ZEROS OF POLYNOMIALS APPROXIMATING ANALYTIC FUNCTIONS

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Abstract. We give some sufficient conditions for the analyticity of an extension of a continuous function f outside a compact set $E \subset \mathbb{C}^N$ in terms of the zeros of polynomials of best uniform or least square approximation to f on the set E.

1. Introduction

The constructive theory of functions discusses the relationship between the regularity of a given function f and the rate of convergence to zero of the sequence

$$\operatorname{dist}(f, \mathcal{A}_n) = \inf\{\mathrm{d}(f, p) \ : \ p \in \mathcal{A}_n\},$$

where $(\mathcal{A}_n)$ is most often a sequence of finite dimensional subspaces of a complete metric space (X, d), which is increasing with respect to inclusion, $\mathcal{A}_k \subset \mathcal{A}_{k+1}$, and whose union $\bigcup_{n\in\mathbb{N}_0} \mathcal{A}_k$ is dense in X.

In the case of polynomial approximation of functions defined on a compact set $E \subset \mathbb{C}$, an answer to the problem is given by the Bernstein-Walsh theorem ([11]), which may be to some extent enriched by the information about the distribution of the zeros of polynomials of best or near best approximation. This was first observed in one-dimensional case by Bernstein [1], who stated the following

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Theorem 1.1. *Let f be a complex-valued continuous function defined on the real interval $I = [-1,\ 1] \subset \{z \in \mathbb{C} : \Re z = 0\}$. Then f admits an analytic continuation onto interior of the ellipse $E_\rho = \{z \in \mathbb{C}\ : |z-1| + |z+1| \leq 2\rho\}$ if and only if all limit points of the zeros of polynomials which are nearest to f in a weighted L^2-norm on I lay outside E_ρ.*

This result was a subject of numerous attempts to generalize it. Pleśniak [6] extended Bernstein's theorem to the case of polynomial approximation in a $L^2(E, \mu)$-norm for an arbitrary compact set $E \subset \mathbb{C}$ and a finite positive measure μ satisfying a *Leja polynomial condition.* Borwein [4] proved Bernstein's theorem for uniform polynomial approximation on the interval $I = [-1,\ 1]$. Later, Blatt and Saff [2] and, independently, Wójcik [12] generalized Borwein's result in the case of polynomial approximation of an analytic function on a compact and *regular* (with respect to the Dirichlet problem) subset $E \subset \mathbb{C}$.

However, all these results concerned the approximation by holomorphic polynomials in one complex variable.

In this paper we give some sufficient conditions for the existence of the holomorphic extension of a function f continuous on a compact set $E \subset \mathbb{C}^N$ in terms of the zeros of polynomials p_n of best uniform approximation (Theorem 3.4) or the zeros of polynomials of least square approximation (Theorem 5.4) on the set E and two constants $\rho(E)$ and $\rho_*(E)$ which depend on the geometry of the set E.

2. Definitions and preliminary remarks

Let $\mathbb{C}^N \ni z \to |z|$ be a fixed norm in $\mathbb{C}^N$. We shall denote by

$$\hat{p_n}(z) = \sum_{|\alpha|=n} a_\alpha z^\alpha \in \mathcal{H}_n(\mathbb{C}^N)$$

the homogeneous part of degree n of the polynomial

$$p_n(z) = \sum_{|\alpha|\leq n} a_\alpha z^\alpha \in \mathcal{P}_n(\mathbb{C}^N),$$

where $\mathcal{P}_n(\mathbb{C}^N) = \{\sum_{|\alpha|\leq n} a_\alpha z^\alpha,\ a_\alpha \in \mathbb{C},\ \alpha \in \mathbb{N}_0^N,\ |\alpha| = \alpha_1 + \cdots + \alpha_N \leq n\}$ is the set of all polynomials of degree at most n, $n \in \mathbb{N}_0$, and

$$\mathcal{H}_k(\mathbb{C}^N) = \{\sum_{|\alpha|=k} a_\alpha z^\alpha,\ a_\alpha \in \mathbb{C},\ \alpha \in \mathbb{N}_0^N, |\alpha| = \alpha_1 + \cdots + \alpha_N = k\}$$

is the set of all homogeneous polynomials of degree k, $k \geq 0$. Here $\mathbb{N}_0$ denotes the set of all nonnegative integers and $z^\alpha = z_1^{\alpha_1} \ldots z_N^{\alpha_N}$.

Moreover, by $\mathcal{P} = \mathcal{P}(\mathbb{C}^N) = \bigcup_{n\geq 0} \mathcal{P}_n(\mathbb{C}^N)$ and $\mathcal{H} = \mathcal{H}(\mathbb{C}^N) = \bigcup_{k\geq 0} \mathcal{H}_k(\mathbb{C}^N)$ we shall denote the set of all polynomials and homogeneous polynomials, respectively.

For a polynomial $p \in \mathcal{P}(\mathbb{C}^N)$ we denote by $\{p = 0\} = \{z \in \mathbb{C}^N,\ p(z) = 0\}$ the set of its zeros.

Definition 2.1. (see e.g. [8]) For a compact subset $E \subset \mathbb{C}^N$ we define the *extremal function*

$$V_E(z) = \sup\{u(z),\ u \in \mathcal{L},\ u_{|E} \leq 0\}$$

where $\mathcal{L}$ denotes the *Lelong class* of plurisubharmonic functions on $\mathbb{C}^N$ satisfying

$$\sup_{z \in \mathbb{C}^N} u(z) - \log^+ |z| < \infty,$$

where $\log^+ |z| = \max\{0, \log |z|\}$.

Note that the definition of the Lelong class does not depend on the choice of a norm $\mathbb{C}^N \ni z \to |z|$.

Remark 2.2. If $\mathbb{C}^N \ni z \to |z| \in \mathbb{C}^N$ is a norm in $\mathbb{C}^N$ and $B = \{z \in \mathbb{C}^N,\ |z-a| \leq r\}$ then $V_B(z) = \log^+ \left|\frac{z-a}{r}\right|$.

Throughout the paper we shall assume that the compact $E \subset \mathbb{C}^N$ is *regular* i.e. the function V_E is continuouos in $\mathbb{C}^N$.

For $R > 1$ and $\delta > 0$ we shall consider the open sets

$$E_R = \{z \in \mathbb{C}^N,\ V_E(z) < \log R\}$$

and

$$E_{(\delta)} = \{z \in \mathbb{C}^N,\ \operatorname{dist}(z, E) < \delta\},$$

where $\operatorname{dist}(z, E) = \inf_{w \in E} |z - w|$.

For a continuous function $f : \mathbb{C}^N \supset E \to \mathbb{C}$ we denote its supremum norm on E by

$$\|f\|_E = \sup_{z \in E} |f(z)|$$

and the error of the polynomial approximation of f on E by

$$E_n(f, E) = \operatorname{dist}_E(f, \mathcal{P}_n(\mathbb{C}^N)) = \inf\{\|f - p\|_E\ :\ p \in \mathcal{P}_n(\mathbb{C}^N)\}.$$

Let $W(E)$ denote the closure of the set $\mathcal{P}(\mathbb{C}^N)$ in the space of continuous functions $\mathcal{C}(E)$ with the norm $f \to \|f\|_E$.

Any polynomial $p_n \in \mathcal{P}_n(\mathbb{C}^N)$ satisfying $E_n(f, E) = \|f - p_n\|_E$ is called a *polynomial of best uniform approximation of f on the set E.* Notice that p_n need not be unique nor $\deg p_n = n$.

Let $A = [a_{ij}] \in \mathbb{C}^{N \times N}$ be a complex matrix. Let us consider the following sets of affine nondegenerate maps in $\mathbb{C}^N$

$$G_0 = \{z \to z + a,\ a \in \mathbb{C}^N\},$$
$$G_1 = \{z \to Az + a,\ |\det A| = 1,\ a \in \mathbb{C}^N\},$$
$$G_2 = \{z \to Az + a,\ \det A \neq 0,\ a \in \mathbb{C}^N\}.$$

Remark 2.3. For any $E \subset \mathbb{C}^N$ and $g \in G_2$ we have

$$V_{g(E)} = V_E \circ g^{-1} \quad \text{and} \quad g(E_R) = (g(E))_R.$$

For $i = 0, 1, 2$, let us define the following classes of polynomials

$$\mathcal{H}_k \circ G_i = \{h \circ g_i,\ h \in \mathcal{H}_k(\mathbb{C}^N),\ g_i \in G_i\},$$
$$\mathcal{P}_k \circ G_i = \{p \circ g_i,\ p \in \mathcal{P}_k(\mathbb{C}^N),\ g_i \in G_i\}.$$

Remark 2.4. Notice that

$$\mathcal{H}_k \subset \mathcal{H}_k \circ G_0 \subset \mathcal{H}_k \circ G_1 \subset \mathcal{H}_k \circ G_2 \subset \mathcal{H}_k \circ G_0 \subset \mathcal{P}_k$$

and

$$\mathcal{P}_k \subset \mathcal{P}_k \circ G_0 \subset \mathcal{P}_k \circ G_1 \subset \mathcal{P}_k \circ G_2 \subset \mathcal{P}_k \circ G_0 \subset \mathcal{P}_k$$

and, in particular, $\deg(h \circ g) = k$ for any $h \in \mathcal{H}_k$, $h \neq 0$, and $g \in G_2$.

Indeed, for any $h \in \mathcal{H}(\mathbb{C}^N)$ and $g_2(z) = Az + a$, the polynomial

$$h_0 = h \circ A \ :\ \mathbb{C}^N \ni z \to h(Az) \in \mathbb{C}$$

is homogeneous and $g_0(z) = z + A^{-1}a$ is a translation, which means that $h \circ g \equiv h_0 \circ g_0 \in \mathcal{H} \circ G_0$.

We shall denote by $\partial^* E$ the closure of the set

$$\bigcup_{\substack{h \in \mathcal{H}\circ G_0 \\ h \neq const}} \{z \in E : |h(z)| = \|h\|_E\}.$$

By the maximum principle we have $\partial^* E \subset \partial E$, where ∂E denotes the topological boundary of E.

We denote by $B(a, r) = \{z \in \mathbb{C}^N,\ |z - a| < r\}$ the open ball of radii r centered at a in a fixed norm of $\mathbb{C}^N$. We put $B(a, 0) = \emptyset$.

Definition 2.5. For a compact set $E \subset \mathbb{C}^N$, we define two constants $\rho(E)$ and $\rho_*(E)$ and two sets E^ρ and E^{ρ_*} as follows:

$$\rho(E) = \sup_{a \in E} \sup\{r \geq 0, B(a, r) \cap \partial E = \emptyset\}$$

$$E^\rho = \{a \in E :\ B(a, \rho(E)) \cap \partial E = \emptyset\}.$$

Similarly, we define

$$\rho_*(E) = \sup_{a \in E^\rho} \sup\{r \geq 0 :\ B(a, r) \cap \partial^* E = \emptyset\},$$

$$E^{\rho_*} = \{a \in E^\rho :\ B(a, \rho_*(E)) \cap \partial^* E = \emptyset\}.$$

Let us note that the sets E^ρ and E^{ρ_*} are nonempty and compact whenever the set $E \subset \mathbb{C}^N$ is nonempty and compact. Moreover, if $\rho(E) = 0$ ($\rho_*(E) = 0$, respectively) then $E^\rho = E$ ($E^{\rho_*} = E^\rho$, respectively).

Since $\partial^* E \subset \partial E$ we have $E \setminus \partial^* E \supset E \setminus \partial E$, which implies the inequality $\rho_*(a) \geq \rho(a)$ for all $a \in E^\rho$ and $\frac{1}{2}\operatorname{diam}(E) \geq \rho_*(E) \geq \rho(E)$.

In general $\partial^* E$ is much smaller than ∂E. For the interval $E = [-1, 1] \subset \mathbb{C}$ we have $\partial_* E = \{-1, 1\}$ whereas $\partial E = E$. In this case we have also $E^\rho = E$ and $\rho(E) = 0$ whereas $E^{\rho_*} = \{0\}$ and $\rho_*(E) = 1$.

Remark 2.6. Notice that the following two inequalities hold
(i) $\operatorname{dist}(a, \mathbb{C}^N \setminus \overline{E_{(\delta)}}) \geq \delta + \rho(E)$ for any $a \in E^\rho$;
(ii) $\operatorname{dist}(a, \partial^* E) \leq \operatorname{diam} E - \rho_*(E)$ for any $a \in E^{\rho_*}$.

It is evident that the definitions of E^ρ, E^{ρ_*}, $\rho(E)$ and $\rho_*(E)$ may depend on the choice of a norm in $\mathbb{C}^N$ if $N > 1$. If E is a closed ball (in the norm of $\mathbb{C}^N$ which has been fixed before defing E^ρ, E^{ρ_*}, $\rho(E)$ and $\rho_*(E)$) i.e. $E = \{z \in \mathbb{C}^N : |z - a| \leq r\}$ then $E^\rho = E^{\rho_*} = \{a\}$ and $\rho(E) = \rho_*(E) = r$.

3. Zeros of polynomials of best uniform approximation

Let E be a compact regular subset of $\mathbb{C}^N$ and let $(p_n)_{n \in \mathbb{N}_0}$, $p_n \in \mathcal{P}_n(\mathbb{C}^N)$, be any sequence of polynomials of best uniform approximation to a nonconstant function $f \in W(E)$.

In the multidimensional case, Siciak ([7] Theorem 10.3) generalized the Bernstein - Walsh theorem as follows:

Theorem 3.1. *A necessary and sufficient condition that a function f be holomorphic in E_R, $R > 1$, is that*

$$\limsup_{n \to \infty} \|f - p_n\|_E^{1/n} \leq \frac{1}{R}.$$

We recall also a one-dimensional Wójcik's result ([12], Theorem 3) concerning the zeros of polynomials of best uniform approximation of a function f.

Theorem 3.2. *Let E be a compact and regular subset of $\mathbb{C}^N$ and let $f \in W(E)$. Let $R > 1$. Then the following conditions are equivalent:*
(i) there exists an extension F of f which is holomorphic in E_R;
(ii) for any $r \in (1, R)$, there exists $A_r \in \mathbb{C}$ such that

$$p_n(z) - A_r \neq 0 \quad \text{for all } n \in \mathbb{N}_0 \quad \text{and } z \in \overline{E_r};$$

(iii) $\limsup_{n \to \infty} |a_n|^{\frac{1}{n}} \leq \dfrac{1}{R d(E)}$,
where a_n is the nth coefficient of the polynomial $p_n(z) = a_n z^n + \cdots + a_0$ and $d(E)$ is the Tchebyshev constant of the set E, i.e. $d(E) = \lim_{n \to \infty} d_n(E)^{1/n}$ where $d_n(E) = \inf\{\|z^n + q\|_E : q \in \mathcal{P}_{n-1}(\mathbb{C}^N)\}$.

In the case of several complex variables, some counterparts of the implications $(i) \implies (iii)$ and $(iii) \implies (i)$ from the above theorem were studied by the author [10] and recently by Bloom [3].

Remark 3.3. *The implication $(i) \implies (ii)$ is valid for any dimension $N \geq 1$.*

Indeed, if f admits a continuation F that is holomorphic in E_R then, by Theorem 3.1, for any $r_1 \in (r, R)$ there exists a sequence of polynomials q_n, $n \in \mathbb{N}_0$, which converges to F uniformly on $\overline{E_{r_1}}$ and

$$\limsup_{n \to \infty} \|F - q_n\|_E^{1/n} \leq \frac{1}{r_1}.$$

This implies that there exist $M > 0$ and $n_0 \in \mathbb{N}_0$ such that

$$\|F - q_n\|_E \leq M r_1^{-n} \quad \text{for all } n \geq n_0.$$

Since $\|f - p_n\|_E \leq \|f - q_n\|_E$ and $F_{|E} = f$, by the Bernstein-Walsh inequality, we have

$$\|p_n - q_n\|_{\overline{E_r}} \leq r^n \|p_n - q_n\|_E \leq 2M\Big(\frac{r}{r_1}\Big)^n,$$

which implies that

$$\|p_n\|_{\overline{E_r}} \leq 2M\Big(\frac{r}{r_1}\Big)^n + \|q_n\|_{\overline{E_r}}.$$

Thus,

$$A_r = 1 + \sup_{n\geq 0} \|p_n\|_{\overline{E_r}} < \infty$$

and we obtain that $p_n(z) - A_r \neq 0$ for all $z \in \overline{E_r}$ and $n \in \mathbb{N}_0$.

Theorem 3.4. *Assume that there exist a norm in $\mathbb{C}^N$, a positive number $\delta > 0$, a complex number A and a positive integer n_0 such that*

$$E_{(\delta)} \cap \bigcup_{n\geq n_0} \{p_n = A\} = \emptyset.$$

If $\delta > diam E - \rho^*(E) - \rho(E)$ then f admits an extension that is holomorphic in E_R with $R = \dfrac{\delta + \rho(E)}{diam E - \rho^*(E)}$.

If E is the closed ball of radii r centred at a point a then we have $\rho(E) = \rho_*(E) = r$. Since $E_{1+\frac{\delta}{r}} = E_{(\delta)}$ we get by Remark 3.3 and Theorem 3.4 the following

Corollary 3.5. *Let E be a closed ball in a norm of $\mathbb{C}^N$ and let $f \in W(E)$. Let $R > 1$. Then the following two conditions are equivalent*

1) f admits an extension that is holomorphic in E_R;

2) for any $r \in (1, R)$ there exists $A_r \in \mathbb{C}$ such that $p_n(z) - A_r \neq 0$ for all $n \in \mathbb{N}$ and all $z \in \overline{E_r}$.

4. Auxiliary lemmas

Following an idea of Bloom [3], for a homogeneous polynomial $h \in \mathcal{H}_n(\mathbb{C}^N)$ we denote by $Tch_E h$ *a Tchebyshev polynomial* associated with h minimizing the norm $\|h + q\|_E$, $q \in \mathcal{P}_{n-1}(\mathbb{C}^N)$ i.e.

$$\|Tch_E h\|_E = \inf\{\|h + q\|_E,\ q \in \mathcal{P}_{n-1}(\mathbb{C}^N)\}.$$

We recall a version of the Bernstein-Walsh-Siciak theorem (see Theorem 3.1) due to Bloom [3]:

Theorem 4.1. *Let E be a compact and regular subset of $\mathbb{C}^N$ and let $f \in W(E)$. Then the following conditions are equivalent:*

1) f admits an extension that is holomorphic in E_R;

2) $\limsup_{n\to\infty} \|f - p_n\|_E^{1/n} \leq \frac{1}{R}$;

3) $\limsup_{n\to\infty} \|Tch_E \hat{p_n}\|_E^{1/n} \leq \frac{1}{R}$,

where (p_n), $p_n \in \mathcal{P}_n(\mathbb{C}^N)$, is a sequence of polynomials of best uniform approximation to f on E. (Let us recall that $\hat{p_n} = 0$ if $\deg p_n < n$.)

We owe the idea of the proof of the next lemma to M. Simkani [9], who first proved it in the case where $E = \{(z,w) \in \mathbb{C}^2, |z| \leq 1,\ |w| \leq 1\}$ and $g(z,w) = (z,w)$ and announced his result during his stay in Cracow in 1991.

Lemma 4.2. *If $\{p_n = 0\} \cap E = \emptyset$ and $\deg p_n = n > 1$ then for any $g \in G_2$ and any point $z_g^* \in \partial^* E$ satisfying $\|(p_n \circ g)\hat{}\circ g^{-1}\|_E = |(p_n \circ g)\hat{}(g^{-1}(z_g^*))|$ we have*

$$\|(p_n \circ g)\hat{}\circ g^{-1}\|_E = |p_n(g(0))| \frac{|z_g^* - g(0)|^n}{\prod_{l=1}^n |z_l - g(0)|}$$

where $\{z_l,\ l = 1,2,\ldots,n\} = \{p_n = 0\} \cap \{(1-\zeta)g(0) + \zeta z_g^*,\ \zeta \in \mathbb{C}\}$.

Proof. By Remark 2.4, for any $g \in G_2$ we have $\deg(p_n \circ g)\hat{}\circ g^{-1} = n$. The regularity of E implies that E is determining for polynomials. In particular, E is determining for the family $\mathcal{H} \circ G_0$, which means that $\|(p_n \circ g)\hat{}\circ g^{-1}\|_E > 0$ and $z_g^* \neq g(0)$. If it were $z_g^* = g(0)$ then $(p_n \circ g)\hat{}g^{-1}(g(0)) = (p_n \circ g)\hat{}(0) = 0$, contradiction.

Let us define $h_g : \mathbb{C} \ni \zeta \to (1-\zeta)g(0) + \zeta z_g^* \in \mathbb{C}^N$. Since $g(z) = Az + g(0)$ we have $g^{-1}(w) = A^{-1}w - A^{-1}g(0) = A^{-1}(w - g(0))$, which gives

$$(g^{-1} \circ h_g)(\zeta) = A^{-1}(\zeta(z_g^* - g(0))) = \zeta A^{-1}(z_g^* - g(0)) = \zeta g^{-1}(z_g^*).$$

Let $(p_n \circ g)(w) = \sum_{|\alpha|\leq n} a_\alpha w^\alpha$, in particular, $(p_n \circ g)\hat{}(w) = \sum_{|\alpha|=n} a_\alpha w^\alpha$.

Next, let us compute

$$(p_n \circ h_g)(\zeta) = (p_n \circ g) \circ (g^{-1} \circ h_g)(\zeta) = \sum_{|\alpha|\leq n} a_\alpha (\zeta g^{-1}(z_g^*))^\alpha$$

$$= \sum_{k=0}^n \Big(\sum_{|\alpha|=k} a_\alpha (g^{-1}(z_g^*))^\alpha\Big)\zeta^k = \Big(\sum_{|\alpha|=n} a_\alpha (g^{-1}(z_g^*))^\alpha\Big) \prod_{l=1}^n (\zeta - \zeta_l),$$

with

$$\Big|\sum_{|\alpha|=n} a_\alpha (g^{-1}(z_g^*))^\alpha\Big| = \|(p_n \circ g)\hat{}\circ g^{-1}\|_E > 0.$$

This implies that

$$|(p_n \circ h_g)(0)| = \Big|\sum_{|\alpha|=n} a_\alpha (g^{-1}(z_g^*))^\alpha\Big| \prod_{l=1}^n |\zeta_l|$$

where

$$\{\zeta_l,\ l = 1,2,\ldots,n\} = \{\zeta \in \mathbb{C} : (p_n \circ h_g)(\zeta) = 0\}$$

(the zeros of the polynomial $p_n \circ h_g$ are to be counted with their multiplicities).

Putting $z_l = (1-\zeta_l)g(0) + \zeta_l z_g^* = \zeta_l(z_g^* - g(0)) + g(0)$ we have

$$|z_l - g(0)| = |\zeta_l| \cdot |z_g^* - g(0)|.$$

Finally, we obtain

$$|p_n(g(0))| = \|(p_n \circ g)\hat{} \circ g^{-1}\|_E \frac{\prod_{l=1}^n |z_l - g(0)|}{|z_g^* - g(0)|^n}$$

where

$$\{z_l,\ l = 1, 2, \dots, n\} = \{p_n = 0\} \cap \{(1-\zeta)g(0) + \zeta z_g^*,\ \zeta \in \mathbb{C}\}.$$

□

Let $p_n \in \mathcal{P}_n(\mathbb{C}^N)$. Since for any $g \in G_2$ and any constant $A \in \mathbb{C}$ the homogeneous parts of the polynomials $((p_n - A) \circ g)\hat{} \circ g^{-1}$ and $Tch_E\hat{p_n}$ coincide we get the following estimate of the norm $\|Tch_E\hat{p_n}\|_E$.

Lemma 4.3. *Let p_n be any polynomial of degree not greater than n. Let A be a complex number such that $p_n(z) - A \neq 0$ for all $z \in E_{(\delta)}$. Then the following estimate holds*

$$\|Tch_E\hat{p_n}\|_E \leq (\|p_n\|_E + |A|)\frac{diamE - \rho_*(E)}{\delta + \rho(E)}.$$

Proof. Applying Remark 2.6 and Lemma 4.2 to the polynomial $z \to p_n(z) - A$ and $g \in G_2$ such that $g(0) \in E^{\rho_*}$ we get the assertion of the lemma in the case of $\deg p_n = n$. If $\deg p_n < n$ then $\hat{p_n} = 0$ and $Tch_E\hat{p_n} = 0$, thus the estimate given by the lemma is valid as well. □

Proof of Theorem 3.4. Let $(p_n)_{n\in\mathbb{N}}$ be any sequence of polynomials of best uniform approximation to the function $f \in W(E)$. If for $\delta > \mathrm{diam}E - \rho^*(E) - \rho(E)$ there exists a complex number A and a positive integer n_0 such that $p_n(z) \neq A$ for all $n \geq n_0$ and all $z \in \overline{E_{(\delta)}}$ then, by Lemma 4.3 and the fact that the norms $\|p_n\|_E$ are bounded, we get

$$\limsup_{n\to\infty} \|Tch_E\hat{p_n}\|_E^{1/n} \leq \frac{\mathrm{diam}E - \rho_*(E)}{\delta + \rho(E)} \ < \ 1.$$

Hence, by Theorem 4.1, f admits an extension that is holomorphic in E_R with $R = \frac{\delta + \rho(E)}{\mathrm{diam}E - \rho_*(E)}$.

5. An application to least square approximation

Let E be a compact and regular subset of $\mathbb{C}^N$ and let μ be a finite Borel measure on E. We say that the pair (E, μ) satisfies the *Bernstein - Markov condition* if for any $\epsilon > 0$ there exists a constant A_ϵ such that

$$\|p\|_E \leq A_\epsilon(1+\epsilon)^{\deg p}\left(\int_E |p(z)|^2 d\mu(z)\right)^{1/2}, \quad \text{for any p} \in \mathcal{P}(\mathbb{C}^N),$$

(see e.g. [5]).

We denote by $L^2_{\mathcal{P}}(E,\mu)$ the closure of the algebra $\mathcal{P}(\mathbb{C}^N)$ in the norm $f \to \|f\|_\mu = \Big(\int_E |f(z)|^2 d\mu(z)\Big)^{1/2}$.

Following an idea of Bloom [3] we introduce the *Tchebyshev polynomial* $Tch_\mu h_n$ associated with the polynomial $h_n \in \mathcal{H}_n(\mathbb{C}^N)$ and the measure μ as follows

$$\|Tch_\mu h_n\|_\mu = \inf\{\|h - q\|_\mu,\ q \in \mathcal{P}_{n-1}(\mathbb{C}^N)\}.$$

The polynomial $Tch_\mu h_n$ is unique since $h_n - Tch_\mu h_n$ is the nearest polynomial to h_n in the subspace $\mathcal{P}_{n-1}(\mathbb{C}^N)$ of the Hilbert space $L^2_{\mathcal{P}}(E,\mu)$.

Let $\alpha : \mathbb{N}_0 \to \mathbb{N}_0^N$ be any bijection satisfying $|\alpha(i)| \le |\alpha(i+1)|$ for any $i \in \mathbb{N}_0$, where $|\alpha| = \alpha_1 + \cdots + \alpha_N$ denotes the length of $\alpha = (\alpha_1, \ldots, \alpha_N)$.

We choose any sequence of polynomials (e_i), $i \in \mathbb{N}_0$ such that

(b1) $\deg e_i = |\alpha(i)|$;

(b2) $\int_E e_i \overline{e_j} d\mu = 0$ if $i \neq j$ and $\int_E |e_i|^2 d\mu = 1$;

(b3) the homogeneous parts of degree n of the polynomials e_i with $|\alpha(i)| = n$ form a basis of $\mathcal{H}_n(\mathbb{C}^N)$.

We may take $e_i(z) := t_i(z)/\|t_i\|_\mu$ where t_i realizes the infimum

$$\|t_i\|_\mu = \inf\{\|z^{\alpha(i)} + \sum_{k=0}^{i-1} c_k z^{\alpha(k)}\|_\mu,\ c_k \in \mathbb{C}\}.$$

Remark 5.1. *Let (e_i) be a sequence of polynomials satisfying conditions b1, b2 and b3 with respect to a measure μ and let $h \in \mathcal{H}_n(\mathbb{C}^N)$. If $h(z) = \sum_{|\alpha(i)|=n} a_i \hat{e}_i(z)$ then $Tch_\mu h_n(z) = \sum_{|\alpha(i)|=n} a_i e_i(z)$. Moreover, the norm*

$$\|Tch_\mu h_n(z)\|_\mu = \Big(\sum_{|\alpha(i)|=n} |a_i|^2\Big)^{1/2}$$

does not depend on the choice of the sequence (e_i).

Let (p_n) be a sequence of any polynomials such that $\deg p_n \le n$. Assume that the pairs (E,μ) and (E,ν) satisfy the Bernstein-Markow condition. We shall consider the Tchebyshev polynomials $Tch_E\hat{p_n}$, $Tch_\mu\hat{p_n}$ and $Tch_\nu\hat{p_n}$ associated with the homogeneous parts of degree n of the polynomial p_n. (Note that $\hat{p_n} = 0$ if $\deg p_n < n$.)

Lemma 5.2. *Let $(p_n)_{n\in\mathbb{N}}$ be a sequence of any polynomials such that $\deg p_n \le n$. Then the following equalities hold*

$$\begin{aligned}\limsup_{n\to\infty} \|Tch_E\hat{p_n}\|_E^{1/n} &= \limsup_{n\to\infty} \|Tch_E\hat{p_n}\|_\mu^{1/n} = \limsup_{n\to\infty} \|Tch_\mu\hat{p_n}\|_\mu^{1/n} \\ &= \limsup_{n\to\infty} \|Tch_\mu\hat{p_n}\|_E^{1/n} = \limsup_{n\to\infty} \|Tch_\mu\hat{p_n}\|_\nu^{1/n} \\ &= \limsup_{n\to\infty} \|Tch_\nu\hat{p_n}\|_\mu^{1/n}.\end{aligned}$$

Proof. Since the pairs (E,μ) and (E,ν) satisfy the Bernstein-Markov condition, by the definition of the Tchebyshev polynomials we have that for any $\epsilon > 0$ there are A_ϵ and B_ϵ such that for any polynomial $p_n \in \mathcal{P}_n(\mathbb{C}^N)$ the following inequalities hold

$$\begin{aligned}\|Tch_E\hat{p_n}\|_E &\leq \|Tch_\mu\hat{p_n}\|_E \leq A_\epsilon(1+\epsilon)^n\|Tch_\mu\hat{p_n}\|_\mu \leq A_\epsilon(1+\epsilon)^n\|Tch_E\hat{p_n}\|_\mu \\ &\leq A_\epsilon(1+\epsilon)^n\mu(E)^{1/2}\|Tch_E\hat{p_n}\|_E\end{aligned}$$

and, similarly,

$$\begin{aligned}\|Tch_E\hat{p_n}\|_E \leq \|Tch_\nu\hat{p_n}\|_E &\leq A_\epsilon(1+\epsilon)^n\|Tch_\nu\hat{p_n}\|_\mu \\ &\leq A_\epsilon(1+\epsilon)^n\mu(E)^{1/2}\|Tch_\nu\hat{p_n}\|_E \\ &\leq A_\epsilon B_\epsilon(1+\epsilon)^{2n}\mu(E)^{1/2}\|Tch_\nu\hat{p_n}\|_\nu \\ &\leq A_\epsilon B_\epsilon(1+\epsilon)^{2n}\mu(E)^{1/2}\|Tch_E\hat{p_n}\|_\nu \\ &\leq A_\epsilon B_\epsilon(1+\epsilon)^{2n}\mu(E)^{1/2}\nu(E)^{1/2}\|Tch_E\hat{p_n}\|_E.\end{aligned}$$

Raising the above inequalities to the power $\frac{1}{n}$ and taking the lim sup we get the assertion of the lemma since $\epsilon > 0$ has been chosen arbitrarily. □

Let (e_i) be a sequence of polynomials satisfying conditions (b1-b3) and let $f \in L^2_{\mathcal{P}}(E,\mu)$. We denote by $\phi_i = \int_E f(z)\overline{e_i(z)}d\mu(z)$ the ith Fourier coefficient and by $f_n(z) = \sum_{k=0}^n \sum_{|\alpha(i)|=k} \phi_i e_i(z)$ the nth partial sum of the Fourier expansion of f.

We give the following counterpart of Theorem 4.1.

Theorem 5.3. *Let E be a compact and regular subset of $\mathbb{C}^N$ and let (E,μ) satisfy the Bernstein-Markov condition. Let $f \in L^2_{\mathcal{P}}(E,\mu)$ and let $R > 1$. The following conditions are equivalent:*

(1) f admits an extension that is holomorphic in E_R;

(2) $\limsup_{i\to\infty} |\phi_i|^{1/|\alpha(i)|} \leq \dfrac{1}{R}$;

(3) $\limsup_{n\to\infty} \|Tch_E\hat{f}_n\|_E \leq \dfrac{1}{R}$.

Proof. In order to prove the implication (1) $\implies$ (2) we use an idea of Zeriahi ([13], Theorem 3.10). Since for i with $|\alpha(i)| = n$ the polynomial $\phi_i e_i(z)$ is orthogonal to the subspace $\mathcal{P}_{n-1}(\mathbb{C}^N)$ we get for all $i \in \mathbb{N}$ with $|\alpha(i)| = n$ the estimate

$$|\phi_i| = |\int_E f\overline{e_i}d\mu| \leq \inf\{|\int_E (f-p)\overline{e_i}d\mu|,\ p \in \mathcal{P}_{n-1}(\mathbb{C}^N)\}.$$

Thus,

$$|\phi_i| \leq \inf\{\|f-p\|_E,\ p \in \mathcal{P}_{n-1}(\mathbb{C}^N)\}.$$

Since f admits an extension that is holomorphic in E_R, appling Theorem 3.1 we obtain $\limsup_{i\to\infty}|\phi_i|^{1/|\alpha(i)|} \leq \frac{1}{R}$.

The implication (2) $\implies$ (1) holds since the Fourier series

$$\sum_{k=0}^{\infty}\sum_{|\alpha(i)|=k}\phi_i e_i(z)$$

converges uniformly on compact subsets of E_R.

Let $r \in (1, R)$. We choose $\epsilon > 0$ and $\rho \in (1, R)$ such that $r(1+\epsilon) < \rho < R$. Since the number of i such that $|\alpha(i)| = n$ is equal to $\binom{N-1+n}{N-1}$ and $\lim_{n\to\infty} \binom{N-1+n}{N-1}^{1/n} = 1$, by (2) we find $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have $\sum_{|\alpha(i)|=n} |\phi_i| \leq 1/\rho$. Thus, by the Bernstein-Walsh inequality and since μ satisfies the Bernstein-Markov condition we get for any $n \geq n_0$ the estimate

$$\sum_{k=n}^{\infty} \| \sum_{|\alpha(i)|=k} \phi_i e_i \|_{\overline{E_r}} \leq \sum_{k=n}^{\infty} r^k \sum_{|\alpha(i)|=k} \|\phi_i e_i\|_E \leq \sum_{k=n}^{\infty} r^k A_\epsilon (1+\epsilon)^k \sum_{|\alpha(i)|=k} |\phi_i|$$
$$\leq \sum_{k=n}^{\infty} A_\epsilon \left(\frac{r(1+\epsilon)}{\rho} \right)^k,$$

A_ϵ being a constant depending only on ϵ.

Since $\frac{r(1+\epsilon)}{\rho} < 1$, the Fourier series $\sum_{k=0}^{\infty} \sum_{|\alpha(i)|=k} \phi_i e_i(z)$ converges uniformly on $\overline{E_r}$.

The equivalence of conditions (2) and (3) follows from Remark 5.1, Lemma 5.2 and the identity

$$\limsup_{i\to\infty} |\phi_i|^{1/|\alpha(i)|} = \limsup_{n\to\infty} \Big(\sum_{|\alpha(i)|=n} |\phi_i|^2 \Big)^{1/2n} = \limsup_{n\to\infty} \|Tch_\mu \hat{f}_n\|_\mu^{1/n}.$$

□

As a corollary from Theorem 5.3 and Lemma 4.3 we get the following

Theorem 5.4. *Let E be a compact and regular subset of $\mathbb{C}^N$ and let the pair (E, μ) satisfy the Bernstein-Markov condition. Let $f \in L^2_{\mathcal{P}}(E, \mu)$. Assume that there exist a norm in $\mathbb{C}^N$, a positive number $\delta > 0$, a complex number A and a positive integer n_0 such that*

$$E_{(\delta)} \cap \bigcup_{n \geq n_0} \{f_n = A\} = \emptyset.$$

If $\delta > diam E - \rho^*(E) - \rho(E)$ then f admits an extension that is holomorphic in E_R with $R = \dfrac{\delta + \rho(E)}{diam E - \rho^*(E)}$.

□

Proceeding similarly as in the proof of Remark 3.3, we also derive a counterpart of Corollary 3.5.

Corollary 5.5. *Let E be a ball in a norm in $\mathbb{C}^N$ and let (E, μ) satisfy the Bernstein - Markov condition. Let $f \in L^2_{\mathcal{P}}(E, \mu)$ and $R > 1$. Then the necessary and sufficient condition that the function f be holomorphic in E_R is that for any $r \in (1, R)$ there exists a complex number A_r such that $f_n(z) \neq A_r$ for all $n \in \mathbb{N}$ and all $z \in \overline{E_r}$.*

References

1. S.N. Bernstein, *On the distribution of the zeros of the polynomials tending to a continuous function positive on a given interval [Russian]*, Complete Works, **I** (1952), 443-451.
2. H.-P. Blatt, E.B.Saff, *Behaviour of zeros of polynomials of near best approximation*, J. Approx. Theory **46** (1986), 323-344.
3. T. Bloom, *Some application of the Robin function to multivariable approximation theory* (to appear).
4. P.B. Borwein, *The relationship between the zeros of best approximations and differentiability*, Proc. Amer. Math. Soc. **92** (1984), 528-532.
5. Nguyen T.V., A. Zeriahi, *Familles de polynômes presque partout bornées*, Bull. Sc. Math. **107** (1983), 81-91.
6. W. Pleśniak, *On the distribution of zeros of the polynomials of best L^2 -approximation to holomorphic functions*, Zeszyty Nauk. Uniw. Jagielloń., Prace Matematyczne **22** (1981), 29-35.
7. J. Siciak, *On some extremal function s and their application in the theory of analytic functions of several complex variables*, Trans. Amer. Math. Soc. **105** (1962), 322-357.
8. J. Siciak, *Extremal plurisubharmonic functions in $\mathbb{C}^n$*, Ann. Pol. Math. **39** (1981), 175-211.
9. M. Simkani, *Private Communication*, Cracow (1991).
10. J. Szczepański, *Indicators of growth of polynomials of best uniform approximation to holomorphic functions on compacta in $\mathbb{C}^N$*, J. Approx. Theory **76** (1994), 233-245.
11. J.L. Walsh, *Interpolation and approximation by rational functions in the complex domain*, Amer. Math. Soc. Coll. Publ. **20** ((Third edition 1960)).
12. A. Wójcik, *On zeros of polynomials of best approximation to holomorphic and C^∞ functions*, Monatsh. Math. **105** (1988), 75-81.
13. A. Zeriahi, *Pluricomplex Green function and approximation of holomorphic functions*, Prépublication no 63 du Laboratoire de Mathématiques Emile Picard, Université Paul Sabatier, Toulouse (1995).

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