Mean dynamical entropy of quantum system tends to infinity in the semiclassical limit

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Abstract

We show that the mean dynamical entropy of a quantum map on the sphere tends logarithmically to infinity in the semiclassical limit. Consequences of this fact for classical dynamical systems are discussed.

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I. INTRODUCTION

Quantum analogues of classically chaotic systems have been an object of intensive investigations for almost twenty years. One has studied statistical properties of the spectra of quantized chaotic systems trying to prove that these systems can be described by suitable ensembles of random matrices (see e.g. [1–3]). In this paper we follow the opposite direction: studying a generic quantum system we find strong arguments which support the conclusion that the dynamical entropy of the corresponding classical system is positive and, actually, arbitrary large. More precisely, we analyze the set of all *structureless* (without geometric or time reversal symmetries) quantum systems [4]. For these systems, described by the ensemble of unitary matrices, we compute the mean dynamical entropy averaged over the Haar measure. We show that it increases logarithmically with the dimension of the Hilbert space where a system lives, and so tends to the infinity in the semiclassical limit.

The dynamical entropy we used depends on the choice of *coherent states*. We prove our assertion for the families of $SU(d)$ coherent states with $d \geq 2$. This includes the well known spin $SU(2)$ coherent states, where the corresponding classical phase space is the two-dimensional sphere S^2. In the present work we only outline the main ideas of the proof putting apart the details of our reasoning to a forthcoming publication.

II. CS-ENTROPY

The attempts to give a quantum analogue of the classical Kolmogorov–Sinai (KS) entropy have a rich history [5–7]. Most of these definitions do not provide a good framework for investigating the *quantum chaos*. Some of them, as the Connes-Narnhofer-Thirring entropy [8] or the Alicki-Fannes [9] entropy, vanish for finite quantum systems and can be rather applied in quantum statistical mechanics. Others do not give the correct semiclassical limit. Most of them are not easy to calculate neither analytically nor numerically, but for very simple cases. In a series of papers [6,10,11] we proposed a new definition of dynamical quantum entropy based on the notion of coherent states, which we would like to recall now. Our approach to quantum entropy is based on the assumption that the knowledge of the time evolution of a quantum state is obtained by performing a sequence of approximate quantum measurements. The possible results of the measurement forms the (coarse-grained) stochastic phase-space being the quantum counterpart of the classical one. The evolution of the system between two subsequent measurements is governed by a unitary matrix.

Let $\mathcal{H}$ be an N-dimensional Hilbert space (which represents the *kinematics* of the quantum system), let U be a unitary operator on $\mathcal{H}$ (which describes the *dynamics*), let Ω be a compact *phase space* endowed with a probability measure m (we shall write dx for $dm(x)$), and let $\Omega \ni x \longrightarrow |x\rangle \in \mathcal{H}$ be a *family of coherent states* (which are related to an *approximated quantum measurement*), i.e., $\int_\Omega |x\rangle\langle x|\, dx = I$ and $x \longrightarrow |x\rangle$ is continuous. Moreover, we assume in this paper that $\langle x|x\rangle \equiv N$.

Let $\mathcal{A} = \{E_1, \ldots, E_k\}$ be a partition of Ω. We define the *coherent states (CS) entropy of U with respect to the partition $\mathcal{A}$* by the formula

$$H(U, \mathcal{A}) := \lim_{n\to\infty}(H_{n+1} - H_n) = \lim_{n\to\infty} \frac{1}{n} H_n \ , \tag{2.1}$$

where the partial entropies H_n are given by

$$H_n := \sum_{i_0,\ldots,i_{n-1}=1}^{k} -P^{CS}(i_0, \ldots, i_{n-1}) \ln P^{CS}(i_0, \ldots, i_{n-1}) \ , \tag{2.2}$$

for $n \in \mathbf{N}$; the probabilities $P^{CS}(i_0, \ldots, i_{n-1})$ of entering the cells $E_{i_0}, \ldots, E_{i_{n-1}}$ are

$$P^{CS}(i_0, \ldots, i_{n-1}) := \int\limits_{E_{i_0}} dx_0 \ldots \int\limits_{E_{i_{n-1}}} dx_{n-1} \prod_{u=1}^{n-1} K_U(x_{u-1}, x_u) \ , \tag{2.3}$$

for $i_j = 1, \ldots, k,\ j = 0, \ldots, n-1$; and the kernel K_U is given by

$$K_U(x, y) := \frac{1}{N} |\langle y|U|x\rangle|^2 \tag{2.4}$$

for $x, y \in \Omega$. The kernel $K_U(x, y)$ may be interpreted as an y-dependent *Husimi distribution* of the transformed state $U|x\rangle$. If U equals to the identity operator I, the quantity $K_I(x, y)$ is called the *overlap* of coherent states $|x\rangle$ and $|y\rangle$.

Note that both sequences in (2.1) are decreasing and the quantity $H_1 = -\sum_{i=1}^{k} m(E_i) \ln m(E_i)$, which does not depend on U, is just the *entropy of the partition* $\mathcal{A}$. We denote it by $H(\mathcal{A})$.

There are two kinds of randomness in our model: the first is connected with the underlying unitary dynamics of the system; the second comes from the approximate measurement process. Accordingly, we divide CS-entropy with respect to a partition into two components: *CS–measurement entropy* and *CS–dynamical entropy*:

$$H_{meas}(\mathcal{A}) := H(I, \mathcal{A}) \ ; \tag{2.5}$$

$$H_{dyn}(U, \mathcal{A}) := H(U, \mathcal{A}) - H_{meas}(\mathcal{A}) \ . \tag{2.6}$$

Finally, we define the partition independent *CS–dynamical entropy of* U as

$$H_{dyn}(U) := \sup_{\mathcal{A}} H_{dyn}(U, \mathcal{A}) \ , \tag{2.7}$$

the supremum being taken over all finite partitions.

It is conjectured that in the semiclassical limit the CS-dynamical entropy tends to the KS-entropy, if the unitary dynamics comes from an appropriate quantization procedure (some results in this direction were proved in [6]). In [10,11] we study the properties of CS-dynamical entropy and present the methods of its numerical computing based on the concept of iterated function systems (IFS). In this work we evaluate the mean value of CS-dynamical entropy $\langle H_{dyn}(U)\rangle_{U(N)}$, taking the average over the unitary matrices $U(N)$ of the circular unitary ensemble (CUE).

III. SU(D) - COHERENT STATES

We study CS-dynamical entropy for the family of $SU(d)$ coherent states, $d \geq 2$ [12]. Let $SU(d) \ni x \longrightarrow T_x \in U(\mathcal{H}_M)$ be the irreducible representation of the group $SU(d)$ in the group of unitary operators acting on Hilbert space $\mathcal{H}_M$, where $dim(\mathcal{H}_M) = N = \binom{M+d-1}{M}$, $M = 1, 2, ...$ We can identify the phase space Ω with the coset space $SU(d)/U(d-1)$, where $U(d-1)$ is the maximal stability subgroup of $SU(d)$ with respect to the reference state $|\kappa\rangle \in \mathcal{H}_M$, i.e., the subgroup of all elements of $SU(d)$ which leave $|\kappa\rangle$ invariant up to a phase factor. The coherent states are defined by $|x\rangle = T_x|\kappa\rangle$ for $x \in SU(d)$. The space Ω which plays the role of the phase space of corresponding classical mechanics is isomorphic to the complex projective space CP^{d-1}. Hence, each point of Ω can be interpreted as a pure quantum state in a d - dimensional complex Hilbert space $\mathbf{C}^d$. One can show that the overlap of two coherent states related to pure quantum states φ and ψ is given by $|\langle\langle\varphi|\psi\rangle\rangle|^{2M}/N$, where $\langle\langle\cdot|\cdot\rangle\rangle$ is the canonical scalar product in $\mathbf{C}^d$ [12]. The *semiclassical limit* is obtained when $M \to \infty$, and M^{-1} plays the role of the relative Planck constant. The above construction may be treated as a particular case of the general construction of group-theoretic coherent states [13].

If $d = 2$, then $\Omega = SU(2)/U(1)$ is simply isomorphic to the two-dimensional sphere S^2, and the coherent states are ordinary *spin coherent states* (see [13] and also [10]). In this case $dim(\mathcal{H}_M) = N = M + 1 = 2j + 1$, where $j = \frac{1}{2}, 1, \frac{3}{2}, ...$ is the spin quantum number, the operators T_x are represented by the Wigner rotation matrices and for the state $|\kappa\rangle$ one usually takes the maximal eigenstate $|j, j\rangle$ of the component J_z of the angular momentum operator.

IV. CONTINUOUS ENTROPY

Computing the CS-dynamical entropy requires the time limit: $n \to \infty$. Surprisingly, one can obtain bounds for this quantity analyzing the *continuous entropy* of U, which depends only on the one-step evolution of the quantum system:

$$H_U := -\int_\Omega \int_\Omega K_U(x,y) \ln K_U(x,y) dx dy \ . \tag{4.1}$$

This quantity is related to the "classical-like" entropy introduced to quantum mechanics by Wherl in [14]. Namely, H_U is equal to the difference of the Wherl entropy of the states $U|x\rangle$ averaged over all x from Ω and $\ln N$ (the latter term follows from the normalization in (2.4)). A similar quantities have been also studied by Schroeck [15] (under the name of stochastic quantum mechanical entropy) and by Korsch and Mirbach [16]. Calculation of continuous entropy is particularly easy for $U = I$ and $SU(d)$ coherent states. In this case $K_I(x,y) = |\langle\kappa|T_y^{-1}T_x|\kappa\rangle|^2/N$ for any points x, y belonging to the phase space Ω and so

$$H_I = -\int_\Omega \int_\Omega \frac{|\langle\kappa|T_{y^{-1}x}|\kappa\rangle|^2}{N} \ln \frac{|\langle\kappa|T_{y^{-1}x}|\kappa\rangle|^2}{N} dx\, dy = -\int_\Omega \frac{|\langle\kappa|T_z|\kappa\rangle|^2}{N} \ln \frac{|\langle\kappa|T_z|\kappa\rangle|^2}{N} dz. \tag{4.2}$$

We can now apply the formula for the overlap of two $SU(d)$ coherent states (see above), which enables us to use the result from Jones [17,18] who calculated generalized mean entropy of pure quantum state in d-dimensional complex Hilbert space. Proceeding in this way we get

$$H_I = -\ln N + M \left[\Psi\left(M+d\right) - \Psi\left(M+1\right)\right] \ , \tag{4.3}$$

where $N = dim(\mathcal{H}_M) = \binom{M+d-1}{M}$ and Ψ is the digamma function, satisfying $\Psi(x+1) = \Psi(x) + \frac{1}{x}$ for every real x. If $d = 2$ the above formula reduces to

$$H_I = -\ln N + \frac{N-1}{N} \ . \tag{4.4}$$

V. BOUNDS FOR CS-DYNAMICAL ENTROPY

Let $\mathcal{A}$ be a partition of Ω and let U be a unitary operator on $\mathcal{H}_M$. Using classical methods from the information theory (see [19], Sect. 2.2) we can prove the following observation:

$$\inf_{\mathcal{A}} \left[H_{n+1}\left(U,\mathcal{A}\right) - H_n\left(U,\mathcal{A}\right) - H\left(\mathcal{A}\right)\right] = H_U \tag{5.1}$$

for each natural n, where the coherent states partial entropies $H_n(U,\mathcal{A})$ are defined in (2.2). Hence and from the definition of CS-dynamical entropy we get

$$H_U + H(\mathcal{A}) \le H(U,\mathcal{A}) \le H(\mathcal{A}) \tag{5.2}$$

and

$$\inf_{\mathcal{A}} [H(U,\mathcal{A}) - H(\mathcal{A})] = H_U \ . \tag{5.3}$$

In fact, the infimum in (5.1) and (5.3) is achieved if the maximal diameter of a member of the partition $\mathcal{A}$ tends to zero. Thus for a sufficiently fine partition the dynamical CS-entropy splits approximately into two parts: the one which depends only on the partition, and the other depending only on the dynamics. Combining the above formulae with the analogous obtained for $U = I$ we conclude that

$$-H_I + H_U \le H_{dyn}(U) \le -H_I \ . \tag{5.4}$$

The famous Lieb conjecture says that for $d = 2$ the Wherl entropy attains its minimum equal to $H_I + \ln N = (N-1)/N$ for any coherent state (4.4) (see [20], and [21] for partial results). We conjecture that this is also true for $d > 2$ and the minimum of the Wehrl entropy for $SU(d)$ is equal to H_I given by Eq.(4.3) plus $\ln N$. This would imply $H_I \le H_U$, and consequently$H_{dyn}(U) \ge 0$ for every unitary matrix U.

As we can see above the quantity H_I decreases approximately as $-\ln N$ and so, if the generalized Lieb conjecture is true, then the entropy $H(U,\mathcal{A})$ is limited from below by $H(\mathcal{A}) - \ln N$. This agrees with the bound obtained by Halliwell for the information of phase space distributions derived from the probabilities for quantum histories [22]. Note, however, that the bound (5.2) seems to be much precise, because, as we will show, $-H_U$ is typically much smaller then $-H_I$.

Averaging Eq.(5.4) over the set of all unitary matrices $U(N)$ with respect to the Haar measure μ we get

$$-H_I + \langle H_U \rangle_{U(N)} \le \langle H_{dyn}(U) \rangle_{U(N)} \le -H_I \ , \tag{5.5}$$

Thus, to obtain the desired bounds for the mean CS-dynamical entropy, it suffices to calculate $\langle H_U \rangle_{U(N)}$. We have

$$\langle H_U \rangle_{U(N)} = - \int\limits_{U(N)} \left(\int\limits_{\Omega} \int\limits_{\Omega} K_U(x,y) \ln K_U(x,y) dx dy \right) d\mu(U) \ . \tag{5.6}$$

Since $K_U(x,y) = |\langle y|U|x\rangle|^2/N = |\langle\kappa|T_y^{-1}UT_x|\kappa\rangle|^2/N$, interchanging the order of integration and using the invariance of the Haar measure we conclude that

$$\begin{aligned}\langle H_U\rangle_{U(N)} &= -\int\limits_{\Omega}\int\limits_{\Omega}\left(\int\limits_{U(N)}\frac{|\langle\kappa|T_y^{-1}UT_x|\kappa\rangle|^2}{N}\ln\frac{|\langle\kappa|T_y^{-1}UT_x|\kappa\rangle|^2}{N}d\mu(U)\right)dxdy\\ &= -\int\limits_{U(N)}\frac{|\langle\kappa|V|\kappa\rangle|^2}{N}\ln\frac{|\langle\kappa|V|\kappa\rangle|^2}{N}d\mu(V)\ . \end{aligned}\tag{5.7}$$

We can calculate the last quantity utilizing the formulae for the distribution of $\langle\kappa|U|\kappa\rangle$ given by Kuś et al. [23]. Otherwise, we can use the already mentioned result of Jones [18]. Applying one of these methods we get the following formula:

$$\langle H_U\rangle_{U(N)} = -\ln N + \Psi(N+1) - \Psi(2)\ . \tag{5.8}$$

Finally from (4.3), (5.5) and (5.8) we obtain the main result of this work: a lower and an upper bound for the mean dynamical entropy

$$l_b\ \le\ \left\langle H_{dyn}^{SU(d)}\right\rangle\ \le\ u_b, \tag{5.9}$$

where

$$\begin{aligned} l_b &= \Psi(N+1) - \Psi(2) - M\left[\Psi(M+d) - \Psi(M+1)\right],\\ u_b &= \ln N - M\left[\Psi(M+d) - \Psi(M+1)\right]. \end{aligned}\tag{5.10}$$

where $N = \binom{M+d-1}{M}$.

In the semiclassical limit $M\to\infty$ we obtain simple approximations for both bounds

$$l_b \sim \ln N - d + \gamma\ ,\quad \text{and}\quad u_b \sim \ln N - d + 1\ , \tag{5.11}$$

where γ is the Euler constant. The difference between an upper bound (which is actually the maximal value of the CS-dynamical entropy!) and a lower one converges to the constant $1-\gamma\ \simeq\ 0.42278$ if $M\to\infty$. Hence the mean value of CS-dynamical entropy tends in the semiclassical limit to the infinity exactly as $\ln N$.

Let us consider the case $d=2$, where the family of spin coherent states is parametrized by the points lying on the two-dimensional sphere S^2. The mean entropy of quantum maps on the sphere is thus bounded by

$$\Psi(N+1) - \Psi(2) - 1 + \frac{1}{N} \le \left\langle H_{dyn}^{SU(2)}\right\rangle_{U(N)} \le \ln N - 1 + \frac{1}{N}. \tag{5.12}$$

FIGURES

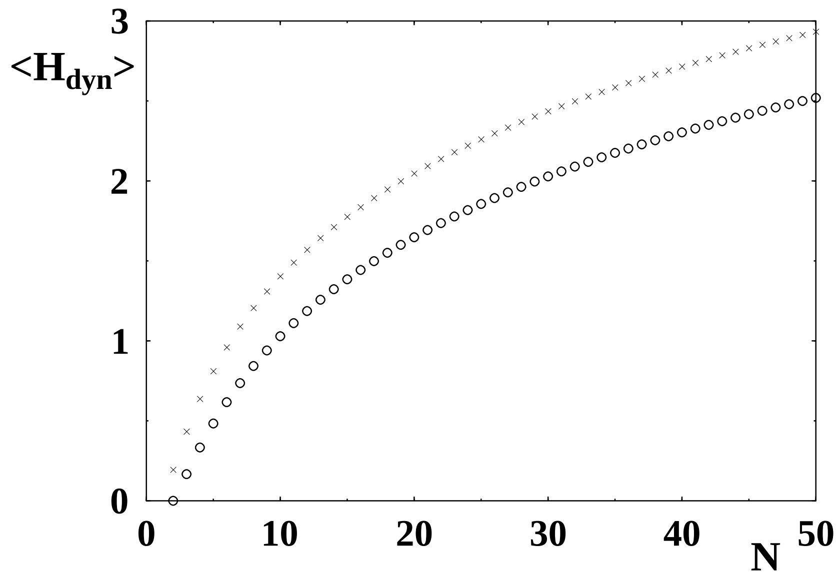

FIG. 1. Upper ($\times$) and lower ($\circ$) bounds for the mean CS-dynamical entropy of unitary matrices representing structureless quantum systems on the sphere as a function of the matrix dimension $N = 2j + 1$.

The dependence of both bounds on the quantum number $N = 2j+1$ is presented in Fig. 1. In the semiclassical limit $N \to \infty$ the mean dynamical entropy diverges in contrast to the CS-dynamical entropy of a given quantum map, which converges to the KS-entropy of the corresponding classical system. Therefore, for sufficiently large N a matrix F representing a given quantum map must differ from a generic (with respect to the Haar measure on $U(N)$) unitary matrix. To visualize the difference we present in Fig. 2 the Husimi function of an exemplary coherent state $|\vartheta, \phi\rangle = |1.6, 3.4\rangle$ transformed once by a Floquet operator F representing the kicked top [2] in classically chaotic regime (a), and by a random unitary matrix U (b). The sphere is represented in the Mercator projection with $0 \le \phi < 2\pi$ and $0 \le \vartheta < \pi$, $t = \cos\vartheta$. In the former case the wave packet remains localized in the vicinity of the classical trajectory, while in the latter, it is entirely delocalized already after one iteration. The same data plotted in the log scale allow one to detect the zeros of Husimi functions [24]. For the quantum map F they form a regular spiral-like structure (c) in contrast to the random distribution over the entire phase space for the unitary map U (d).

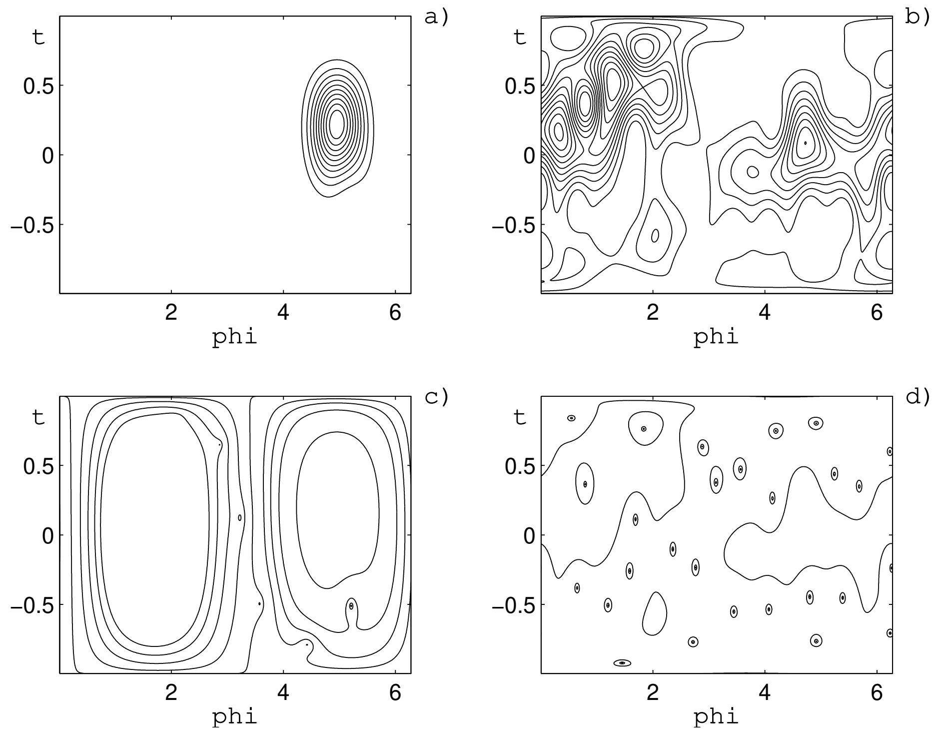

FIG. 2. Contour plot of the Husimi function of an exemplary coherent state transformed by the quantum kicked top model (a) and by a generic random matrix (b) of size $N = 30$. Observe qualitative differences in the distribution of the zeros of Husimi function visible in figures (c) and (d), respectively, obtained from the same data using a log scale for the contour heights.

VI. CONCLUSIONS

Obtained estimation (5.9) allows us to conclude that a quantum system represented by a typical unitary matrix from CUE ensemble is characterized by positive dynamical entropy, which is only insignificantly smaller than the maximal diverging with $M \sim 1/\hbar$. In other words, a generic quantum system is almost as chaotic, as possible. We prove this for $SU(d)$ coherent states, but the method seems to work also in the general case, i.e., for coherent states defined on arbitrary homogenous compact manifold, as well as for the orthogonal and symplectic ensembles.

At a first glance this result seems to be paradoxical as the KS-entropy of a classical map is finite and the CS-dynamical entropy of the corresponding quantum system tends to this value in the semiclassical limit. Hence for a Hilbert space of the sufficiently large dimension matrices representing a quantum analogue of a given classical chaotic system can not be typical. Their entropy is substantially smaller than the CUE average, even though many other statistics (level spacing distribution, spectral rigidity, etc.) conform to the predictions of random matrix theory.

However, this need not contradict the general believe that quantum analogues of classically chaotic systems might be represented by a typical unitary matrix. The paradox can be resolved if we assume that strongly chaotic systems dominate less chaotic ones in the 'space' of classical systems defined on the corresponding symplectic manifold. Thus, our results provide a strong argument in favor of ubiquity of chaos in the classical mechanics.

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