same classical system, distinguished only by different values of the relative Planck constant (spin length).

Let us start the discussion analyzing periodically time-dependent quantum systems. Generally speaking, JPD P_1 corresponds to fully chaotic systems with (generalized) time reversal invariance, while spectrum characterized by P_2 provides an evidence that such a symmetry has been broken. Let us consider the composed ensemble D6. A single orthogonal matrices O pertaining to HOE does not often appear alone in the theory, nevertheless the compositions $O * D_2 * O^T$ are crucial for many important models. Consider an exemplary periodically kicked system described by a Hamiltonian $H = H_0 + kV \sum_n \delta(t - nT)$. Its free evolution is represented by $U_1 = \exp(TH_0)$ and the perturbation term can be written as $U_2 = \exp(ikV)$, where V is a symmetric operator and k is the perturbation strength. It is natural to represent the system in the eigenbasis of H_0 so the unitary matrix $D_1 = \exp(iTH_0)$ is diagonal. Orthogonal rotation O allows one to change the basis into eigenbasis of V and obtain eigenvalues of U_2. Note that discussed ensemble $D_1 * O * D_2 * O^T$ corresponds just to the Floquet operator

$$F = e^{itH} e^{ikV} \tag{3.12}$$

of such a system. We can therefore expect that if both operators H and V sufficiently do not commute (so as to assure that the transition matrix O is generic in sense of μ_O), than for generic values of the parameters t and k the operators $\exp(itH)$ and $\exp(ikV)$ are "relatively random" [18] and the system described by Floquet operator F is chaotic. In fact the structure (3.12) is typical to several models for quantum chaos discussed in the literature (kicked rotator [19], kicked top [20,4], kicked Harper model [21]).

In ensemble D6 it is assumed that the diagonal matrices D_i are random. In the simplest chaotic kicked top model F_1, defined by the angular momentum operators J_x, J_y, J_z acting on $2j + 1$ dimensional Hilbert space as: $F_1 = \exp(itJ_z)\exp(ikJ_x^2/2j)$ [20], the diagonal matrix D_2 reads $(D_2)_{lm} = \delta_l^m \exp(ikl^2/2j)$. Due to the factor l^2 in the exponent for a generic value of the parameter k the diagonal elements of the matrix D_2 are pseudorandom [22] what assures the COE-like spectral fluctuations of the orthogonal top $F1$.

To observe the P_1 JPD of eigenvalues characteristic to the composed ensemble D6 it is therefore sufficient, if at least one of the matrices D_1 and D_2 is pseudorandom. On the other hand, if both diagonal matrices $(D_i)_{ll}$ have the structure $\exp(ikl)$, the resulting operator F_1' does not pertain to COE, what corresponds to the integrability of the kicked top with $V' = J_x$ [4].

In order to get a CUE spectrum it is necessary to break the time reversal symmetry (or any generalized antiunitary symmetry) [4]. As follows from example D8 this can be done by adding additional unitary term generated by a kick - perturbation $\tilde{V}$ not commuting with H nor with V. This scheme corresponds exactly to the so–called unitary kicked top given by [20]

$$F_2 = e^{ik_1 J_x^2/2j} e^{itJ_z} e^{ik_2 J_y^2/2j}, \tag{3.13}$$

with $k_1 \neq k_2$ (and arbitrary order of unitary factors), or CUE version of kicked rotator [19].

According to remark f) the systems which can be brought to a symmetric COE -like structure by a similarity transformation display spectra described by P_1 JPD. Therefore example B5 represented by $S_1 S_2 \sim S_1^{1/2} S_2 S_1^{1/2}$ leads to COE spectrum, in contrast to example C1: $S_1 S_2 D$, for which such a transformation is not possible. In the same spirit it is sufficient to modify slightly the system (3.13) into $F_3 = e^{ik_1 J_x^2/2j} e^{itJ_z} e^{ik_2 J_x^2/2j}$, or $F_4 = e^{ikJ_x^2/2j} e^{itJ_z} e^{ikJ_y^2/2j}$, so as it recovers the generalized antiunitary symmetry and its spectrum pertains to COE.

Any "unitary" top F_u, without time-reversal symmetry, may be artificially made symmetric by adding the same sequences F_u of perturbation in the reverse order. Therefore $F = F_u F_u^T$ displays COE like fluctuations of the spectra. Mathematical theory of time reversible and irreversible tops is given in [20], while some further examples where numerically studied in [23].

A product of two symmetric random matrices $S_1 S_2$ arises in the theory of chaotic scattering [11,12,24]. Its spectrum obeys COE statistics, as follows from the example B5. The same statistics is characteristic to several versions of quantized Baker map [25–27], which is also represented by a product of two symmetric matrices $B = \mathcal{F}_1 \mathcal{F}_2$, although both matrices $\mathcal{F}_1$ and $\mathcal{F}_2$, defined via Fourier matrices, do not show the properties of COE.

As a last example let us consider the piecewise affine transformation of the torus, which can be quantized as [27] $T = D_2 \mathcal{F}^\dagger D_1 \mathcal{F}$. Diagonal matrices, of the type discussed above, $(D_1)_{ll} = \exp(ial^2)$ are pseudorandom for a generic value of the parameter a. Albeit the symmetric Fourier matrix $\mathcal{F}$ is not typical to COE, the structure of T resembles the ensemble D12, and its spectrum confers to the predictions of CUE.

IV. CONCLUDING REMARKS

Let us conclude our paper with the following, summarizing remarks. Various statistical properties of products of random matrices can be interesting when studying quantum chaotic systems influenced by symmetry breaking perturbations. We showed that using our results we can predict properties of spectra of a large class of periodically driven model systems (kicked tops).

From the mathematical point of view our investigations leave many questions open. Not in all cases we were able to calculate the resulting probability distribution of the composite ensembles. In fact it was possible only in those cases where the distribution coincided with one of the "classical" ones (COE, CUE, CPE, HOE). In some cases for which we did not find the probability distribution of the ensemble we were nevertheless able to give the corresponding distribution of the eigenvalues, from which the most popular statistical measure of quantum chaotic systems, namely the distribution of neighboring levels, is easily calculable. For some other composed ensembles we provided numerical evidence for their distribution of eigenvalues applying efficient methods of constructing of random unitary ensembles of all canonical ensembles. Further investigation should resolve the problem of the full probability distributions for these composed examples and find analytical arguments for distributions of eigenvalues founded numerically.

ACKNOWLEDGMENTS

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APPENDIX A: RANDOM ORTHOGONAL MATRICES

The distribution of eigenvalues in the ensemble of random orthogonal matrices can be found in the books of Girko [14,28]. We shall give here the relevant result for completeness. The distribution density of matrices in the ensemble is the (normalized) Haar measure on the orthogonal group $o(N)$. The simpler situation occurs for N odd. In this case among the eigenvalues there is one (say ϕ_0) equal 1 or -1. The rest of eigenphases can be grouped into pairs $(\phi_i, -\phi_i), -\pi < \phi_i < \pi,\ i = 1, \dots, (N-1)/2$. With the probability 1 they are not degenerate and distributed independently of eigenvectors. The joint probability distribution of eigenphases reads

$$P(\phi_1, \dots, \phi_{(N-1)/2}, \pm 1) = \mathcal{N} \prod_{n=1}^{(N-1)/2} \left((1 \pm 1) \sin^2 \frac{\phi_n}{2} + (1 \mp 1) \cos^2 \frac{\phi_n}{2} \right) |\sin \phi_n| \prod_{k<n} \sin^2 \frac{\phi_n - \phi_k}{2} \sin^2 \frac{\phi_n + \phi_k}{2}, \quad \text{(A1)}$$

where the last argument ± 1 and the alternative signs in the rest of the formula refer to $\phi_0 = 1$ or $\phi_0 = -1$. For the slightly more complicated case of N even consult the above cited books of Girko.

In order to generate numerically a random orthogonal matrix typical of HOE we employed a parametrisation of the orthogonal group defined by Hurwitz in the classical paper [29] published exactly one hundred years ago. An arbitrary N dimensional orthogonal matrix O may be written as a product of $N(N-1)/2$ elementary orthogonal rotations in two-dimensional subspaces. The matrix of such an elementary orthogonal rotation will be denoted by $F^{(i,j)}(\psi)$. The only nonzero elements of $F^{(i,j)}$ are

$$\begin{aligned} F^{(i,j)}_{kk} &= 1, \quad k = 1, \dots, N; \quad k \neq i, j \\ F^{(i,j)}_{ii} &= \cos\psi, \quad F^{(i,j)}_{ij} = \sin\psi, \\ F^{(i,j)}_{ji} &= -\sin\psi, \quad F^{(i,j)}_{jj} = \cos\psi. \end{aligned} \quad \text{(A2)}$$

From these transformations one constructs the following $N-1$ composite orthogonal rotations

$$
\begin{aligned}
F_1 &= F^{(N-1,N)}(\psi_{01}),\\
F_2 &= F^{(N-2,N-1)}(\psi_{12})F^{(N-1,N)}(\psi_{02}),\\
F_3 &= F^{(N-3,N-2)}(\psi_{23})F^{(N-2,N-1)}(\psi_{13})F^{(N-1,N)}(\psi_{03}),\\
&\dots\\
F_{N-1} &= F^{(1,2)}(\psi_{N-2,N-1})F^{(2,3)}(\psi_{N-3,N-1})\dots F^{(N-1,N)}(\psi_{0,N-1}),
\end{aligned}
\tag{A3}
$$

and finally forms the orthogonal transformation O as

$$
O = F_1 F_2 F_3 \dots F_{N-1}. \tag{A4}
$$

Uniform distribution with respect to the Haar measure on the orthogonal group is achieved if the generalized Euler angles ψ_{0s} are uniformly distributed in the interval $0 \leq \psi_{0s} < 2\pi$, and the remaining angles ψ_{rs} (for $r > 0$) are taken from the interval $[0, \pi]$ according to the measure $d\mu_r = (\sin\psi_{rs})^r d\psi_{rs}$ [29]. An alternative way to generate random orthogonal matrices was recently proposed by Heiss [30]. Random orthogonal matrices may also be obtained as eigenvectors of real random symmetric matrices typical to Gaussian orthogonal ensemble.

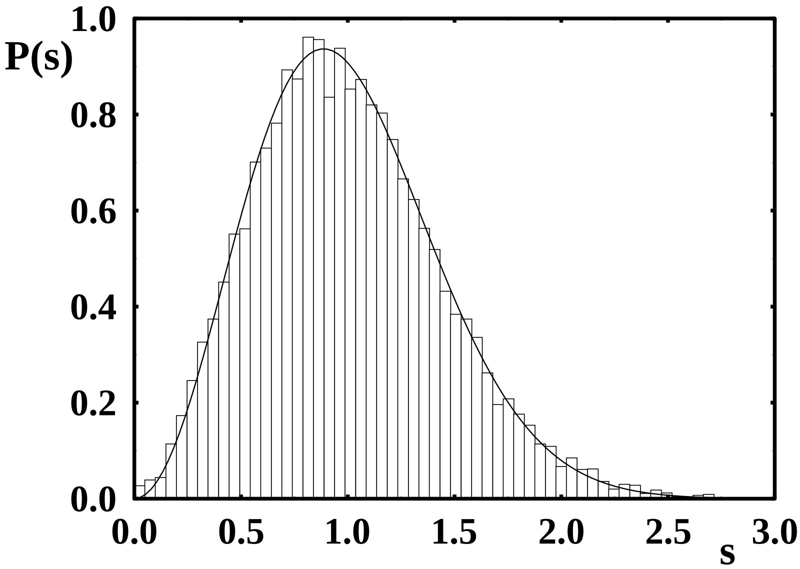

FIG. 5. Level spacing distribution $P(s)$ for an ensemble of orthogonal matrices distributed uniformly with respect to the Haar measure (histogram) may be approximated by the CUE formula (solid line).

Figure 5 presents the level spacing distribution obtained of 20000 random orthogonal matrices obtained by Hurwitz parametrisation for $N = 41$. Numerical data suggest that statistical properties of the spectra of random orthogonal matrices are close to the CUE predictions.

APPENDIX B: RANDOM UNITARY MATRICES

In our earlier paper [15] we have also used the Hurwitz [29] parametrisation to generate random unitary matrices. We present it here in details for completeness of the present paper and since in the text of [15] a slightly different, not yet verified algorithm appeared (the numerical calculation, however, were based on the prescription given below).

An arbitrary unitary transformation U can be composed from elementary unitary transformations in two-dimensional subspaces. The matrix of such an elementary unitary transformation will be denoted by $E^{(i,j)}(\phi, \psi, \chi)$. The only nonzero elements of $E^{(i,j)}$ are

$$
\begin{aligned}
E^{(i,j)}_{kk} &= 1, \quad k = 1, \dots, N; \quad k \neq i, j\\
E^{(i,j)}_{ii} &= \cos\phi e^{i\psi},\\
E^{(i,j)}_{ij} &= \sin\phi e^{i\chi},
\end{aligned}
\tag{B1}
$$

$$E_{ji}^{(i,j)} = -\sin\phi e^{-i\chi},$$
$$E_{jj}^{(i,j)} = \cos\phi e^{-i\psi}.$$

From the above elementary unitary transformations one constructs the following $N-1$ composite rotations

$$\begin{aligned} E_1 &= E^{(N-1,N)}(\phi_{01}, \psi_{01}, \chi_1), \\ E_2 &= E^{(N-2,N-1)}(\phi_{12}, \psi_{12}, 0) E^{(N-1,N)}(\phi_{02}, \psi_{02}, \chi_2), \\ E_3 &= E^{(N-3,N-2)}(\phi_{23}, \psi_{23}, 0) E^{(N-2,N-1)}(\phi_{13}, \psi_{13}, 0) E^{(N-1,N)}(\phi_{03}, \psi_{03}, \chi_3), \\ &\dots \\ E_{N-1} &= E^{(1,2)}(\phi_{N-2,N-1}, \psi_{N-2,N-1}, 0) E^{(2,3)}(\phi_{N-3,N-1}, \psi_{N-3,N-1}, 0) \dots \\ &\quad \dots E^{(N-1,N)}(\phi_{0,N-1}, \psi_{0,N-1}, \chi_{N-1}), \end{aligned} \tag{B2}$$

and finally forms the unitary transformation U as

$$U = e^{i\alpha} E_1 E_2 E_3 \dots E_{N-1}. \tag{B3}$$

The angles $\alpha, \phi_{rs}, \psi_{rs}$, and χ_{1s} are taken uniformly from the intervals

$$0 \le \psi_{rs} < 2\pi\delta, \quad 0 \le \chi_{1s} < 2\pi\delta, \quad 0 \le \alpha < 2\pi\delta, \tag{B4}$$

whereas

$$\phi_{rs} = \arcsin(\xi_{rs}^{1/2r}), \quad r = 1, 2, \dots, N-1, \tag{B5}$$

with ξ_{rs} uniformly distributed in

$$0 \le \xi_{rs} < \delta. \tag{B6}$$

If the parameter δ is set to unity then the obtained matrix is drawn from the Circular Unitary Ensemble [29].

In order to obtain a a family of ensembles interpolating between diagonal matrices of CPE and generic unitary matrix typical of CUE we construct a product $U_\delta = D\hat{U}_\delta$. Diagonal matrix D is typical of CPE, while the matrix $\hat{U}_\delta$ is obtained according the above procedure with real parameter $\delta \in (0,1)$ determining the intervals in Eq. (B4) and (B6). Varying the value of this parameter from zero to unity one obtains a continuous interpolation between CPE and CUE [31].

Random unitary matrices may be also constructed by taking N eigenvectors of random Hermitian matrix pertaining to the Gaussian unitary ensemble. In this procedure one must specify N arbitrary phases of each eigenvector. This method, albeit simple, does not allow to control parameters of the interpolating ensemble as it is possible for the Hurwitz algorithm discussed above.

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