ON THE EXISTENCE OF A SOLUTION TO STOCHASTIC NAVIER-STOKES EQUATIONS

Marek Capiński and Szymon Peszat

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Abstract. We consider stochastic Navier–Stokes equations on a possibly unbounded domain $\mathcal{O} \subseteq \mathbb{R}^d$, where d is equal to 2 or 3. First we prove the existence of a martingale solution for the initial value being a probability measure on the space of square integrable $\mathbb{R}^d$-valued functions. Then we show the existence of a spatially homogeneous solution to the equation on the whole $\mathbb{R}^d$, driven by a spatially homogeneous Wiener random field.

0. Introduction

Let $\mathcal{O}$ be a possibly unbounded open subset of $\mathbb{R}^d$, where d is equal to 2 or 3. We assume that $\mathcal{O}$ is connected with the boundary $\partial\mathcal{O}$ of class C^2. Let us fix a bounded time interval $[0,T]$. The paper is concerned with the existence of a martingal solution to the following system of stochastic Navier–Stokes equations

$$\begin{gathered} \partial_t u - \nu\Delta u + \langle u, \nabla u\rangle + \nabla p = F(t,u) + G(t,u)\dot{W}, \\ \operatorname{div} u = 0 \end{gathered} \tag{0.1}$$

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considered with the Dirichlet boundary condition $u|_{\partial\mathcal{O}} = 0$ in case $\mathcal{O} \neq \mathbb{R}^d$. In (0.1), $u = (u_1, \dots, u_d)$ and p are unknown random fields on $[0,T] \times \mathcal{O}$, ν is a strictly positive constant, and W is a cylindrical Wiener process on a Hilbert space $\mathcal{H}$. We interpret $u(t,x)$ and $p(t,x)$ as the velocity and pressure of an incompressible fluid at point x at time t. Then $F(t,u) + G(t,u)\dot{W}$ is the density of the force per unit volume. The force has the deterministic and random parts which depend on u.

There is an extensive literature concerning stochastic fluid mechanics. Stochastic Navier–Stokes equations in a bounded domain were investigated in a number of papers. In [3] the existence of a solution was proved for a constant diffusion coefficient and a one-dimensional Wiener process. The case of a more general diffusion term was studied using different methods in [2], [8], [9], [11], and [21]. The case of an unbounded domain, the present paper is concerned with, has only been considered in the deterministic setup. Some other problems for the stochastic Navier–Stokes equations were discussed in [12], [13], [20], [26], [33], and [34]. The one-dimensional case, the stochastic Burgers equation, was dealt with in [14] and [15]. Stochastic Korteweg–de Vries equation has been recently investigated in [5]. Two dimensional stochastic Euler equations were studied in [4], [7] and [10].

In the paper we consider two cases. First we assume that the initial value is a probability measure on the space of square integrable function on $\mathcal{O}$. Then we consider the Navier–Stokes equations on $\mathbb{R}^d$ with homogeneous Wiener random field and homogeneous initial data. In the former case our Theorem 1.1 provides sufficient conditions for the existence of a martingale solution to (0.1). We formulate the theorem in a rather abstract and general form. Then we apply it to the problem (0.1) with F and G being Nemitskii operators, and the noise W being either an $L^2(\mathcal{O}; \mathbb{R}^m)$-valued Wiener process, or, for $\mathcal{O} = \mathbb{R}^d$, a finite system of independent spatially homogeneous Wiener random fields. In the case of the initial data being a spatially homogeneous probability measure, we assume that F and G are Nemitskii operators. The constructed solution lives in a weighted Sobolev space.

The fact that the domain is unbounded creates a difficulty. This is concerned with the embedding of the Sobolev space of functions with square integrable gradient into the L^2-space, which, unlike in the bounded case, is not compact. Compactness is crucial in proving the existence of solution and so it is necessary to introduce spaces with weights. Notice that if the initial value is a nontrivial spatially homogeneous probability measure, then it is not supported on the space of square integrable functions. This is another reason for introducing weighted spaces. This, however, brings in another difficulty. The nonlinear term $\langle u, \nabla \rangle u$, which can be tackled as being orthogonal to the solution, has no longer this property in spaces with weights. Unbounded domain also forces some modification of the construction of a solution. The standard method for bounded domains uses Galerkin approximations obtained by solving projected equations on the finite-dimensional spaces spanned by the eigenfunctions of the Laplace operator. In our approach we adapt the Galerkin method employing special bases of spaces of square integrable functions with weights. These bases have to be Schauder bases of some weighted Sobolev and L^q spaces.

In the proof of the existence of a spatially homogeneous solution to (0.1) we employ a beautiful idea of Vishik and Fursikov from the paper [37] dealing with a statistical solution to deterministic Navier–Stokes equations in $\mathbb{R}^d$ with a spatially homogeneous initial value. Namely, they perturb (0.1) by adding the term $\varepsilon|u|^4u$, they replace the pressure by $-\varepsilon^{-1}\mathrm{div}\, u$, and the resulting problem has a unique solution. The limit passage $\varepsilon \to 0$ for a statistical solution is possible due to some uniform estimates, where the properties of spatially homogeneous measures play a crucial role. We combine their approach with that of [11], where a method of solving stochastic Navier–Stokes equations in a bounded domain was given. Thus we prove the existence of a spatially homogeneous solution to (0.1) by solving first the following auxiliary equation

$$\partial_t u^\varepsilon - \nu\Delta u^\varepsilon + \langle u^\varepsilon, \nabla\rangle u^\varepsilon + \varepsilon|u^\varepsilon|^4u^\varepsilon - \frac{1}{\varepsilon}\nabla \mathrm{div}\, u^\varepsilon = F(t, u^\varepsilon) + G(t, u^\varepsilon)\dot{W}. \tag{0.2}$$

In Section 5 we prove the existence of a solution to (0.2). Then, in Section 6, we prove the existence of a spatially homogeneous solution to (0.1) by making use of the weak relative compactness of $\{u^\varepsilon, \varepsilon > 0\}$.

1. Notation and formulation of the main results

Let $\mathfrak{U} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ be a filtered probability space, and let $\mathcal{H}$ be a real separable Hilbert space.

Definition 1.1. By an *$(\mathcal{F}_t)$-adapted cylindrical Wiener process on $\mathcal{H}$* we understand a mapping $W : [0,T] \times \mathcal{H} \to L^2(\Omega, \mathcal{F}, \mathbb{P})$ satisfying the following conditions:

(i) for each $t \in [0,T]$, $W(t,\cdot)$ is a bounded linear operator acting from $\mathcal{H}$ into $L^2(\Omega, \mathcal{F}, \mathbb{P})$,

(ii) for all $t \in [0,T]$, and $\psi, \varphi \in \mathcal{H}$, $\mathbb{E}W(t,\psi)W(t,\varphi) = t\langle\psi,\varphi\rangle_{\mathcal{H}}$,

(iii) for each $\psi \in \mathcal{H}$, $\{W(t,\psi)\}_{t\in[0,T]}$ is a real valued $(\mathcal{F}_t)$-adapted Wiener process such that $W(t,\psi)$ has covariance $t|\psi|^2_{\mathcal{H}}$.

Let V be a separable Hilbert space. We denote by $\mathcal{P}_T(V)$ the space of all $\mathcal{B}([0,T]) \times \mathcal{F}$-measurable V-valued processes σ such that:

(i) $\sigma(t)$ is $\mathcal{F}_t$-measurable for $t \in [0,T]$,

(ii) $\mathbb{E}\int_0^T |\sigma(t)|^2_V\, \mathrm{d}t < \infty$.

Let us denote by $L_{(\mathrm{HS})}(\mathcal{H}, V)$ the space of all Hilbert–Schmidt operators from $\mathcal{H}$ into V. Let $\sigma \in \mathcal{P}_T(L_{(\mathrm{HS})}(\mathcal{H}, V))$, and let W be an $(\mathcal{F}_t)$-adapted cylindrical Wiener process on $\mathcal{H}$. Then for $t \in [0,T]$ we define the stochastic integral $\int_0^t \sigma(s)\,\mathrm{d}W(s)$ as the $L^2(\Omega, \mathcal{F}_t; V, \mathbb{P})$-limit of the series $\sum \int_0^t \sigma(s)\psi_k\, \mathrm{d}W(s,\psi_k)$, where $\{\psi_k\}$ is an orthonormal basis of $\mathcal{H}$, and each term is understood as Itô's integral with respect to the real valued Wiener process $W(\cdot,\psi_k)$. Note that $\int_0^t \sigma(s)\,\mathrm{d}W(s)$, $t \in [0,T]$, is a continuous martingale in V, with 0 mean and second moment satisfying

$$\mathbb{E}\left|\int_0^t \sigma(s)\,\mathrm{d}W(s)\right|^2_V = \mathbb{E}\int_0^t ||\sigma(s)||^2_{L_{(\mathrm{HS})}(\mathcal{H},V)}\,\mathrm{d}s.$$

Clearly, the stochastic integral does not depend on the particular choice of the orthonormal basis of $\mathcal{H}$.

Let $\rho \in \mathbb{R}$. We write

$$\vartheta_\rho(x) = \prod_{i=1}^{d} (1 + |x_i|^2)^{-\rho}, \qquad x = (x_1, \dots, x_d) \in \mathbb{R}^d.$$

For brevity we denote by L^q_ρ and $\mathbb{L}^q_\rho$ the weighted spaces $L^q(\mathcal{O}, \vartheta_\rho(x)\,\mathrm{d}x)$ and $L^q(\mathcal{O}, \vartheta_\rho(x)\,\mathrm{d}x; \mathbb{R}^d)$, respectively.

Let us denote by $C^\infty_{0,\mathrm{div}}(\mathcal{O};\mathbb{R}^d)$ the space consisting of all $v \in C^\infty_0(\mathcal{O};\mathbb{R}^d)$ such that $\operatorname{div} v = 0$, and let H_ρ be the closure of $C^\infty_{0,\mathrm{div}}(\mathcal{O};\mathbb{R}^d)$ in $\mathbb{L}^2_\rho$. The scalar product and norm on L^2_ρ, H_ρ and $\mathbb{L}^2_\rho$ are denoted by $\langle\cdot,\cdot\rangle_\rho$ and $|\cdot|_\rho$. Let us define the Sobolev spaces W^1_ρ and H^1_ρ as the completions of $C^\infty_0(\mathcal{O};\mathbb{R}^d)$ and $C^\infty_{0,\mathrm{div}}(\mathcal{O};\mathbb{R}^d)$ with respect to the norm

$$\left(\int_{\mathcal{O}} \Big(\sum_{i=1}^{d} |\partial_{x_i} v(x)|^2 + |v(x)|^2\Big) \vartheta_\rho(x)\,\mathrm{d}x\right)^{1/2}.$$

Then we define W^{-1}_ρ and H^{-1}_ρ as the dual spaces $(W^1_\rho)^*$ and $(H^1_\rho)^*$, where the duality is given by the identification of $(\mathbb{L}^2_\rho)^*$ with $\mathbb{L}^2_\rho$ and $(H_\rho)^*$ with H_ρ. The norm on W^r_ρ and H^r_ρ, $r = -1, 1$, is denoted by $||\cdot||_{r,\rho}$.

Throughout the paper we adopt the convention that if $\rho = 0$ then we write L^q, $\mathbb{L}^q$, W^r, H^r, and $||\cdot||_r$ of instead of L^q_0, $\mathbb{L}^q_0$, W^r_0, H^r_0, and $||\cdot||_{r,0}$. The products on L^2, $\mathbb{L}^2$, H, and $\mathbb{R}^d$ are denoted by the same symbol $\langle\cdot,\cdot\rangle$. Occasionally, $\langle\cdot,\cdot\rangle_\rho$ stands for the canonical bilinear form on $W^{-1}_\rho \times W^1_\rho$.

In the paper F and G map $[0,T] \times \mathbb{L}^2_\rho$ into W^{-1}_ρ and $L_{(\mathrm{HS})}(\mathcal{H}, \mathbb{L}^2_\rho)$, respectively. They are supposed to be measurable, weakly continuous with respect to the second variable, and having linear growth. Here by the weak continuity of F and G we understand that for all $t \in [0,T]$, $\psi \in C^\infty_0(\mathcal{O};\mathbb{R}^d)$, and $\varphi \in \mathcal{H}$, the functions $u \mapsto \langle F(t,u), \psi\rangle_\rho$ and $u \mapsto \langle G(t,u)\varphi, \psi\rangle_\rho$ are continuous.

Definition 1.2. Let $\rho \in \mathbb{R}$, and let U_0 be a probability measure on H_ρ. By an *H_ρ-valued martingale solution to* (0.1) *starting from* U_0 we understand a triple $(\mathfrak{U}, W, u)$ consisting of a filtered probability space $\mathfrak{U} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$, an $(\mathcal{F}_t)$-adapted cylindrical Wiener process W on $\mathcal{H}$, and a measurable $(\mathcal{F}_t)$-adapted stochastic process u such that:

(i) $u \in L^p(\Omega, \mathcal{F}, \mathbb{P}; L^\infty(0,T; H_\rho))$ for every $p \in [1,\infty)$,
(ii) $u \in L^2(\Omega \times [0,T], \mathcal{F} \times \mathcal{B}([0,T]), \mathbb{P} \times \mathrm{d}t; H^1_\rho)$,
(iii) $u(0)$ has the law U_0,
(iv) for all $t \in [0,T]$ and $\psi \in C^\infty_0(\mathcal{O};\mathbb{R}^d)$ with $\operatorname{div}(\psi\vartheta_\rho) = 0$ we have

$$
\begin{aligned}
\langle u(t),\psi\rangle_\rho = \langle u(0),\psi\rangle_\rho &+ \nu\int_0^t \langle u(s),\Delta(\vartheta_\rho\psi)\rangle\,\mathrm{d}s \\
&+ \int_0^t \Big\{\sum_{i=1}^d \big\langle u(s)u_i(s),\partial_{x_i}(\psi\vartheta_\rho)\big\rangle + \langle F(s,u(s)),\psi\rangle_\rho\Big\}\,\mathrm{d}s \\
&+ \Big\langle \int_0^t G(s,u(s))\,\mathrm{d}W(s),\psi\Big\rangle_\rho .
\end{aligned}
$$

Remark 1.1. In Definition 1.2 we make a specific choice of the space and the class of testing functions. The incompressibility condition $\operatorname{div} u = 0$ is contained in the requirement that u belong to the divergence-free space H_ρ. Since $\operatorname{div} u = 0$, the term $\langle u, \nabla u_i\rangle$ in (0.1) has been replaced by $\operatorname{div}(uu_i)$.

The form of the equation appearing in (iv) is explained by the following heuristic argument. If we take the inner product in $\mathbb{L}^2_\rho$ of both sides of (0.1) with ψ such that $\operatorname{div}(\psi\vartheta_\rho) = 0$, then the pressure vanishes and after integrating by parts we obtain (iv). Going back, suppose that u is sufficiently regular. We move all the terms of the equation in condition (iv) of Definition 1.2 on one side and integrate by parts. Then we arrive at the relation of the form $\langle\Theta(t;u),\psi\rangle_\rho = 0$ with a certain Θ. Since we have the following orthogonal decomposition

$$
\mathbb{L}^2_\rho = \overline{\{\nabla p : p\in C_0^\infty(\mathcal{O};\mathbb{R}^d)\}} \oplus \overline{\{\psi\in C_0^\infty(\mathcal{O};\mathbb{R}^d) :\ \operatorname{div}(\psi\vartheta_\rho) = 0\}},
$$

we conclude that Θ is in the form of the gradient of a certain function. Hence formally $\frac{\mathrm{d}}{\mathrm{d}t}\Theta(t;u) = \nabla p(t)$, however, similarly as in the bounded domain case, Θ need not to be absolutely continuous with respect to the time variable.

The main result concerning the case of U_0 being concentrated on $H = H_0$ is the following theorem.

Theorem 1.1. *Let U_0 be a probability measure on H satisfying*

$$
\int_H |v|^p\,\mathrm{d}U_0(v) < \infty \qquad \textit{for every } p\in[1,\infty).
$$

Assume that $F : [0,T]\times\mathbb{L}^2 \to W^{-1}$ and $G : [0,T]\times\mathbb{L}^2 \to L_{(\mathrm{HS})}(\mathcal{H},\mathbb{L}^2)$ are measurable, weakly continuous with respect to the second variable, and satisfy

$$
\|F(t,v)\|_{-1} + \|G(t,v)\|_{L_{(\mathrm{HS})}(\mathcal{H},\mathbb{L}^2)} \le C(1+|v|)
$$

with some constant C independent of u. Then there exists an H-valued martingale solution to (0.1) *starting from U_0.*

Remark 1.2. In the 2-dimensional case, under the condition of Lipschitz continuity of $F : [0,T]\times\mathbb{L}^2 \to W^{-1}$ and $G : [0,T]\times\mathbb{L}^2 \to L_{(\mathrm{HS})}(\mathcal{H},\mathbb{L}^2)$ with respect to the second variable, we can prove that the solution is unique. Due to the estimate of Proposition 2.2(iii) we can proceed in exactly the same way as for bounded domains, see [33]. Namely, let us consider the difference $z(t) = u_1(t) - u_2(t)$ of two solutions with the same initial value and let us apply Itô's formula to $|z(t)|^2\xi(t)$, where $\xi(t) = \exp\{-C\int_0^t ||u_1(s)||_1^2 \, \mathrm{d}s\}$ with C large enough. Straightforward estimations lead to

$$\mathbb{E}(|z(t)|^2\xi(t)) \le C\int_0^t \mathbb{E}(|z(s)|^2\xi(s))\,\mathrm{d}s,$$

which by means of Gronwall's lemma implies $z(t) = 0$ a.s. as $\mathbb{P}(\xi < \infty) = 1$.

Let $W^Q = (W_1^Q, \dots, W_m^Q)$ be an $L^2(\mathcal{O}; \mathbb{R}^m)$-valued Wiener process with a nuclear covariance operator Q. Then, see e.g. [17], W can be viewed as a cylindrical Wiener process on the space $\mathcal{H}_Q = \operatorname{Range} Q^{1/2}$ equipped with the scalar product

$$\langle \psi, \varphi\rangle_{\mathcal{H}_Q} = \langle u, v\rangle_{L^2(\mathcal{O};\mathbb{R}^m)},$$

where $u, v \in \left(\operatorname{Ker} Q^{1/2}\right)^\perp \subseteq L^2(\mathcal{O}; \mathbb{R}^m)$ are such that $\psi = Q^{1/2}u$, $\varphi = Q^{1/2}v$. Since $Q^{1/2}$ is Hilbert–Schmidt, it is given be an integral kernel $\mathcal{Q}$, that is

$$Q^{1/2}\psi(x) = \Big(\sum_{j=1}^m \int_{\mathcal{O}} \mathcal{Q}_{1,j}(x,y)\psi_j(y)\,\mathrm{d}y, \dots, \sum_{j=1}^m \int_{\mathcal{O}} \mathcal{Q}_{m,j}(x,y)\psi_j(y)\,\mathrm{d}y\Big).$$

Now let W^Q and $W^{\tilde{Q}}$ be two independent $L^2(\mathcal{O}; \mathbb{R}^m)$-valued Wiener processes with the covariance operators Q and $\tilde{Q}$, respectively. We denote by $\mathcal{Q}$ and $\tilde{\mathcal{Q}}$ the integral kernels corresponding to $Q^{1/2}$ and $\tilde{Q}^{1/2}$. Let

$$f, f_i, g_j, \tilde{g}_j : [0,T] \times \mathcal{O} \times \mathbb{R}^d \to \mathbb{R}^d, \quad i = 1, \dots, d, \quad j = 1, \dots, m.$$

Consider the following system of stochastic Navier–Stokes equations

$$\begin{aligned} \partial_t u - \nu\Delta u + \langle u, \nabla\rangle u + \nabla p &= f(t,x,u) + \sum_{i=1}^d \frac{\mathrm{d}}{\mathrm{d}x_i} f_i(t,x,u) \\ &\quad + \sum_{j=1}^m \{g_j(t,x,u)\dot{W}_j^Q(t) + \tilde{g}_j(t,x,u)\dot{W}_j^{\tilde{Q}}(t)\}, \\ \operatorname{div} u &= 0, \end{aligned} \tag{1.3}$$

with the Dirichlet boundary condition $u|_{\partial\mathcal{O}} = 0$ in case $\mathcal{O} \neq \mathbb{R}^d$. In (1.3), $\mathrm{d}/\mathrm{d}x_i f_i(t,x,u)$ denotes the total derivative with respect to the x_i-variable.

Since W^Q and $W^{\tilde{Q}}$ are independent, $W = (W^Q, W^{\tilde{Q}})$ is a cylindrical Wiener space on $\mathcal{H} = \mathcal{H}_Q \times \mathcal{H}_{\tilde{Q}}$. By a solution to (1.3) we understand a solution to the problem (0.1) with F and G defined for $x \in \mathcal{O}$, $v : \mathcal{O} \to \mathbb{R}^d$, and $\psi = (\psi^Q, \psi^{\tilde{Q}}) = ((\psi_1^Q, \dots, \psi_m^Q), (\psi_1^{\tilde{Q}}, \dots, \psi_m^{\tilde{Q}})) \in \mathcal{H}$ by the formulas

$$
\begin{aligned}
F(t,v)(x) &= f(t,x,v(x)) + \sum_{i=1}^{d} \frac{\mathrm{d}}{\mathrm{d}x_i} f_i(t,x,v(x)), \\
(G(t,v)\psi)(x) &= \sum_{j=1}^{m} \{ g_j(t,x,v(x))\psi_j^Q(x) + \tilde{g}_j(t,x,v(x))\psi_j^{\tilde{Q}}(x) \}.
\end{aligned}
\tag{1.4}
$$

In fact, we will show in Section 4 that under the conditions of the theorem below, F and G satisfy the conditions of Theorem 1.1, and consequently we will obtain the existence of a martingale solution to (1.3).

Theorem 1.2. *Assume that:*

(i) *U_0 is a probability measure on H such that $\int_H |v|^p \,\mathrm{d}U_0(v) < \infty$ for every $p \in [1, \infty)$,*

(ii) *the integral kernels $\mathcal{Q}$ and $\tilde{\mathcal{Q}}$ satisfy the estimate*

$$
\sum_{i,j=1}^{m} \left[\sup_{x \in \mathcal{O}} \int_{\mathcal{O}} \mathcal{Q}_{i,j}^2(x,y)\,\mathrm{d}y + \int_{\mathcal{O}}\int_{\mathcal{O}} \tilde{\mathcal{Q}}_{i,j}^2(x,y)\,\mathrm{d}x\,\mathrm{d}y \right] < \infty,
$$

(iii) *$f, f_i, g_j, \tilde{g}_j$, are measurable functions of t, x and v, continuous with respect to v, and there exist a constant L and a function $\gamma \in L^2$ such that for all t, x, v one has*

$$
\begin{aligned}
|f(t,x,v)| + \sum_{i=1}^{d} |f_i(t,x,v)| + \sum_{j=1}^{m} |g_j(t,x,v)| &\le L(\gamma(x) + |v|), \\
\sum_{j=1}^{m} |\tilde{g}_j(t,x,v)| &\le L.
\end{aligned}
$$

Then there exists a martingale H-valued solution to (1.3) *starting from U_0.*

Remark 1.3. Let F and G be given by (1.4). Recall that $\mathcal{O}$ may be unbounded. Thus, in particular constant functions need not be square integrable. The reason why we have introduced the function γ is that the assumptions of Theorem 1.2 should guarantee that F and G take values in the spaces W^{-1} and $L_{(\mathrm{HS})}(\mathcal{H}, \mathbb{L}^2)$. Now note that from (iii) we have that $g_j(t,\cdot,v(\cdot)) \in \mathbb{L}^2$ and $\tilde{g}_j(t,\cdot,v(\cdot)) \in \mathbb{L}^\infty$ for any $v \in \mathbb{L}^2$. Then, see the proof of Theorem 1.2, the condition (ii) guarantees that the multiplication operator

$$
M(v)\psi = \sum_{j=1}^{m} v\psi_j
$$

acts continuously from $\mathbb{L}^2$ into $L_{(\mathrm{HS})}(\mathcal{H}_Q, \mathbb{L}^2)$, and from $\mathbb{L}^\infty$ into $L_{(\mathrm{HS})}(\mathcal{H}_{\tilde{Q}}, \mathbb{L}^2)$.

The remaining part of this section is devoted to the Navier–Stokes equations on $\mathcal{O} = \mathbb{R}^d$ driven by an $\mathbb{R}^m$-valued spatially homogeneous Wiener random field. This field can be viewed as an example of the abstract cylindrical Wiener process introduced earlier.

First we set $\tau_h v(x) = v(x+h)$ for $v \in L^2_\rho$ or $v \in \mathbb{L}^2_\rho$, and $x \in \mathbb{R}^d$. A measure U on L^2_ρ, or $\mathbb{L}^2_\rho$ is said to be *spatially homogeneous* if and only if $U = U \circ \tau_{-h}$ for every $h \in \mathbb{R}^d$. For $\rho > 1/2$ there are nontrivial spatially homogeneous probability measures on L^2_ρ and $\mathbb{L}^2_\rho$ whereas for $\rho \le 1/2$ only the measure concentrated on the constant 0-function is spatially homogeneous. A random field v on $[0,T] \times \mathbb{R}^d$ is *spatially homogeneous* if for all $t \in [0,T]$, $x_1, \ldots, x_n \in \mathbb{R}^d$ and $h \in \mathbb{R}^d$ the random vectors $\big(v(t,x_1), \ldots, v(t,x_n)\big)$ and $\big(v(t,x_1+h), \ldots, v(t,x_n+h)\big)$ have the same distribution. Note that if $v(t,\cdot)$ is a measurable spatially homogeneous random field such that $\mathbb{E}\,|v(t,x_0)|^2 < \infty$ for a certain $x_0 \in \mathbb{R}^d$, then

$$\mathbb{E}\,|v(t,x)|^2 = \mathbb{E}\,|v(t,x_0)|^2 < \infty \quad \text{for every } x \in \mathbb{R}^d,$$

and consequently, with probability one $v(t,\cdot)$ belongs to L^2_ρ for any $\rho > 1/2$. In this case $v(t,\cdot)$ is spatially homogeneous if and only if its law is spatially homogeneous.

Definition 1.3. Let $\mathfrak{U} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ be a filtered probability space. By an $\mathbb{R}^m$*-valued spatially homogeneous Wiener random field on* $[0,T] \times \mathbb{R}^d$ we understand a system $\mathcal{W}_j$, $j = 1, \ldots, m$, of independent measurable real valued spatially homogeneous random fields on $[0,T] \times \mathbb{R}^d$ such that each $\mathcal{W}_j$ is Gaussian and Wiener, that is:

(i) The random vector $\big(\mathcal{W}_j(t_1,x_1), \ldots, \mathcal{W}_j(t_n,x_n)\big)$ is Gaussian for an arbitrary finite sequence $(t_1,x_1), \ldots, (t_n,x_n) \in [0,T] \times \mathbb{R}^d$.
(ii) For each $x \in \mathbb{R}^d$, $\{\mathcal{W}_j(t,x)\}_{t\in[0,T]}$ is a real valued Wiener process with respect to the filtration $(\mathcal{F}_t)$.

Let $\mathcal{W}$ be an $\mathbb{R}^m$-valued spatially homogeneous Wiener random field on $\mathbb{R}^d$. Then, see [32], there are symmetric positive finite measures μ_j, $j = 1, \ldots, m$, on $\mathbb{R}^d$ such that for all $j = 1, \ldots, m$, $t, s \in [0,T]$, and $x, y \in \mathbb{R}^d$ one has

$$\mathbb{E}\,\mathcal{W}_j(t,x)\mathcal{W}_j(s,y) = t \wedge s\, \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \exp\{i\langle x-y, z\rangle\}\, \mathrm{d}\mu_j(z).$$

We call $\mu = (\mu_1, \ldots, \mu_m)$ the *spectral measure of* $\mathcal{W}$. Clearly, the distribution of $\mathcal{W}$ is uniquely determined by its spectral measure.

Stochastic evolution equations driven by a homogeneous Wiener process were studied in [19] and [32]. In [19] the spectral measure of the noise is assumed to be absolutely continuous with respect to Lebesgue measure. In our paper we do not need this assumption. In the absolutely continuous case the typical densities are $a_j \exp(-b_j|x|^{r_j})$, $a_j, b_j, r_j > 0$, and $a_j(b_j + |x|^2)^{-r_j}$, $a_j, b_j > 0$ and $r_j > d/2$. In the case where we have no density, a simple example is obtained by taking μ_j equal to the Dirac measure δ_0. Then $\mathcal{W}_j(t,x)$ does not depend on x and is a one dimensional Brownian motion. Some interesting examples can be constructed by considering the combinations of the above or the measures concentrated on manifolds.

Let $\mathcal{W}$ be an $\mathbb{R}^m$-valued spatially homogeneous Wiener random field on $\mathbb{R}^d$ with a spectral measure μ. Let us denote by $L^2_{(\mathrm{s})}(\mathbb{R}^d,\mu_j)$ the subspace of $L^2(\mathbb{R}^d,\mu_j;\mathbb{C})$ consisting of all ξ such that $\xi(x)=\overline{\xi(-x)}$, $x\in\mathbb{R}^d$. Set

$$\mathcal{H}_\mu=\{\widehat{\xi\mu}=(\widehat{\xi_1\mu_1},\dots,\widehat{\xi_m\mu_m}):\ \xi_j\in L^2_{(\mathrm{s})}(\mathbb{R}^d,\mu_j)\},$$

where $\widehat{v}$ denotes the Fourier transform of v. Then as each μ_j is finite we have $\mathcal{H}_\mu\subset C_{\mathrm{b}}(\mathbb{R}^d;\mathbb{R}^m)$. Clearly, $\mathcal{H}_\mu$ equipped with the norm

$$\langle\widehat{\xi\mu},\widehat{\eta\mu}\rangle_{\mathcal{H}_\mu}=\sum_{j=1}^m\int_{\mathbb{R}^d}\xi_j(x)\overline{\eta_j(x)}\,\mathrm{d}\mu_j(x)$$

is a real separable Hilbert space. Then, see [32],

$$W(t,\psi)=\sum_{j=1}^m\int_{\mathbb{R}^d}\mathcal{W}_j(t,x)\psi_j(x)\,\mathrm{d}x,\qquad \psi=(\psi_1,\dots,\psi_m)\in\mathcal{H}_\mu \tag{1.5}$$

is a cylindrical Wiener process on $\mathcal{H}_\mu$. Note that having a cylindrical Wiener process W on $\mathcal{H}_\mu$ we can easily find an $\mathbb{R}^m$-valued spatially homogeneous Wiener random field $\mathcal{W}$ with spectral measure μ such that (1.5) is fulfilled.

Let $\mathcal{W}$ be an $\mathbb{R}^m$-valued spatially homogeneous Wiener random field. Let

$$f,f_i,g_j:[0,T]\times\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}^d,\quad i=1,\dots,d,\quad j=1,\dots,m,$$

and let us consider the system of stochastic Navier–Stokes equations on $\mathbb{R}^d$ driven by $\mathcal{W}$,

$$\begin{aligned}\partial_t u-\nu\Delta u+\langle u,\nabla\rangle u+\nabla p&=f(t,x,u)+\sum_{i=1}^d\frac{\mathrm{d}}{\mathrm{d}x_i}f_i(t,x,u)\\&\quad+\sum_{j=1}^m g_j(t,x,u)\dot{\mathcal{W}}_j,\\ \operatorname{div}u&=0.\end{aligned}\tag{1.6}$$

As in the case of (1.3), by a solution to (1.6) we understand a solution to the problem (0.1) with $\mathcal{H}=\mathcal{H}_\mu$, and F and G given by the formulas

$$\begin{aligned}F(t,v)(x)&=f(t,x,v(x))+\sum_{i=1}^d\frac{\mathrm{d}}{\mathrm{d}x_i}f_i(t,x,v(x)),\\ \big(G(t,v)\widehat{\xi\mu}\big)(x)&=\sum_{j=1}^m g_j(t,x,v(x))\widehat{\xi_j\mu_j}(x).\end{aligned}\tag{1.7}$$

It is worth pointing out that the stochastic integral term in Definition 1.2 can be interpreted as

$$\sum_{i=1}^{d}\sum_{j=1}^{m}\int_{\mathbb{R}^d}\int_0^t g_{j,i}(s,x,u(s,x))\mathcal{W}_j(\,\mathrm{d}s,x)\psi_i(x)\vartheta_\rho(x)\,\mathrm{d}x,$$

where $\mathcal{W}_j(\,\mathrm{d}s,x)$ means that we integrate in Itô's sense with respect to the real valued Wiener process $\mathcal{W}_j(\cdot,x)$, see [32, Section 1.2] for more details.

Theorem 1.3. *Assume that:*

(i) *U_0 is a probability measure on H such that $\int_H |v|^p\,\mathrm{d}U_0(v) < \infty$ for every $p \in [1,\infty)$,*

(ii) *f, f_i, g_j, are measurable functions of t, x and v, continuous with respect to v, and there exist a constant L and a function $\gamma \in L^2$ such that for all t, x, v one has*

$$|f(t,x,v)| + \sum_{i=1}^{d}|f_i(t,x,v)| + \sum_{j=1}^{m}|g_j(t,x,v)| \le L(\gamma(x) + |v|).$$

Then there exists a martingale H-valued solution to (1.6) *starting from U_0.*

Our last result is concerned with spatially homogeneous solutions. As in Theorem 1.3 we assume that $\mathcal{O} = \mathbb{R}^d$, $\mathcal{W}$ is an $\mathbb{R}^m$-valued spatially homogeneous Wiener random field, and F, G are given by (1.7). The initial value need not be concentrated on H. Thus we have to take $\rho > 1/2$ and the result is not covered by Theorem 1.1. We assume that $\rho < 1$ for some technical reasons, see Theorem C.1.

Theorem 1.4. *Let $1/2 < \rho < 1$, and let U_0 be a homogeneous probability measure on H_ρ satisfying*

$$\int_{H_\rho} |v|_\rho^p\,\mathrm{d}U_0(v) < \infty \qquad \text{for every } p \in [1,\infty).$$

Assume that the functions f, f_i and g_i are measurable and independent of the x-variable. Furthermore, assume that there exists a constant L such that for all $t \in [0,T]$, and $v, z \in \mathbb{R}^d$ and i, j one has

$$\begin{aligned}|f(t,v) - f(t,z)| + |f_i(t,v) - f_i(t,z)| + |g_j(t,v) - g_j(t,z)| &\le L|v-z|,\\ |f(t,v)| + |f_i(t,v)| + |g_j(t,v)| &\le L(1+|v|).\end{aligned}$$

Then there exists an H_ρ-valued martingale solution to (1.6) *starting from U_0. Moreover, the random field u is spatially homogeneous.*

2. Auxiliary results

For any $q \in [2,\infty)$ we denote by S the semigroup on $\mathbb{L}^q_0$ generated by $A=\nu\Delta$ with the Dirichlet boundary conditions. Then S is holomorphic on any $\mathbb{L}^q_0$-space. Moreover, for arbitrary $\rho \geq 0$ and $q\in[2,\infty)$, S has the unique extension to a holomorphic semigroup on $\mathbb{L}^q_\rho$. To see this we set $i_\rho v = \vartheta_{-\rho/2} v$. Obviously, for every $q\in[1,\infty)$, i_ρ is an isometry between $\mathbb{L}^q_0$ and $\mathbb{L}^q_{q\rho/2}$. Consider now the operator $\tilde{A}$ given by

$$\tilde{A} = i^{-1}_{\frac{2\rho}{q}} A i_{\frac{2\rho}{q}} = A + \nu \sum_{j=1}^{d} \left(\vartheta_{\frac{\rho}{q}} \partial_{x_j} \vartheta_{\frac{-\rho}{q}}\right)\partial_{x_j} + \nu \vartheta_{\frac{\rho}{q}} \Delta \vartheta_{\frac{-\rho}{q}}.$$

Since $\tilde{A}$ is a uniformly elliptic operator with coefficients from $C^\infty_{\mathrm{b}}(\mathbb{R}^d)$, it generates a holomorphic semigroup $\tilde{S}$ on $\mathbb{L}^p_0$. Let $S_\rho = i_{2\rho/q}\tilde{S} i^{-1}_{2\rho/q}$. Then S_ρ is holomorphic on $\mathbb{L}^q_\rho$ as it is unitarily equivalent to $\tilde{S}$. Note that A is the generator of S_ρ, and that S_ρ is an extension of S. In what follows we will use the same symbol to denote the semigroup on $\mathbb{L}^q_0$ and its extension to $\mathbb{L}^q_\rho$.

It is easy to check that each H_ρ is invariant with respect to S. Consequently, the restriction of S to H_ρ is also a holomorphic semigroup. It will cause no confusion if we use the same letter for the semigroup and its restriction to H_ρ. Similarly, we denote by A the generator of S on $\mathbb{L}^2_\rho$ and H_ρ.

Let λ belong to the resolvent set of A. Using the same methods as in [32, Section 3] one may easily show that the Sobolev spaces W^1_ρ and H^1_ρ introduced in Section 1 are identical with the domain of $(-A+\lambda\mathrm{I})^{\frac{1}{2}}$ in $\mathbb{L}^2_\rho$ and H_ρ, respectively.

For any positive real r we define W^r_ρ and H^r_ρ as the domain of $(-A+\lambda\mathrm{I})^{\frac{r}{2}}$, where A is considered as the generator of a holomorphic semigroup on $\mathbb{L}^2_\rho$ and H_ρ. The spaces W^r_ρ and H^r_ρ are equipped with the graph norm $||\cdot||_{r,\rho}$. Identifying $(\mathbb{L}^2_\rho)^*$ with $\mathbb{L}^2_\rho$, and $(H_\rho)^*$ with H_ρ we set $W^{-r}_\rho = (W^r_\rho)^*$ and $H^{-r}_\rho = (H^r_\rho)^*$. In the paper we write W^r, H^r, and $||\cdot||_r$ instead of W^r_0, H^r_0, and $||\cdot||_{r,0}$.

Let $\rho\in\mathbb{R}$ and $\alpha\in[0,1)$. Set

$$\mathcal{J}^\rho_\alpha(v)(t) = \int_0^t (t-s)^{-\alpha} S(t-s) v(s)\,\mathrm{d}s, \quad v\in L^1(0,T;\mathbb{L}^1_\rho),\ t\in[0,T]. \tag{2.8}$$

The following propositions play an important role in the proof of the existence of a martingale solution. Their proofs are postponed into Appendix A.

Proposition 2.1. *Let $0\leq\rho<\kappa$, and let $\mathcal{J}^\rho_\alpha$ be given by (2.8).*
(i) *For an arbitrary $p>1$, $\mathcal{J}^\rho_0$ can be extended to a compact operator acting from $L^p(0,T;W^{-1}_\rho)$ into $L^2(0,T;\mathbb{L}^2_\kappa)$.*
(ii) *For arbitrary $\alpha\in[0,1)$ and $p\in(1,\infty)$ such that $\alpha+p^{-1}<1$, $\mathcal{J}^\rho_\alpha$ is a compact operator from $L^p(0,T;\mathbb{L}^2_\rho)$ into $L^2(0,T;\mathbb{L}^2_\kappa)$.*

(iii) *If $5\rho/3 < \kappa$, then $\mathcal{J}_0^\rho$ is a compact operator acting from $L^{6/5}(0,T;\mathbb{L}_\rho^{6/5})$ into $L^2(0,T;\mathbb{L}_\kappa^2)$.*

For $u \in C_0^\infty(\mathcal{O};\mathbb{R}^d)$ and $x \in \mathbb{R}^d$ write

$$B_0(v)(x) = \Big(\sum_{i=1}^d \partial_{x_i}(v_i v_1)(x), \dots, \sum_{i=1}^d \partial_{x_i}(v_i v_d)(x)\Big). \tag{2.9}$$

Proposition 2.2. *Let B_0 be given by (2.9). Then:*
(i) *for any $\rho \in \mathbb{R}$, B_0 has a unique extension to a continuous mapping B acting from $\mathbb{L}_\rho^4$ into W_ρ^{-1}. Moreover, there exists a constant C such that*

$$||B(v)||_{-1,\rho} \le C|v|_{\mathbb{L}_\rho^4}^2 \qquad \textit{for } v \in \mathbb{L}_\rho^4.$$

(ii) *B_0 has a unique extension to a continuous mapping B acting from W^1 into W^{-1}. Moreover, there exists a constant C such that*

$$||B(v)||_{-1} \le C|v|^{\frac{1}{2}}||v||_1^{\frac{3}{2}} \qquad \textit{for } v \in W^1.$$

(iii) *If $d = 2$ then there is a constant C such that*

$$||B(v)||_{-1} \le C|v|||v||_1 \qquad \textit{for } v \in W^1.$$

3. Proof of Theorem 1.1

Let B be the mapping constructed in Proposition 2.2. A martingale solution to (0.1) is found using a modification of the classical Galerkin method. The idea is this: consider a sequence of equations in which F, B, and G are replaced by $\Pi_n F$, $\Pi_n B(\Pi_n \cdot)$ and $\Pi_n G$ for a properly chosen system of projections $\{\Pi_n\}$. The goal is to show that the laws of the sequence of solutions are tight in a suitable space and to extract a convergent subsequence. Skorokhod's theorem allows us to replace the convergence of laws by pointwise convergence of the stochastic processes after changing the underlying probability space. Hence we can find that the limit is a solution to our equation.

By Theorem B.1 from Appendix B there exists an orthonormal basis $\{e_k\}$ of H such that $\{e_k\} \subset C_{\mathrm{b}}^\infty(\mathcal{O};\mathbb{R}^d) \cap H^2$, $\{e_k\}$ is a Schauder basis of H^1, and

$$\sup_n ||\Pi_n||_{L(H^1,H^1)} < \infty. \tag{3.10}$$

Let $n \in \mathbb{N}$. As $\{e_k\} \subset H^1 = (H^{-1})^*$ we can extend each Π_n to H^{-1} putting

$$\Pi_n v = \sum_{k=1}^n \langle u, e_k \rangle e_k,$$

where $\langle\cdot,\cdot\rangle$ is now considered as the dual form on $H^{-1}\times H$. Note that

$$\begin{aligned}||\Pi_n||_{L(H^{-1},H^{-1})} &= \sup_{||v||_{-1}\le 1}||\Pi_n v||_{-1} = \sup_{||u||_1\le 1}\sup_{||v||_{-1}\le 1}|\langle \Pi_n v,u\rangle| \\ &= \sup_{||u||_1\le 1}\sup_{||v||_{-1}\le 1}|\langle v,\Pi_n u\rangle| \le ||\Pi_n||_{L(H^1,H^1)}.\end{aligned}$$

Hence, from (3.10), we have

$$\sup_n ||\Pi_n||_{L(H^{-1},H^{-1})}<\infty. \tag{3.11}$$

For $t\in[0,T]$ and $v\in H$ we set $F^{(n)}(t,v)=\Pi_n F(t,v)$, $G^{(n)}(t,v)=\Pi_n G(t,v)$, and $B^{(n)}(v)=\Pi_n B(\Pi_n v)$.

Let $\mathfrak{U}=(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\in[0,T]},\mathbb{P})$ be a filtered probability space, let W be an $(\mathcal{F}_t)$-adapted cylindrical Wiener process on $\mathcal{H}$, and let $u(0)$ be an $\mathcal{F}_0$-measurable H-valued random element with the law U_0. Consider the following stochastic evolution equation

$$\begin{aligned}\mathrm{d}u^{(n)} =&\big(Au^{(n)}+B^{(n)}(u^{(n)})+F^{(n)}(t,u^{(n)})\big)\,\mathrm{d}t \\ &+\ G^{(n)}(t,u^{(n)})\,\mathrm{d}W, \\ u^{(n)}(0) =&\Pi_n u(0).\end{aligned} \tag{3.12}$$

By a solution to (3.12) we mean the so-called *mild solution*, that is a measurable $(\mathcal{F}_t)$-adapted process $u^{(n)}$, having continuous trajectories in H and such that for $t\in[0,T]$,

$$\begin{aligned}u^{(n)}(t) = S(t)u^{(n)}(0) &+ \int_0^t S(t-s)\big\{B^{(n)}(u^{(n)}(s))+F^{(n)}(s,u^{(n)}(s))\big\}\,\mathrm{d}s \\ &+\int_0^t S(t-s)G^{(n)}(s,u^{(n)}(s))\,\mathrm{d}W(s).\end{aligned}$$

It is known that, see [17], that every mild solution is also a *weak solution*, that is

$$\begin{aligned}\langle u^{(n)}(t),\psi\rangle = \langle u^{(n)}(0),\psi\rangle &+ \int_0^t \langle u^{(n)}(s),A\psi\rangle\,\mathrm{d}s \\ &+\int_0^t\langle B^{(n)}(u^{(n)}(s))+F^{(n)}(s,u^{(n)}(s)),\psi\rangle\,\mathrm{d}s \\ &+\Big\langle\int_0^t G^{(n)}(s,u^{(n)}(s))\,\mathrm{d}W(s),\psi\Big\rangle\end{aligned}$$

for all $t\in[0,T]$ and $\psi\in\mathrm{Dom}\,A^*=\mathrm{Dom}\,A$. Note that $C_0^\infty(\mathcal{O};\mathbb{R}^d)\subset\mathrm{Dom}\,A$.

Lemma 3.1. *Assume that there are constants L and K such that for all $t \in [0,T]$, and $u, v \in \mathbb{L}^2$ we have*

$$||F(t,v) - F(t,u)||_{-1} + ||G(t,v) - G(t,u)||_{L_{(\mathrm{HS})}(\mathcal{H},\mathbb{L}^2)} \leq L|v-u|,$$
$$||F(t,v)||_{-1} + ||G(t,v)||_{L_{(\mathrm{HS})}(\mathcal{H},\mathbb{L}^2)} \leq K(1+|v|).$$

Then for every $n \in \mathbb{N}$ there exists a unique solution $u^{(n)}$ to (3.12). Moreover, for any $p \geq 2$,

$$\mathbb{E} \sup_{t\in[0,T]} |u^{(n)}(t)|^p \leq C(p) \tag{3.13}$$

and

$$\mathbb{E}\left(|u^{(n)}(t)|^2 + \int_0^t ||u^{(n)}(s)||_1^2 \, \mathrm{d}s\right) \leq C, \tag{3.14}$$

the constants $C(p)$ and C being independent of n, L, and $t \in [0,T]$.

Proof. We consider (3.12) as an equation on H with locally Lipschitz nonlinearities. Then it is easy to show that for an arbitrary probability space there exists a unique, possibly exploding, mild solution $u^{(n)}$ of (3.12). Since $u^{(n)}(0) \in \operatorname{Dom} A$, and $B^{(n)}$, $F^{(n)}$ are continuous mapping from H into $\operatorname{Dom} A$, and since $G^{(n)}$ is a continuous mapping from H into $L_{(\mathrm{HS})}(\mathcal{H}, \operatorname{Dom} A)$, $u^{(n)}$ is a local *strong solution* to (3.12). In particular, $u^{(n)}$ has continuous trajectories in $\operatorname{Dom} A$. Set

$$\Psi_n(t) = Au^{(n)}(t) + B^{(n)}(u^{(n)}(t)) + F^{(n)}(t, u^{(n)}(t)).$$

We locally have

$$\mathrm{d}u^{(n)} = \Psi_n \, \mathrm{d}t + G^{(n)}(t, u^{(n)}) \, \mathrm{d}W.$$

We now show (3.14) which guarantees that $u^{(n)}$ is a global solution to (3.12). Let $p \geq 2$. Applying Itô's formula to the function $H_p(v) = |v|^p$ we obtain

$$\mathrm{d}|u^{(n)}|^p = \Phi_p(t, u^{(n)}) \, \mathrm{d}t + H_p'(u^{(n)}) G^{(n)}(t, u^{(n)}) \, \mathrm{d}W,$$

where

$$\Phi_p(t,v) := \langle H_p'(v), \Psi_n(t)\rangle + \frac{1}{2} \operatorname{Trace} H_p''(v) G^{(n)}(t,v) G^{(n)}(t,v)^*.$$

Then we have

$$\Phi_p(t,v) \leq p|v|^{p-2}\Big(\langle Av, v\rangle + |\langle F^{(n)}(t,v), v\rangle| + |\langle B^{(n)}(v), v)\rangle|$$
$$+ \frac{p-1}{2} ||G^{(n)}(t,v)||^2_{L_{(\mathrm{HS})}(\mathcal{H},H)}\Big).$$

As A is self-adjoint, for $v \in \mathrm{Dom}\, A$ we have $\langle Av, v\rangle = -\nu ||v||_1^2$. Integrating by parts and using the fact that $\mathrm{div}\, v = 0$ we obtain $\langle B^{(n)}(v), v\rangle = 0$. From (3.11) we have

$$\begin{aligned}|\langle F^{(n)}(t,v), v\rangle| &+ \frac{p-1}{2}||(G^{(n)}(t,v))||^2_{L_{(\mathrm{HS})}(\mathcal{H},H)}\\ &\le c_1(1+|v|)||v||_1 + c_2(1+|v|^2)\\ &\le \frac{\nu}{2}||v||_1^2 + c_3(1+|v|^2).\end{aligned}$$

Hence

$$\Phi_p(t,v) \le p|v|^{p-2}\big(-\frac{\nu}{2}||v||_1^2 + c_3(1+|v|^2)\big).$$

Now let $p = 2$. Then taking the mathematical expectation we get

$$\mathbb{E}\left(|u^{(n)}(t)|^2 + \nu\int_0^t ||u^{(n)}(s)||_1^2\,\mathrm{d}s\right) \le \mathbb{E}|u(0)|^2 + 2c_3t + 2c_3\,\mathbb{E}\int_0^t |u^{(n)}(s)|^2\,\mathrm{d}s.$$

Applying Gronwall's lemma we obtain (3.14).

Now let $p \ge 2$. Then using the maximal inequality for martingales we obtain

$$\begin{aligned}\mathbb{E}\sup_{s\in[0,t]}|u^{(n)}(s)|^{2p} &\le 3\,\mathbb{E}|u^{(n)}(0)|^{2p}\\ &\quad + 3\left(c_3p\,\mathbb{E}\int_0^t |u^{(n)}(s)|^{p-2}(1+|u^{(n)}(s)|^2)\,\mathrm{d}s\right)^2\\ &\quad + c_4\,\mathbb{E}\int_0^t ||H_p'(u^{(n)}(s))G^{(n)}(s,u^{(n)}(s))||^2_{L_{(\mathrm{HS})}(\mathcal{H},H)}\,\mathrm{d}s\\ &\le c_5 + 3\mathbb{E}|u^{(n)}(0)|^{2p} + c_5\,\mathbb{E}\int_0^t |u^{(n)}(s)|^{2p}\,\mathrm{d}s,\end{aligned}$$

c_5 being independent of n and $t \in [0,T]$. Hence Gronwall's lemma gives (3.13). $\square$

Proof of Theorem 1.1. We first prove the theorem under the additional assumption that F, G satisfy the conditions of Lemma 3.1.

For each n let $u^{(n)}$ be the solution to (3.12). Let us fix $\alpha \in (1/2, 1)$ and $p > 2$ such that $\alpha + p^{-1} < 1$. Recall that the operators $\mathcal{J}_\alpha^\rho$, $\rho \in \mathbb{R}$ were introduced in Section 2. In this section we write $\mathcal{J}_0$ and $\mathcal{J}_\alpha$ instead of $\mathcal{J}_0^0$ and $\mathcal{J}_\alpha^0$. Let

$$Y_n^\alpha(t) = \frac{\sin(\alpha-1)\pi}{\pi}\int_0^t (t-s)^{\alpha-1}S(t-s)G^{(n)}(s,u^{(n)}(s))\,\mathrm{d}W(s).$$

Then by the factorization result from [16, Proposition 1] we have

$$\int_0^t S(t-s)G^{(n)}(s,u^{(n)}(s))\,\mathrm{d}W(s) = \mathcal{J}_\alpha(Y_n^\alpha)(t),$$

and consequently

$$u^{(n)}(t) = S(t)\Pi_n u(0) + \mathcal{J}_0\big(B^{(n)}(u^{(n)}) + F^{(n)}(\cdot, u^{(n)})\big)(t) + \mathcal{J}_\alpha(Y_n^\alpha)(t).$$

Our first goal is to show that the laws of $u^{(n)}$, $n \in \mathbb{N}$, are tight in $L^2(0,T;H_\kappa)$, where $\kappa > 0$. Let us fix such a κ. Clearly, the sequence $\{S(\cdot)\Pi_n u(0)\}$ converges, and it is enough to show that the laws of the remaining sequence

$$y_n = \mathcal{J}_0\big(B^{(n)}(u^{(n)}) + F^{(n)}(\cdot, u^{(n)})\big) + \mathcal{J}_\alpha(Y_n^\alpha)$$

are tight. To this end, for each $\delta > 0$ we will find a relatively compact set Λ_δ in $L^2(0,T;H_\kappa)$ such that

$$\mathbb{P}(y_n \in \Lambda_\delta) \geq 1 - \delta \qquad \text{for every } n. \tag{3.15}$$

We put $\Lambda_\delta = \Lambda(R)$ for R sufficiently large, where

$$\Lambda(R) = \mathcal{J}_0(K_1(R)) + \mathcal{J}_\alpha(K_2(R)),$$

$K_i(R)$ being balls with radius R in $L^{5/4}(0,T;H^{-1})$ and $L^p(0,T;H)$, respectively. Note that, by Proposition 2.1(i) and (ii), $\Lambda(R)$ is compact. Thus to prove that for each $\delta > 0$ there exists an R such that (3.15) holds it is enough to show that

$$\sup_n \mathbb{E}\int_0^T ||B^{(n)}(u^{(n)}(t))||_{-1}^{\frac{5}{4}}\, dt < \infty, \tag{3.16}$$

$$\sup_n \mathbb{E}\int_0^T ||F^{(n)}(t, u^{(n)}(t))||_{-1}^{\frac{5}{4}}\, dt < \infty, \tag{3.17}$$

$$\sup_n \mathbb{E}\int_0^T |Y_n^\alpha(t)|^p\, dt < \infty. \tag{3.18}$$

To show (3.16) note that, by (3.11), Proposition 2.2(ii), and (3.10) we have

$$\begin{aligned} ||B^{(n)}(u^{(n)})||_{-1} &\leq c_1\, ||B(\Pi_n u^{(n)})||_{-1} \leq c_2\, |\Pi_n u^{(n)}(t)|^{\frac{1}{2}}\ \ ||\Pi_n u^{(n)}(t)||_1^{\frac{3}{2}} \\ &\leq c_3\, |u^{(n)}(t)|^{\frac{1}{2}}\ \ ||u^{(n)}(t)||_1^{\frac{3}{2}}. \end{aligned}$$

Hence

$$\begin{aligned} \mathbb{E}\int_0^T ||B^{(n)}(u^{(n)}(t))||_{-1}^{\frac{5}{4}}\, dt &\leq c_3\, \mathbb{E}\int_0^T |u^{(n)}(t)|^{\frac{5}{8}} ||u^{(n)}(t)||_1^{\frac{15}{8}}\, dt \\ &\leq c_3\, \mathbb{E}|u^{(n)}|_{L^\infty(0,T;H)}^{\frac{5}{8}} |u^{(n)}|_{L^{\frac{15}{8}}(0,T;H_1)}^{\frac{15}{8}} \\ &\leq c_3\, \left(\mathbb{E}|u^{(n)}|_{L^\infty(0,T;H)}^{10}\right)^{\frac{1}{16}} \left(\mathbb{E}|u^{(n)}|_{L^{\frac{15}{8}}(0,T;H_1)}^{2}\right)^{\frac{15}{16}}, \end{aligned}$$

and applying (3.13) and (3.14) we obtain (3.16).

Clearly (3.17) follows from (3.11), and (3.14). To show (3.18) note that

$$\begin{aligned}\mathbb{E}\int_0^T &|Y_n^\alpha(t)|^p\,\mathrm{d}t\\ &= \int_0^T \mathbb{E}\Big|\frac{\sin(\alpha-1)\pi}{\pi}\int_0^t (t-s)^{\alpha-1}S(t-s)G^{(n)}(s,u^{(n)}(s))\,\mathrm{d}W(s)\Big|^p\,\mathrm{d}t\\ &\le c_4\int_0^T \mathbb{E}\Big(\int_0^t (t-s)^{2\alpha-2}||S(t-s)G^{(n)}(s,u^{(n)}(s))||^2_{L_{(\mathrm{HS})}(\mathcal{H},H)}\,\mathrm{d}s\Big)^{\frac{p}{2}}\,\mathrm{d}t.\end{aligned}$$

Thus (3.18) follows from (3.13).

Let $\mathcal{L}(u^{(n)})$ denote the law of $u^{(n)}$ on $L^2(0,T;\mathbb{L}^2_\kappa)$. According to (3.15) the sequence $\mathcal{L}(u^{(n)})$, $n\in\mathbb{N}$, is tight, and hence, by the Prokhorov theorem, it is relatively weakly compact. Let $\tilde{\mathcal{H}}$ be a Hilbert space such that $\mathcal{H}\hookrightarrow\tilde{\mathcal{H}}$ with a Hilbert–Schmidt imbedding. Then since W is a process with continuous trajectories in $\tilde{\mathcal{H}}$, the sequence $\mathcal{L}(u^{(n)},W)$, $n\in\mathbb{N}$ of the laws of $(u^{(n)},W)$ on $L^2(0,T;\mathbb{L}^2_\kappa)\times C([0,T];\tilde{\mathcal{H}})$ is also relatively weakly compact. So there is a probability P^0 on $L^2(0,T;\mathbb{L}^2_\kappa)\times C([0,T];\tilde{\mathcal{H}})$ such that $\mathcal{L}(u^{(n_l)},W)$ converges weakly to P^0 for a subsequence $\{n_l\}$, which we will denote by $\{n\}$. By the Skorokhod imbedding, see [24], there exists a probability space $(\Omega^1,\mathcal{F}^1,(\mathcal{F}^1_t),\mathbb{P}^1)$ and a sequence of random elements $\{v^{(n)},W^{(n)}\}$ on Ω^1 such that the laws of $(u^{(n)},W)$ and $(v^{(n)},W^{(n)})$ are the same, and $(v^{(n)},W^{(n)})$ converges $\mathbb{P}^1$-almost surely in $L^2(0,T;\mathbb{L}^2_\kappa)\times C([0,T];\tilde{\mathcal{H}})$ to a certain random element $(\tilde{u},\tilde{W})$. Note that for every n, $W^{(n)}$ is a cylindrical Wiener processes on $\mathcal{H}$, and $v^{(n)}$ is the solution of (3.12) driven by $W^{(n)}$. Taking a subsequence we may assume that $v^{(n)}(t,x)$ converges to $\tilde{u}(t,x)$, $\mathbb{P}^1\times\mathrm{d}t\times\mathrm{d}x$ - almost surely. Thus, by (3.13) and Fatou's lemma we obtain that for all $p,q\in[1,\infty)$,

$$\mathbb{E}^1\left(\int_0^T|\tilde{u}(t)|^q\,\mathrm{d}t\right)^{\frac{p}{q}} \le \liminf_{n\to\infty}\mathbb{E}^1\left(\int_0^T|v^{(n)}(t)|^q\,\mathrm{d}t\right)^{\frac{p}{q}}\le CT^{\frac{p}{q}}<\infty.$$

Hence, we have

$$\tilde{u}\in\bigcap_{p\in[1,\infty)}L^p(\Omega^1,\mathcal{F}^1,\mathbb{P}^1;L^\infty(0,T;H)). \tag{3.19}$$

Now since for every q the sequence $|v^{(n)}-\tilde{u}|^q_{L^q(0,T;H)}$ is uniformly integrable, we have

$$\mathbb{E}^1\int_0^T|v^{(n)}(t)-\tilde{u}(t)|^q\,\mathrm{d}t\to 0\quad\text{for } q\in[1,\infty), \tag{3.20}$$

and, in particular, we may assume that

$$\int_0^T|v^{(n)}(t)-\tilde{u}(t)|^2\,\mathrm{d}t\to 0,\quad \mathbb{P}^1\text{ - a.s.}.$$

Now, as $\{v^{(n)}\}$ is bounded in $L^2(\Omega^1 \times [0,T], \mathcal{F}^1 \times \mathcal{B}([0,T]), \mathbb{P}^1 \times \,\mathrm{d}t; H^1)$, we have

$$\tilde{u} \in L^2(\Omega^1 \times [0,T], \mathcal{F}^1 \times \mathcal{B}([0,T]), \mathbb{P}^1 \times \,\mathrm{d}t; H^1). \tag{3.21}$$

To this end we note that the embedding $H^1 \hookrightarrow H_\kappa$ is compact, hence, see [31, Lemma 2.2], there exist an orthonormal basis $\{e_k^\kappa\}$ of H_κ and a sequence $\{\lambda_k\}$ such that for any $v \in H_\kappa$ we have

$$v \in H^1 \iff \sum_{k=1}^{\infty} \langle v, e_k^\kappa \rangle_\kappa^2 \lambda_k^2 < \infty,$$

and, if $v \in H^1$ then

$$||v||_1^2 = \sum_{k=1}^{\infty} \langle v, e_k^\kappa \rangle_\kappa^2 \lambda_k^2.$$

Applying Fatou's lemma we obtain

$$\mathbb{E}^1 \int_0^T \sum_{k=1}^{\infty} \langle \tilde{u}(t), e_k^\kappa \rangle_\kappa^2 \lambda_k^2 \,\mathrm{d}t \le \liminf_n \mathbb{E}^1 \int_0^T \sum_{k=1}^{\infty} \langle v^{(n)}(t), e_k^\kappa \rangle_\kappa^2 \lambda_k^2 \,\mathrm{d}t < \infty.$$

Taking into account (3.19) and (3.21) we see that $\tilde{u}$ satisfies the regularity conditions (i) and (ii) of Definition 1.2.

For $n \in \mathbb{N}$ we write

$$\mathcal{M}_n(t) = \int_0^t G^{(n)}(s, v^{(n)}(s)) \,\mathrm{d}W^{(n)}(s), \quad t \in [0,T].$$

Then $\mathcal{M}_n$ is a square integrable martingale in H, and its quadratic variation is given by

$$\ll \mathcal{M}_n \gg (t) = \int_0^t G^{(n)}(s, v^{(n)}(s)) G^{(n)}(s, v^{(n)}(s))^* \,\mathrm{d}s.$$

From (3.20) we have

$$\lim_{n,m\to\infty} \mathbb{E}^1 \int_0^T ||G^{(n)}(t, v^{(n)}(t)) - G^{(m)}(t, v^{(m)}(t))||^2_{L_{(\mathrm{HS})}(\mathcal{H},H)} \,\mathrm{d}t = 0.$$

Since there is a constant C such that for all n, m,

$$\begin{aligned}\mathbb{E}^1 \sup_{t\in[0,T]} &|\mathcal{M}_n(t) - \mathcal{M}_m(t)|^2 \\ &\le C\, \mathbb{E}^1 \int_0^T ||G^{(n)}(t, v^{(n)}(t)) - G^{(m)}(t, v^{(m)}(t))||^2_{L_{(\mathrm{HS})}(\mathcal{H},H)} \,\mathrm{d}t,\end{aligned}$$

the sequence $\mathcal{M}_n$, $n \in \mathbb{N}$, converges in $L^2(\Omega^1, \mathcal{F}^1, \mathbb{P}^1; C([0,T]; H))$ to a certain martingale $\mathcal{M}$. Taking if necessary a subsequence we may assume that the convergence is also $\mathbb{P}^1$ - almost surely. It is easy to see that

$$\ll \mathcal{M} \gg (t) = \int_0^t \Pi G(s, \tilde{u}(s)) G^*(s, \tilde{u}(s)) \Pi \, \mathrm{d}s, \quad t \in [0,T],$$

where Π is the projection of $\mathbb{L}^2$ onto H. Using now the representation theorem [17, Theorem 8.2] we can find a filtered probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t), \tilde{\mathbb{P}})$ and an $(\mathcal{F}^1_t \times \tilde{\mathcal{F}}_t)$-adapted cylindrical Wiener process $\mathbf{W}$ on $\mathcal{H}$, such that

$$M(t, \omega, \tilde{\omega}) = \int_0^t \Pi G(s, u(s, \omega, \tilde{\omega})) \, \mathrm{d}\mathbf{W}(s, \omega, \tilde{\omega}),$$

where $M(t, \omega, \tilde{\omega}) = \mathcal{M}(t, \omega)$, and $u(s, \omega, \tilde{\omega}) = \tilde{u}(s, \omega)$. Clearly, the process u is adapted to the filtration $(\mathcal{F}^1_t \times \tilde{\mathcal{F}}_t)$, $u(0)$ has the law U_0, and u satisfies the conditions (i) and (ii) of Definition 1.2. What is left is to show that u satisfies the integral equation in (iv). Fix $t \in [0,T]$ and $\psi \in C_0^\infty(\mathcal{O}; \mathbb{R}^d)$ satisfying $\operatorname{div} \psi = 0$. Set $v^{(n)}(t, \omega, \tilde{\omega}) = v^{(n)}(t, \omega)$. Since F and G are continuous we have that $\mathbb{P}^1 \times \tilde{\mathbb{P}}$ - almost surely:

$$\langle v^{(n)}(t), \psi \rangle \to \langle u(t), \psi \rangle, \qquad \langle \Pi_n v^{(n)}(0), \psi \rangle \to \langle u(0), \psi \rangle,$$

$$\int_0^t \Big\{ \langle v^{(n)}(s), \nu \Delta \psi \rangle + \sum_{i=1}^d \big\langle (\Pi_n v^{(n)}(s) (\Pi_n v^{(n)})_i(s), \partial_{x_i} \Pi_n \psi \big\rangle \Big\} \, \mathrm{d}s$$
$$\to \int_0^t \Big\{ \langle u(s), \nu \Delta \psi \rangle + \sum_{i=1}^d \big\langle u(s) u_i(s), \partial_{x_i} \psi \big\rangle \Big\} \, \mathrm{d}s,$$

$$\int_0^t \langle F(s, v^{(n)}(s)), \Pi_n \psi \rangle \, \mathrm{d}s \to \int_0^t \langle F(s, u(s)), \psi \rangle \, \mathrm{d}s,$$

$$\Big\langle \int_0^t G^{(n)}(s, v^{(n)}(s)) \, \mathrm{d}W^{(n)}(s), \psi \Big\rangle$$
$$\to \Big\langle \Pi \int_0^t G(t, u(s)) \, \mathrm{d}\mathbf{W}(s), \psi \Big\rangle = \Big\langle \int_0^t G(t, u(s)) \, \mathrm{d}\mathbf{W}(s), \psi \Big\rangle.$$

Thus the desired conclusion follows from the fact that each $v^{(n)}$ is a solution to (3.12).

Let us note that the solution constructed satisfies the estimates

$$\mathbb{E} \sup_{t \in [0,T]} |u(t)|^p \le C(p),$$

$$\mathbb{E}\left(|u(t)|^2 + \int_0^t ||u(s)||_1^2 \, \mathrm{d}s\right) \le C,$$

with $C(p)$, C independent of the Lipschitz constant L for F and G.

Finally, to get rid of the Lipschitz condition, we approximate continuous F and G by Lipschitz mappings $F_{(l)}$ and $G_{(l)}$ with the same growth constant K. Thus we obtain a sequence of solutions $u_{(l)}$ and using the same argument as in the first step of the proof we construct the required solution. □

4. Proofs of Theorems 1.2 and 1.3

Proof of Theorem 1.2. It is enough to show that the mappings F and G given by (1.4) satisfy the assumptions of Theorem 1.1. Clearly, F maps $[0,T]\times\mathbb{L}^2$ into W^{-1}. It has linear growth, and it is weakly measurable, and hence measurable. Since f and f_i are continuous with respect to the third variable, F is weakly continuous with respect to v. Similarly, as g_j and $\tilde{g}_j$ are measurable and continuous with respect to v, we have measurability and weak continuity of G. In order to show that G is Hilbert–Schmidt and its norm satisfies the linear growth estimate from Theorem 1.1 we take two orthonormal bases $\{e_k\}$ and $\{\tilde{e}_k\}$ of $(\mathrm{Ker}\, Q^{1/2})^\perp$ and $(\mathrm{Ker}\, \tilde{Q}^{1/2})^\perp$, respectively. Then for $T \in [0,T]$ and $v \in \mathbb{L}^2$ we have

$$\begin{aligned}
&\sum_k \int_{\mathcal{O}} |G(t,v)(Q^{1/2}e_k, \tilde{Q}^{1/2}\tilde{e}_k)(x)|^2 \, \mathrm{d}x \\
&\qquad = \sum_k \int_{\mathcal{O}} \Big| \sum_{i,j=1}^m \int_{\mathcal{O}} \{ g_j(t,x,v(x)) \mathcal{Q}_{i,j}(x,y)(e_k(y))_i \\
&\qquad\quad + \tilde{g}_j(t,x,v(x)) \tilde{\mathcal{Q}}_{i,j}(x,y)(\tilde{e}_k(y))_i \} \, \mathrm{d}y \Big|^2 \, \mathrm{d}x \\
&\qquad \le C_1 \sum_{i,j=1}^m \int_{\mathcal{O}} \int_{\mathcal{O}} \{ |g_j(t,x,v(x))|^2 \, \mathcal{Q}_{i,j}^2(x,y) \\
&\qquad\quad + |\tilde{g}_j(t,x,v(x))|^2 \, \tilde{\mathcal{Q}}_{i,j}^2(x,y) \} \, \mathrm{d}x \, \mathrm{d}y \\
&\qquad \le C_2(|v|^2+1) \sum_{i,j=1}^m \left[\sup_{x\in\mathcal{O}} \int_{\mathcal{O}} \mathcal{Q}_{i,j}^2(x,y) \, \mathrm{d}y + \int_{\mathcal{O}} \int_{\mathcal{O}} \tilde{\mathcal{Q}}_{i,j}^2(x,y) \, \mathrm{d}x \, \mathrm{d}y \right].
\end{aligned}$$

Hence there is a constant $C < \infty$ such that for all $t \in [0,T]$ and $v \in \mathbb{L}^2$ one has

$$||G(t,v)||_{L_{(\mathrm{HS})}(\mathcal{H},\mathbb{L}^2)} \le C(1+|v|),$$

which is the desired conclusion. □

Let $\rho \in \mathbb{R}$. For $\psi \in \mathcal{H}_\mu$ and a function $v : \mathbb{R}^d \to \mathbb{R}^m$ we write

$$M(v)\psi = \sum_{j=1}^m v_j \psi_j.$$

For a proof of the lemma below see [32, Lemma 4.1(ii)].

Lemma 4.1. *For every $\rho \in \mathbb{R}$, M is a bounded linear operator acting from $\mathbb{L}^2_\rho$ into the space of Hilbert–Schmidt operators $L_{(\mathrm{HS})}(\mathcal{H}_\mu, L^2_\rho)$.*

Proof of Theorem 1.3. Using the arguments from the proof of Theorem 1.2 one can easily show that the mapping F given by (1.7) satisfies the condition from Theorem 1.1. Let

$$\mathcal{G}_i(t,v)(x) = \left(g^i_1(t,x,v(x)),\ldots,g^i_m(t,x,v(x))\right),$$

where g^i_j denotes the i^{th}-coordinate of g_j. Note that $\mathcal{G} = (\mathcal{G}_1,\ldots,\mathcal{G}_d)$ is measurable and weakly continuous with respect to v mapping from $[0,T]\times\mathbb{L}^2$ into the m-fold product $\mathbb{L}^2\times\ldots\times\mathbb{L}^2$. Thus the desired properties of G follows from Lemma 4.1 and the representation

$$G(t,v)\psi = (M(\mathcal{G}_1(t,v))\psi,\ldots,M(\mathcal{G}_d(t,v))\psi).$$

□

5. Solution of the auxiliary equation

In this section we deal with equation (0.2) with fixed $\varepsilon > 0$. Recall that here $\mathcal{O} = \mathbb{R}^d$, $\mathcal{W}$ is an $\mathbb{R}^m$-valued spatially homogeneous Wiener random field with a spectral measure μ, and we identify $\mathcal{W}$ with a cylindrical Wiener process W on $\mathcal{H}_\mu$. Let us fix $1/2 < \rho < 1$. We assume that the nonlinear mappings F and G are given by (1.7). Since we will pass to the limit $\varepsilon \to 0$, it is sufficient to consider $\varepsilon \in (0,1]$, which will be convenient for a technical reason. Let us first explain what we mean by a solution to (0.2).

Definition 5.1. Let U_0 be a probability measure on $\mathbb{L}^2_\rho$. By an $\mathbb{L}^2_\rho$*-valued martingale solution to* (0.2) *starting from* U_0 we understand a triple $(\mathfrak{U}, W, u)$ consisting of a filtered probability space $\mathfrak{U} = (\Omega,\mathcal{F},(\mathcal{F}_t)_{t\in[0,T]},\mathbb{P})$, $(\mathcal{F}_t)$-adapted cylindrical Wiener process W on $\mathcal{H}_\mu$, and an adapted stochastic process u such that:

(i) $u \in L^p(\Omega,\mathcal{F},\mathbb{P};L^\infty(0,T;\mathbb{L}^2_\rho))$ for every $p\in[1,\infty)$,
(ii) $u \in L^2(\Omega\times[0,T],\mathcal{F}\times\mathcal{B}([0,T]),\mathbb{P}\times\,\mathrm{d}t;W^1_\rho)\cap L^6(\Omega,\mathcal{F},\mathbb{P};L^6(0,T;\mathbb{L}^6_\rho))$,
(iii) $u(0)$ has the law U_0,
(iv) for all $t\in[0,T]$ and $\psi\in C_0^\infty(\mathbb{R}^d;\mathbb{R}^d)$,

$$\begin{aligned}
\langle u(t),\psi\rangle_\rho = \langle u(0),\psi\rangle_\rho &+ \nu\int_0^t \langle u(s),\Delta(\vartheta_\rho\psi)\rangle\,\mathrm{d}s \\
&+ \int_0^t \sum_{i=1}^d \big\langle u(s)u_i, \partial_{x_i}(\psi\vartheta_\rho)\big\rangle\,\mathrm{d}s \\
&+ \int_0^t \Big\{-\langle\varepsilon|u(s)|^4u(s),\psi\rangle_\rho + \frac{1}{\varepsilon}\big\langle u(s),\nabla\mathrm{div}\,(\psi\vartheta_\rho)\big\rangle\Big\}\,\mathrm{d}s \\
&+ \int_0^t \langle F(s,u(s)),\psi\rangle_\rho\,\mathrm{d}s + \Big\langle \int_0^t G(s,u(s))\,\mathrm{d}W(s),\psi\Big\rangle_\rho .
\end{aligned}$$

Note that if $(\mathfrak{U},W,u)$ is a martingale solution to (0.2) then for each ψ from $C_0^\infty(\mathbb{R}^d;\mathbb{R}^d)$ the real valued process $\langle u(t),\psi\rangle_\rho$, $t\in[0,T]$ has continuous trajectories.

Definition 5.2. Let U_0 be a probability measure on $\mathbb{L}^2_\rho$. We say that equation (0.2) has a *unique martingale solution* starting from U_0 if for any two $\mathbb{L}^2_\rho$-valued martingale solutions $(\mathfrak{U}, W, u)$ and $(\tilde{\mathfrak{U}}, \tilde{W}, \tilde{u})$ to (0.2) starting from U_0, and for all $t \in [0,T]$ and $\psi \in C_0^\infty(\mathbb{R}^d;\mathbb{R}^d)$ the random variables $\langle u(t), \psi\rangle_\rho$ and $\langle \tilde{u}(t), \psi\rangle_\rho$ have the same distribution.

The main existence result of this section is

Theorem 5.1. *Let U_0 be a probability measure on $\mathbb{L}^2_\rho$ such that*

$$\int_{\mathbb{L}^2_\rho} |v|^p_\rho \,\mathrm{d}U_0(v) < \infty \qquad \text{for every } p \in [1,\infty).$$

Assume that the functions f, f_i and g_j appearing in (1.7) *satisfy the condition from Theorem 1.4. Then there exists a unique $\mathbb{L}^2_\rho$-valued martingale solution $(\mathfrak{U}, W, u)$ to* (0.2) *starting from U_0.*

As in the proof of Theorem 1.1 we shall use a modification of the Galerkin method. Let us denote by $\mathcal{S}^l$ the space of all $v \in C^l(\mathbb{R}^d;\mathbb{R}^d)$ such that

$$\sup_{|\alpha|\le l}\sup_{x\in\mathbb{R}^d} |(1+|x|^2)^n \partial^\alpha v(x)| < \infty \qquad \text{for all } n \in \mathbb{N}.$$

By Theorem C.1 from Appendix C, there exists an orthonormal basis $\{e_k\}$ of $\mathbb{L}^2_\rho$, such that $\{e_k\} \subset \mathcal{S}^3$, and $\{e_k\}$ is a Schauder basis of all spaces $\mathbb{L}^q_\rho$, $2 \le q \le 6$, and W^1_ρ. Let Π_n be the orthogonal projection onto the spaces $\mathrm{span}\{e_1, \dots, e_n\}$. Then by Lemma C.1 we have

$$\sup_n \big(||\Pi_n||_{L(W^1_\rho, W^1_\rho)} + ||\Pi_n||_{L(\mathbb{L}^q_\rho, \mathbb{L}^q_\rho)} \big) < \infty, \tag{5.22}$$

for any $2 \le q \le 6$. Thus, see the proof of (3.11) we also have

$$\sup_n ||\Pi_n||_{L(W^{-1}_\rho, W^{-1}_\rho)} < \infty. \tag{5.23}$$

Using the arguments from the proof of Theorem 1.3 one can easily prove the following lemma.

Lemma 5.1. *Assume that f, f_i and g_j satisfy the condition in Theorem 1.4. Then there is a constant K such that for all $t \in [0,T]$ and $v, u \in \mathbb{L}^2_\rho$ one has*

$$\begin{aligned} ||F(t,v) - F(t,u)||_{-1,\rho} + ||G(t,v) - G(t,u)||_{L_{(\mathrm{HS})}(\mathcal{H}_\mu, \mathbb{L}^2_\rho)} &\le K|v-u|_\rho, \\ ||F(t,v)||_{-1,\rho} + ||G(t,v)||_{L_{(\mathrm{HS})}(\mathcal{H}_\mu, \mathbb{L}^2_\rho)} &\le K(1+|v|_\rho). \end{aligned}$$

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ be a filtered probability space. Assume that W is an $(\mathcal{F}_t)$-adapted cylindrical Wiener process on $\mathcal{H}_\mu$, and that $u(0)$ is an $\mathcal{F}_0$-measurable random variable in $\mathbb{L}^2_\rho$ with the law U_0. For $v \in C^3_{\mathrm{b}}(\mathbb{R}^d;\mathbb{R}^d)$ we write

$$D(v) = \frac{1}{\varepsilon}\nabla \mathrm{div}\, v \quad \text{and} \quad K(v) = -\varepsilon |v|^4 v. \tag{5.24}$$

Write $B^{(n)}(\cdot) = \Pi_n B(\Pi_n \cdot)$, $K^{(n)}(\cdot) = \Pi_n K(\Pi_n \cdot)$, $D^{(n)}(\cdot) = \Pi_n D(\Pi_n \cdot)$, $F^{(n)} = \Pi_n F$, and $G^{(n)} = \Pi_n G$.

Consider the following stochastic evolution equation

$$
\begin{aligned}
\mathrm{d}u^{(n)} =& \big(Au^{(n)} + B^{(n)}(u^{(n)}) + K^{(n)}(u^{(n)}) + D^{(n)}(u^{(n)}) \\
&+ F^{(n)}(t, u^{(n)})\big)\,\mathrm{d}t + G^{(n)}(t, u^{(n)})\,\mathrm{d}W, \\
u^{(n)}(0) =& \Pi_n u(0).
\end{aligned}
\tag{5.25}
$$

Lemma 5.2. *For every $n \in \mathbb{N}$ there exists a strong solution $u^{(n)}$ to (5.25). Moreover, for any $p \geq 2$,*

$$
\sup_n \mathbb{E} \sup_{t\in[0,T]} |u^{(n)}(t)|_\rho^p < \infty
\tag{5.26}
$$

and

$$
\begin{aligned}
\mathbb{E}\Big(|u^{(n)}(t)|_\rho^2 + \int_0^t ||u^{(n)}(s)||_{1,\rho}^2\,\mathrm{d}s + \frac{1}{\varepsilon}\int_0^t |\mathrm{div}\,\Pi_n u^{(n)}(s)|_\rho^2\,\mathrm{d}s \\
+ \varepsilon \int_0^t |\Pi_n u^{(n)}(s)|_{\mathbb{L}_\rho^6}^6\,\mathrm{d}s\Big) \leq \frac{C}{\varepsilon^2}\exp\Big(\frac{C}{\varepsilon}\Big),
\end{aligned}
\tag{5.27}
$$

the constant C being independent of n, ε and $t \in [0,T]$.

Proof. Using the arguments from the proof of Lemma 3.1 we can obtain the existence of a local strong solution $u^{(n)}$. Thus it is enough to show (5.27). In the proof of the theorem we adopt the convention that all constants $c_1, c_2, \ldots$ are independent of n and ε. Applying Itô's formula to the function $H(v) = |v|_\rho^2$ we obtain

$$
\begin{aligned}
|u^{(n)}(t)|_\rho^2 = |u^{(n)}(0)|_\rho^2 + \int_0^t \Phi(s, u^{(n)}(s))\,\mathrm{d}t \\
+ \int_0^t 2\big\langle u^{(n)}(s), G^{(n)}(s, u^{(n)}(s))\,\mathrm{d}W(s)\big\rangle_\rho.
\end{aligned}
\tag{5.28}
$$

where

$$
\begin{aligned}
\Phi(s, u^{(n)}) =& ||G^{(n)}(t, u^{(n)})||_{L_{(\mathrm{HS})}(\mathcal{H}_\mu, \mathbb{L}_\rho^2)}^2 \\
&+ \big\langle (Au^{(n)} + B^{(n)}(u^{(n)}) + D^{(n)}(u^{(n)}) + K^{(n)}(u^{(n)}) + F^{(n)}(s, u^{(n)})), u^{(n)}\big\rangle_\rho.
\end{aligned}
$$

Now, let $v \in \mathrm{Dom}\,A$. Then

$$
\begin{aligned}
\langle Av + D^{(n)}(v), v\rangle_\rho &= \big\langle \nu\Delta v + \frac{1}{\varepsilon}\Pi_n \nabla \mathrm{div}\,\Pi_n v, v\big\rangle_\rho \\
&\leq -\nu||v||_{1,\rho}^2 + c_1|v|_\rho||v||_{1,\rho} - \frac{1}{\varepsilon}|\mathrm{div}\,\Pi_n v|_\rho^2 + \frac{c_2}{\varepsilon}|\mathrm{div}\,\Pi_n v|_\rho|\Pi_n v|_\rho.
\end{aligned}
$$

Clearly,

$$-\nu||v||_{1,\rho}^2 + c_1|v|_\rho||v||_{1,\rho} \le -\frac{\nu}{2}||v||_{1,\rho}^2 + c_3|v|_\rho^2,$$

and

$$-\frac{1}{\varepsilon}|\mathrm{div}\,\Pi_n v|_\rho^2 + c_2\frac{1}{\varepsilon}|\mathrm{div}\,\Pi_n v|_\rho|\Pi_n v|_\rho \le -\frac{1}{2\varepsilon}|\mathrm{div}\,\Pi_n v|_\rho^2 + \frac{c_4}{\varepsilon}|v|_\rho^2.$$

Hence

$$\langle Av + D^{(n)}(v), v\rangle_\rho \le -\frac{\nu}{2}||v||_{1,\rho}^2 - \frac{1}{2\varepsilon}|\mathrm{div}\,\Pi_n v|_\rho^2 + c_5|v|_\rho^2$$

By Proposition 2.2(i) we have

$$\begin{aligned}\langle B^{(n)}(v) + K^{(n)}(v), v\rangle_\rho &\le c_6|\Pi_n v|_{\mathbb{L}_\rho^4}^2||\Pi_n v||_{1,\rho} - \varepsilon|\Pi_n v|_{\mathbb{L}_\rho^6}^6 \\ &\le c_7|\Pi_n v|_{\mathbb{L}_\rho^6}^4 + \frac{\nu}{8}||v||_{1,\rho}^2 - \varepsilon|\Pi_n v|_{\mathbb{L}_\rho^6}^6.\end{aligned}$$

Using now the elementary inequality

$$c_7 r^4 - \frac{\varepsilon}{2}r^6 \le \frac{c_8}{\varepsilon^2}, \quad r \in \mathbb{R},$$

we get

$$\langle B^{(n)}(v) + K^{(n)}(v), v\rangle_\rho \le \frac{c_8}{\varepsilon^2} + \frac{\nu}{8}||v||_{1,\rho}^2 - \frac{\varepsilon}{2}|\Pi_n v|_{\mathbb{L}_\rho^6}^6.$$

From Lemma 5.1 and (5.22) we have

$$\begin{aligned}|\langle F^{(n)}(t,v), v\rangle_\rho| + ||(G^{(n)}(t,v))||_{L_{(\mathrm{HS})}(\mathcal{H}_\mu,\mathbb{L}_\rho^2)}^2 &\le c_8(|v|_\rho^2 + ||v||_{1,\rho}|v|_\rho + 1) \\ &\le c_9|v|_\rho^2 + \frac{\nu}{8}||v||_{1,\rho}^2 + c_9.\end{aligned}$$

Adding up, we obtain

$$\Phi(t,v) \le -\frac{\nu}{4}||v||_{1,\rho}^2 - \frac{1}{\varepsilon}|\mathrm{div}\,\Pi_n v|_\rho^2 - \frac{\varepsilon}{2}|\Pi_n v|_{\mathbb{L}_\rho^6}^6 + \frac{c_{10}}{\varepsilon}|v|_\rho^2 + \frac{c_{10}}{\varepsilon^2}.$$

Thus taking the mathematical expectation in (5.28) we get

$$\begin{aligned}&\mathbb{E}\Big(|u^{(n)}(t)|_\rho^2 + \frac{\nu}{4}\int_0^t ||u^{(n)}(s)||_{1,\rho}^2\,\mathrm{d}s + \frac{1}{\varepsilon}\int_0^t |\mathrm{div}\,\Pi_n u^{(n)}(s)|_\rho^2\,\mathrm{d}s \\ &\quad + \frac{\varepsilon}{2}\int_0^t |\Pi_n u^{(n)}(s)|_{\mathbb{L}_\rho^6}^6\,\mathrm{d}s\Big) \le \mathbb{E}\,|u(0)|_\rho^2 + \frac{c_{10}}{\varepsilon}\int_0^t \mathbb{E}\,|u^{(n)}(s)|_\rho^2\,\mathrm{d}s + \frac{c_{10}}{\varepsilon^2}t.\end{aligned}$$

Applying Gronwall's lemma we obtain (5.27).

Proceeding as above, but using Itô's formula to the function $|v|_\rho^p$ for $p \ge 2$ one can obtain (5.26), see the proof of (3.13). □

Proof of Theorem 5.1 - existence. The proof is based on the same idea as the proof of Theorem 1.1. For each n let $u^{(n)}$ be the solution to (5.24). The first step is to show that the sequence $\mathcal{L}(u^{(n)})$, $n \in \mathbb{N}$, of the laws of $u^{(n)}$ is tight in $L^2(0,T;H_\kappa)$ for a fixed $\kappa > 5\rho/3$. Write

$$\begin{aligned} y_n(t) &= S(t)u^{(n)}(0) + \int_0^t S(t-s)\left\{B^{(n)}(u^{(n)}(s)) + F^{(n)}(s,u^{(n)}(s))\right\} \mathrm{d}s \\ &\quad + \int_0^t S(t-s)G^{(n)}(s,u^{(n)}(s))\, \mathrm{d}W(s), \\ z_n(t) &= \int_0^t S(t-s)\left\{D^{(n)}(u^{(n)}(s)) + K^{(n)}(s,u^{(n)}(s))\right\} \mathrm{d}s \\ &= \mathcal{J}_0^\rho(D^{(n)}(u^{(n)}))(t) + \mathcal{J}_0^\rho(K^{(n)}(u^{(n)}))(t). \end{aligned}$$

Then $u^{(n)} = y_n + z_n$. Using the arguments from the proof of Theorem 1.1 we can show the tightness of the laws $\mathcal{L}(y_n)$, $n \in \mathbb{N}$, and it remains to prove the tightness of $\mathcal{L}(z_n)$, $n \in \mathbb{N}$. Recall, see Proposition 2.1 (i) and (iii), that $\mathcal{J}_0^\rho$ is a compact operator acting from $L^2(0,T;W_\rho^{-1})$ into $L^2(0,T;\mathbb{L}_\kappa^2)$, and from $L^{6/5}(0,T;\mathbb{L}_\rho^{6/5})$ into $L^2(0,T;\mathbb{L}_\kappa^2)$. Thus it is enough to show that

$$\sup_n \mathbb{E}\int_0^T ||D^{(n)}(u^{(n)}(t))||_{-1,\rho}^2 \,\mathrm{d}t < \infty, \tag{5.29}$$

$$\sup_n \mathbb{E}\int_0^T |K^{(n)}(u^{(n)}(t))|_{\mathbb{L}_\rho^{\frac{6}{5}}}^{\frac{6}{5}} \,\mathrm{d}t < \infty. \tag{5.30}$$

From (5.22) and (5.23) there is a constant c such that for every n,

$$||D^{(n)}(u^{(n)}(t))||_{-1,\rho} = \frac{1}{\varepsilon}||\Pi_m \nabla \mathrm{div}\, \Pi_n u^{(n)}(t)||_{-1,\rho} \le \frac{c}{\varepsilon}||u^{(n)}||_{1,\rho},$$

and (5.29) follows from (5.27). Similarly,

$$|K^{(n)}(u^{(n)}(t))|_{\mathbb{L}_\rho^{\frac{6}{5}}}^{\frac{6}{5}} = \varepsilon^{\frac{6}{5}}|\Pi_n\left\{|\Pi_n u^{(n)}(t)|^4 \Pi_n u^{(n)}(t)\right\}|_{\mathbb{L}_\rho^{\frac{6}{5}}}^{\frac{6}{5}} \le c\varepsilon^{\frac{6}{5}}|\Pi_n u^{(n)}(t)|_{\mathbb{L}_\rho^6}^6,$$

and (5.30) follows from (5.27). Having the tightness of $\{\mathcal{L}(u^{(n)})\}$ in $L^2(0,T;\mathbb{L}_\kappa^2)$, and using the Skorokhod imbedding and the arguments from the proof of Theorem 1.1 we may construct a filtered probability space $(\Omega^1, \mathcal{F}^1, (\mathcal{F}_t^1), \mathbb{P}^1)$, and a sequence $(v^{(n)}, W^{(n)})$ such that for any n, $W^{(n)}$ is cylindrical Wiener process on $\mathcal{H}_\mu$ defined on Ω^1, $v^{(n)}$ is a solution to (5.24) driven by $W^{(n)}$, and $v^{(n)}$, $n \in \mathbb{N}$, converges $\mathbb{P}^1 \times \mathrm{d}t \times \mathrm{d}x$ - almost surely to a certain u. Then u is adapted, and $u(0)$ has the law U_0. Using now Fatou's lemma we obtain

$$u \in L^p(\Omega^1; \mathcal{F}^1, \mathbb{P}^1; L^\infty(0,T;\mathbb{L}_\rho^2)) \quad \text{for } p \in [1,\infty).$$

The uniform integrability of $|v^{(n)} - u|^q_{L^q(0,T;\mathbb{L}^2_\rho)}$, $n \in \mathbb{N}$, gives that

$$\lim_{n\to\infty} \mathbb{E}^1 \int_0^T |v^{(n)}(t) - u(t)|^q_{\mathbb{L}^2_\rho} \,\mathrm{d}t = 0 \quad \text{for } q \in [2,\infty). \tag{5.31}$$

Note that

$$\lim_{n\to\infty} \mathbb{E}^1 \int_0^T |\Pi_n v^{(n)}(t) - u(t)|^2_\rho \,\mathrm{d}t = 0.$$

Thus taking if necessary a subsequence we may assume that

$$\Pi_n v^{(n)} \to u, \ \mathbb{P}^1 \times \mathrm{d}t \times \mathrm{d}x \ - \ \text{almost surely}.$$

Consequently, (5.27) and Fatou's lemma yield

$$u \in L^6(\Omega^1, \mathbb{P}^1, \mathcal{F}^1; L^6(0,T;\mathbb{L}^6_\rho)).$$

Moreover, as for any $2 \le q < 6$ the sequence $|\Pi_n v^{(n)} - u|^q_{\mathbb{L}^q_\rho}$, $n \in \mathbb{N}$, is uniformly integrable we have

$$\lim_{n\to\infty} \mathbb{E}^1 \int_0^T |\Pi_n v^{(n)}(t) - u(t)|^q_{\mathbb{L}^q_\rho} \,\mathrm{d}t = 0 \quad \text{for } 2 \le q < 6. \tag{5.32}$$

Let us observe now that the martingales

$$\mathcal{M}_n(t) = \int_0^t G^{(n)}(s, v^{(n)}(s))\,\mathrm{d}W^{(n)}(s), \quad t \in [0,T], \ n \in \mathbb{N},$$

converge in $L^2(\Omega^1, \mathcal{F}^1, \mathbb{P}^1; C([0,T];\mathbb{L}^2_\rho))$ to a martingale $\mathcal{M}$ satisfying

$$\ll \mathcal{M} \gg (t) = \int_0^t G(s,u(s))G(s,u(s))^* \,\mathrm{d}s, \quad t \in [0,T].$$

Taking into account the representation theorem we may assume that $\mathcal{M}$ is of the form

$$\mathcal{M}(t) = \int_0^t G(s,u(s))\,\mathrm{d}\mathbf{W}(s), \quad t \in [0,T],$$

where $\mathbf{W}$ is a cylindrical Wiener process on $\mathcal{H}_\mu$. Let us now fix ψ from $C_0^\infty(\mathbb{R}^d;\mathbb{R}^d)$. It remains to note that (5.31) and (5.32) guarantee the convergence of each term appearing in the weak form of (5.24) to the corresponding term in (iv) from Definition 5.1. $\square$

Proof of Theorem 5.1 - uniqueness. We prove pathwise uniqueness, which by [24, Section 4.3] implies uniqueness in law. Take two solutions u^ε and v^ε with the same initial value and write $w^\varepsilon = u^\varepsilon - v^\varepsilon$. As in the proof of the existence we fix an orthonormal basis $\{e_k\}$ of $\mathbb{L}^2_\rho$, such that $\{e_k\} \subset \mathcal{S}^3$ and $\{e_k\}$ is a Schauder basis of all spaces $\mathbb{L}^q_\rho$, $2 \le q \le 6$, and W^1_ρ. Let Π_n be the orthogonal projection onto the space spanned on $\{e_1, \dots, e_n\}$. Using Itô's formula and integrating by parts we get

$$
\begin{aligned}
\frac{1}{2}|\Pi_n w^\varepsilon(t)|^2_\rho = {} & \nu \int_0^t \big\langle \Delta(\vartheta_\rho \Pi_n w^\varepsilon(s)), w^\varepsilon(s)\big\rangle \, \mathrm{d}s \\
& + \frac{1}{\varepsilon}\int_0^t \big\langle \nabla \mathrm{div}\,\big(\vartheta_\rho \Pi_n w^\varepsilon(s)\big), w^\varepsilon(s)\big\rangle \, \mathrm{d}s \\
& + \int_0^t \sum_{i=1}^d \big\langle \partial_{x_i}\big(\vartheta_\rho \Pi_n w^\varepsilon(s)\big), u_i^\varepsilon(s) u^\varepsilon(s) - v_i^\varepsilon(s) v^\varepsilon(s)\big\rangle \, \mathrm{d}s \\
& - \varepsilon \int_0^t \big\langle \Pi_n w^\varepsilon(s), |u^\varepsilon(s)|^4 u^\varepsilon(s) - |v^\varepsilon(s)|^4 v^\varepsilon(s)\big\rangle_\rho \, \mathrm{d}s \\
& + \int_0^t \big\langle \Pi_n w^\varepsilon(s), F(s,u^\varepsilon(s)) - F(s,v^\varepsilon(s))\big\rangle_\rho \, \mathrm{d}s \\
& + \frac{1}{2}\int_0^t ||\Pi_n(G(s,u^\varepsilon(s)) - G(s,v^\varepsilon(s)))||^2_{L_{(\mathrm{HS})}(\mathcal{H}_\mu, \mathbb{L}^2_\rho)} \, \mathrm{d}s \\
& + \int_0^t \big\langle \Pi_n w^\varepsilon(s), \big(G(s,u^\varepsilon(s)) - G(s,v^\varepsilon(s))\big)\, \mathrm{d}W(s)\big\rangle_\rho .
\end{aligned}
$$

We can again integrate by parts, and due to the regularity properties of solutions and since we deal with a basis which is a Schauder basis in all relevant spaces we may pass to the limit with $n \to \infty$. Then the first term on the right hand side of the resulting equation can be estimated by

$$
-\frac{\nu}{2}\int_0^t ||w^\varepsilon(s)||^2_{1,\rho} \, \mathrm{d}s + C_1 \int_0^t |w^\varepsilon(s)|^2_\rho \, \mathrm{d}s.
$$

The second term is equal to

$$
-\frac{1}{\varepsilon}\int_0^t \int_{\mathbb{R}^d} \mathrm{div}\,(\vartheta_\rho(x) w^\varepsilon(s,x)) \mathrm{div}\, w^\varepsilon(s,x) \, \mathrm{d}x \, \mathrm{d}s
$$

and hence can be estimated by $C_2 \varepsilon^{-1} \int_0^t |w^\varepsilon(s)|^2_\rho \, \mathrm{d}s$. We estimate the third term by

$$
-\frac{\nu}{4}\int_0^t ||w^\varepsilon(s)||^2_{1,\rho} \, \mathrm{d}s + C_3 \int_0^t \int_{\mathbb{R}^d} |w^\varepsilon(s,x)|^2 (|u^\varepsilon(s,x)|^2 + |v^\varepsilon(s,x)|^2) \vartheta_\rho(x) \, \mathrm{d}x \, \mathrm{d}s
$$

$$
\begin{aligned}
\le {} & -\frac{\nu}{4}\int_0^t ||w^\varepsilon(s)||^2_{1,\rho} \, \mathrm{d}s + \frac{C_4}{\varepsilon}\int_0^t |w^\varepsilon(s)|^2_\rho \, \mathrm{d}s \\
& + \frac{\varepsilon}{2}\int_0^t \int_{\mathbb{R}^d} |w^\varepsilon(s,x)|^2 (|u^\varepsilon(s,x)|^4 + |v^\varepsilon(s,x)|^4) \vartheta_\rho(x) \, \mathrm{d}x \, \mathrm{d}s.
\end{aligned}
$$

The fourth term is equal to

$$-\,\varepsilon\int_0^t\int_{\mathbb{R}^d}\big\{|u^\varepsilon(s,x)|^6+|v^\varepsilon(s,x)|^6 \\ -\ (|u^\varepsilon(s,x)|^4+|v^\varepsilon(s,x)|^4)\langle u^\varepsilon(s,x),v^\varepsilon(s,x)\rangle\vartheta_\rho(x)\big\}\,\mathrm{d}x\,\mathrm{d}s$$

and due to the inequality $a^6+b^6\geq 2^{-1}(a^4+b^4)(a^2+b^2)$ this term is less or equal to

$$-\frac{\varepsilon}{2}\int_0^t\int_{\mathbb{R}^d}(|u^\varepsilon(s,x)|^4+|v^\varepsilon(s,x)|^4)|w^\varepsilon(s,x)|^2\vartheta_\rho(x)\,\mathrm{d}x\,\mathrm{d}s.$$

So the sum of the first four terms is less or equal to

$$C(\varepsilon)\int_0^t|w^\varepsilon(s)|_\rho^2\,\mathrm{d}s-\frac{\nu}{4}\int_0^t||w^\varepsilon(s)||_{1,\rho}^2\,\mathrm{d}s,$$

$C(\varepsilon)$ being independent of t. For the next two terms we make use of the Lipschitz condition on F and G, and we easily estimate them by

$$C_5\int_0^t|w^\varepsilon(s)|_\rho^2\,\mathrm{d}s+\frac{\nu}{4}\int_0^t||w^\varepsilon(s)||_{1,\rho}^2\,\mathrm{d}s.$$

Therefore, putting the above together and taking the mathematical expectation we get

$$\mathbb{E}\,|w^\varepsilon(t)|_\rho^2\leq(C(\varepsilon)+C_5)\int_0^t\mathbb{E}\,|w^\varepsilon(s)|_\rho^2\,\mathrm{d}s,$$

and consequently by Gronwall's lemma, $w^\varepsilon=0$ a.s., which is the desired conclusion. $\square$

As a consequence of the uniqueness of the solution to (0.2) and the fact that the coefficients of (0.2) do not depend on the space variable we have

Proposition 5.1. *Assume that the conditions of Theorem 5.1 are fulfilled. Additionally, we assume that the initial measure U_0 is spatially homogeneous. Then the solution is a spatially homogeneous process.*

Proof. Let U_0 be a spatially homogeneous measure on $\mathbb{L}^2_\rho$, and let $(\mathfrak{U},W,u^\varepsilon)$ be a solution to (0.2) starting from U_0. Since we have the uniqueness in law, it is enough to observe that for any $h\in\mathbb{R}^d$, $(\mathfrak{U},\tau_hW,\tau_hu^\varepsilon)$ is a solution to (0.2) starting from $U_0\circ\tau_h=U_0$. Clearly, $\tau_h\mathcal{W}(t)$, $t\in[0,T]$, is an $\mathbb{R}^m$-valued spatially homogeneous Wiener random field with the spectral measure μ. Thus $\tau_hW=\tau_h\mathcal{W}$ is a cylindrical Wiener process on $\mathcal{H}_\mu$. Since F and G do not depend on x-variable, one has $\tau_hF(s,u^\varepsilon(s))=F(s,\tau_hu^\varepsilon(s))$, and

$$\tau_h\int_0^tG(s,u^\varepsilon(s))\,\mathrm{d}W(s)=\int_0^tG(s,\tau_hu^\varepsilon(s))\,\mathrm{d}(\tau_hW)(s).$$

From this, and from the fact that $\langle\tau_hu^\varepsilon(t),\psi\rangle_\rho=\langle u^\varepsilon(t),\tau_{-h}(\vartheta_\rho\psi)\rangle$ one can easily deduce the desired conclusion. $\square$

6. Limit passage

The limit passage requires some estimates, uniform with respect to ε, for the solution u^ε to (0.2), see Lemma 6.2. In their proof we will make use the fact that for any t the law of $u^\varepsilon(t)$ is a spatially homogeneous probability measure. For further references we present the following lemma, whose proof is given in [38], see Théoréme 2.1, p. 537.

Lemma 6.1. *Let $z \in C(\mathbb{R}^{(d+1)d}; \mathbb{R})$, $\rho > 1/2$, and let U be a spatially homogeneous probability measure on W^1_ρ. Let*

$$Z(u)(x) = z(u(x), \nabla u) \qquad \text{for } u \in W^1_\rho,\ x \in \mathbb{R}^d.$$

Suppose that $Z(u) \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ for every $u \in W^1_\rho$, and assume that the mappings

$$u \mapsto \int_{\mathbb{R}^d} Z(u(x))\psi(x)\,\mathrm{d}x, \quad u \mapsto \int_{\mathbb{R}^d} |Z(u(x))|\psi(x)\,\mathrm{d}x$$

are U-integrable for any $\psi \in C^\infty_0(\mathbb{R}^d)$. Then there is a constant C such that for any $\psi \in L^1(\mathbb{R}^d)$ one has

$$\int_{W^1_\rho}\int_{\mathbb{R}^d} Z(u(x))\psi(x)\,\mathrm{d}x\,\mathrm{d}U(u) = C\int_{\mathbb{R}^d}\psi(x)\,\mathrm{d}x.$$

Lemma 6.2. *Assume that the conditions of Theorem 5.1 are fulfilled. Additionally, we assume that the initial measure U_0 is spatially homogeneous. Let $\varepsilon \in (0,1]$, and let u^ε be a solution to (0.2) starting from U_0. Then there is a constant C independent of ε such that for any $p \geq 2$,*

$$\mathbb{E}\sup_{t\in[0,T]}|u^\varepsilon(t)|^p_\rho \leq C \tag{6.33}$$

and

$$\mathbb{E}\Big(|u^\varepsilon(t)|^2_\rho + \int_0^t ||u^\varepsilon(s)||^2_{1,\rho}\,\mathrm{d}s + \frac{1}{\varepsilon}\int_0^t|\mathrm{div}\, u^\varepsilon|^2_\rho\,\mathrm{d}s + \varepsilon\int_0^t |u^\varepsilon|^6_{\mathbb{L}^6_\rho}\,\mathrm{d}s\Big) \leq C. \tag{6.34}$$

Proof. Using Itô's lemma and limit arguments, see the proof of the uniqueness, we get

$$\begin{aligned}
\frac{1}{2}|u^\varepsilon(t)|^2_\rho &= \frac{1}{2}|u^\varepsilon(0)|^2_\rho - \nu\sum_{i=1}^d\int_0^t\big\langle\partial_{x_i}(\vartheta_\rho u^\varepsilon(s)), \partial_{x_i}u^\varepsilon(s)\big\rangle\,\mathrm{d}s\\
&\quad - \frac{1}{\varepsilon}\int_0^t\big\langle\mathrm{div}\,(\vartheta_\rho u^\varepsilon(s)), \mathrm{div}\,u^\varepsilon(s)\big\rangle\,\mathrm{d}s\\
&\quad - \int_0^t\Big\{\sum_{i=1}^d\big\langle u^\varepsilon(s), \partial_{x_i}\big(u^\varepsilon_i(s)u^\varepsilon(s)\big)\big\rangle_\rho - \varepsilon|u^\varepsilon(s)|^6_{\mathbb{L}^6_\rho}\Big\}\,\mathrm{d}s\\
&\quad + \int_0^t\Big\{\big\langle u^\varepsilon(s), F(s,u^\varepsilon(s))\big\rangle_\rho + \frac{1}{2}||G(s,u^\varepsilon(s))||^2_{L_{(\mathrm{HS})}(\mathcal{H}_\mu,\mathbb{L}^2_\rho)}\Big\}\,\mathrm{d}s\\
&\quad + \int_0^t\big\langle u^\varepsilon(s), G(s,u^\varepsilon(s))\,\mathrm{d}W(s)\big\rangle_\rho.
\end{aligned}$$

The first integral on the right hand side is estimated by

$$-\frac{\nu}{2}\int_0^t ||u^\varepsilon(s)||_{1,\rho}^2\,\mathrm{d}s + C_1\int_0^t |u^\varepsilon(s)|_\rho^2\,\mathrm{d}s.$$

The second is equal to

$$-\frac{1}{\varepsilon}\int_0^t\int_{\mathbb{R}^d}\Big\{|\mathrm{div}\,u^\varepsilon(s,x)|^2\vartheta_\rho(x) + u^\varepsilon(s,x)\mathrm{div}\,u^\varepsilon(s,x)\sum_{i=1}^d \partial_{x_i}\vartheta_\rho(x)\Big\}\,\mathrm{d}x\,\mathrm{d}s.$$

Integrating by parts we can show that the first term of the third integral is equal to

$$\frac{1}{2}\int_0^t\int_{\mathbb{R}^d}\Big\{\mathrm{div}\,u^\varepsilon(s,x)|u^\varepsilon(s,x)|^2\vartheta_\rho(x) - |u^\varepsilon(s,x)|^2\sum_{i=1}^d u_i^\varepsilon(s,x)\partial_{x_i}\vartheta_\rho(x)\Big\}\,\mathrm{d}x\,\mathrm{d}s,$$

and next we use the Schwartz and Young inequalities to get

$$\Big|\int_{\mathbb{R}^d}|u^\varepsilon(s,x)|^2\mathrm{div}\,u^\varepsilon(s,x)\vartheta_\rho(x)\,\mathrm{d}x\Big| \le \frac{1}{2\varepsilon}|\mathrm{div}\,u^\varepsilon(s)|_\rho^2 + \frac{\varepsilon}{2}|u^\varepsilon(s)|_{L_\rho^6}^6 + C_2|u^\varepsilon(s)|_\rho^2.$$

By the linear growth of F and G we estimate the forth integral by

$$C_3 + C_4\int_0^t |u^\varepsilon(s)|_\rho^2\,\mathrm{d}s + \frac{\nu}{4}\int_0^t ||u^\varepsilon(s)||_{1,\rho}^2\,\mathrm{d}s.$$

Thus taking the mathematical expectation we have

$$\mathbb{E}\Big(|u^\varepsilon(t)|_\rho^2 + \int_0^t\Big\{\frac{\nu}{4}||u^\varepsilon(s)||_{1,\rho}^2 + \frac{1}{2\varepsilon}|\mathrm{div}\,u^\varepsilon(s)|_\rho^2 + \frac{\varepsilon}{2}|u^\varepsilon(s)|_\rho^6\Big\}\,\mathrm{d}s\Big) \tag{6.35}$$

$$\le \mathbb{E}|u^\varepsilon(0)|_\rho^2 + C_5 + C_6\,\mathbb{E}\int_0^t |u^\varepsilon(s)|_\rho^2\,\mathrm{d}s$$
$$-\,\mathbb{E}\int_0^t\int_{\mathbb{R}^d}\sum_{i=1}^d\Big\{\frac{1}{2}|u^\varepsilon(s,x)|^2u_i^\varepsilon(s,x) + \frac{1}{\varepsilon}\mathrm{div}\,u^\varepsilon(s,x)u_i^\varepsilon(s,x)\Big\}\partial_{x_i}\vartheta_\rho(x)\,\mathrm{d}x\,\mathrm{d}s.$$

By Proposition 5.1, the laws $\mathcal{L}(u^\varepsilon(s))$ of $u^\varepsilon(s)$, $s\ge 0$, are spatially homogeneous. Taking into account the regularity of u^ε we see that each $\mathcal{L}(u^\varepsilon(s))$ is supported on $W_\rho^1\cap\mathbb{L}_\rho^6$. Hence, by Lemma 6.1 we have

$$\mathbb{E}\int_{\mathbb{R}^d}\sum_{i=1}^d\Big\{\frac{1}{2}|u^\varepsilon(s,x)|^2u_i^\varepsilon(s,x) + \frac{1}{\varepsilon}\mathrm{div}\,u^\varepsilon(s,x)u_i^\varepsilon(s,x)\Big\}\partial_{x_i}\vartheta_\rho(x)\,\mathrm{d}x$$

$$= C \int_{\mathbb{R}^d} \sum_{i=1}^{d} \partial_{x_i} \vartheta_\rho(x)\,\mathrm{d}x = 0,$$

and (6.35) and Gronwall's lemma give (6.34). Applying Itô's formula to the function $u \mapsto |\Pi_n u|_\rho^p$, and then letting $n \to \infty$ one can prove (6.33). Almost all calculations in the proofs of (6.33) and (6.34) are the same. However in the case of (6.33) one should apply a maximal inequality for the stochastic term, as in the proofs of (3.13) and (5.26). □

Proof of Theorem 1.4. Let $\kappa > 5\rho/3$, $\alpha \in (1/2, 1)$, and let u^ε be a solution to (0.2). The estimates in Lemma 6.1 enables us to show the tightness of $\mathcal{L}(u^\varepsilon)$, $\varepsilon \in (0,1]$ in the space $L^2(0,T;\mathbb{L}^2_\kappa)$. For, we have the equivalence between mild and weak solutions to stochastic evolution equations, see [17]. Thus u^ε is also a mild solution to (0.2), and

$$\begin{aligned} u^\varepsilon(t) = S(t)u(0) &+ \mathcal{J}_0^\rho\big(B(u^\varepsilon) + F(\cdot, u^\varepsilon)\big)(t) + \mathcal{J}_0^\rho\big(D^\varepsilon(u^\varepsilon) + K^\varepsilon(u^\varepsilon)\big)(t) \\ &+ \mathcal{J}_\alpha^\rho(Y_\varepsilon^\alpha)(t), \end{aligned}$$

where: $D^\varepsilon(u^\varepsilon) = \varepsilon^{-1}\nabla \mathrm{div}\, u^\varepsilon$, $K^\varepsilon(u^\varepsilon) = -\varepsilon|u^\varepsilon|^4 u^\varepsilon$, and $\mathcal{J}_0^\rho$ and $\mathcal{J}_\alpha^\rho$ are defined in Section 2, and

$$Y_\varepsilon^\alpha(t) = \frac{\sin(\alpha-1)\pi}{\pi} \int_0^t (t-s)^{\alpha-1} S(t-s) G(s, u^\varepsilon(s))\,\mathrm{d}W(s).$$

Using estimates (6.33) and (6.34) and arguments from the proof of Theorem 1.1 one can show the tightness of $\mathcal{J}_0^\rho(B(u^\varepsilon) + F(\cdot, u^\varepsilon))$, and $\mathcal{J}_\alpha^\rho(Y_\varepsilon^\alpha)$, $\varepsilon \in (0,1]$. Using the same estimates and arguments from the proof of Theorem 5.1, see (5.29) and (5.30) one can show the tightness of the remaining sequence $\mathcal{J}_0^\rho(D^\varepsilon(u^\varepsilon) + K^\varepsilon(u^\varepsilon))$, $\varepsilon \in (0,1]$. Thus by Prokhorov's and Skorokhod's theorems there exist a filtered probability space $(\Omega^1, \mathcal{F}^1, (\mathcal{F}_t^1), \mathbb{P}^1)$, and sequences $\varepsilon_n \downarrow 0$, $(v^{(n)}, W^{(n)})$, such that every $v^{(n)}$ is a solution to (0.2) with $\varepsilon = \varepsilon_n$, driven by a cylindrical process $W^{(n)}$. Moreover, there is a process u such that

$$v^{(n)} \to u, \quad \mathbb{P}^1 \times \mathrm{d}t \times \mathrm{d}x \text{ - almost surely}$$

and

$$\lim_{n\to\infty} \int_0^T |v^{(n)}(t) - u(t)|_\rho^p\,\mathrm{d}t = 0 \quad \text{for } p \in [2,\infty). \tag{6.36}$$

We have to show now that u is a divergence free process. This is a consequence of that fact that the sequence

$$\frac{1}{\varepsilon}\,\mathbb{E}^1 \int_0^T |\mathrm{div}\, v^{(n)}(t)|_\rho^2\,\mathrm{d}t, \qquad n \in \mathbb{N},$$

is bounded, see (6.34). For, we first note that the divergence operator acts from $\mathbb{L}^2_\rho$ into W^{-1}_ρ. It is easy to see that H_ρ is its kernel, and that $H^1_\rho = W^1_\rho \cap H$. Hence it is enough to observe that for every $\psi \in C_0^\infty(\mathbb{R}^d;\mathbb{R}^d)$,

$$\begin{aligned}\mathbb{E}^1 \int_0^T \langle \operatorname{div} u(t), \psi\rangle_\rho^2 \,\mathrm{d}t &= \mathbb{E}^1 \int_0^T \langle u(t), \nabla(\psi\vartheta_\rho)\rangle^2 \,\mathrm{d}t \\ &\le \limsup_{n\to\infty} \mathbb{E}^1 \int_0^T \langle v^{(n)}(t), \nabla(\psi\vartheta_\rho)\rangle^2 \,\mathrm{d}t \\ &\le \limsup_{n\to\infty} \varepsilon_n \sup_m \frac{1}{\varepsilon_m} \mathbb{E}^1 \int_0^T \langle v^{(m)}(t), \nabla(\psi\vartheta_\rho)\rangle^2 \,\mathrm{d}t = 0.\end{aligned}$$

Thus u satisfies the desired regularity properties from Definition 1.2. Clearly $u(0)$ has the law U_0. Using now the representation theorem we may assume that

$$\int_0^t G^{(n)}(s, v^{(n)}(s))\,\mathrm{d}W^{(n)}(s) \to \int_0^t G(s, u(s))\,\mathrm{d}\mathbf{W}(s),$$

where $\mathbf{W}$ is a cylindrical Wiener process on $\mathcal{H}_\mu$, and the convergence is in the space $L^2(\Omega^1, \mathcal{F}^1, \mathbb{P}^1; C([0,T]; H_\rho))$.

It remains to show that the integral equation from Definition 1.2(v) is fulfilled for any $\psi \in C_0^\infty(\mathbb{R}^d;\mathbb{R}^d)$ such that $\operatorname{div}(\psi\vartheta_\rho) = 0$. We have to show the convergence of seven terms. Five of them are very similar to those listed at the and of the proof of Theorem 1.1. The differences are that now we have no projections, and that the scalar product on H is replaced by the scalar product on $\mathbb{L}^2_\rho$. Obviously, (6.36) guarantees the convergence of these terms in $L^2(\Omega^1, \mathcal{F}^1, \mathbb{P}^1)$. What is left is to show that for the two remaining terms we have:

$$\lim_{n\to\infty} \frac{1}{\varepsilon_n} \mathbb{E}^1 \int_0^T \langle \nabla \mathrm{div} v^{(n)}(t), \psi\rangle_\rho \,\mathrm{d}t = 0 \tag{6.37}$$

and

$$\lim_{n\to\infty} \varepsilon_n \mathbb{E}^1 \int_0^T \langle |v^{(n)}(t)|^4 v^{(n)}(t), \psi\rangle_\rho \,\mathrm{d}t = 0. \tag{6.38}$$

Since $\operatorname{div}(\psi\vartheta_\rho) = 0$ we have $\langle \nabla \mathrm{div} v^{(n)}(t), \psi\rangle_\rho \equiv 0$, and consequently (6.37). Now we prove (6.38). Since ψ is bounded it is enough to show that

$$\lim_{n\to\infty} \varepsilon_n \mathbb{E}^1 \int_0^T |v^{(n)}(t)|^5_{\mathbb{L}^5_\rho} \,\mathrm{d}t = 0.$$

To this end for $p \in [2,\infty)$ set

$$\mathcal{L}_p = L^p(\Omega^1 \times [0,T] \times \mathbb{R}^d, \mathcal{F}^1 \times \mathcal{B}([0,T]) \times \mathcal{B}(\mathbb{R}^d), \mathbb{P}^1 \times \,\mathrm{d}t \times \vartheta_\rho(x)\,\mathrm{d}x; \mathbb{R}^d).$$

We have to show that $\varepsilon_n^{\frac{1}{5}}|v^{(n)}|_{\mathcal{L}_5} \to 0$ as $n \to \infty$. By (6.34), the sequence $\varepsilon_n^{\frac{1}{6}}|v^{(n)}|_{\mathcal{L}_5}$, $n \in \mathbb{N}$ is bounded. Since $\mathcal{L}_p$ is an L^p-space with respect to a finite measure, there is a constant c such that

$$\begin{aligned}\limsup_{n\to\infty} \varepsilon_n^{\frac{1}{5}}|v^{(n)}|_{\mathcal{L}_5} &\le c \limsup_{n\to\infty} \varepsilon_n^{\frac{1}{5}}|v^{(n)}|_{\mathcal{L}_6} \le c \limsup_{n\to\infty} \varepsilon_n^{\frac{1}{5}-\frac{1}{6}} \varepsilon_n^{\frac{1}{6}}|v^{(n)}|_{\mathcal{L}_6} \\ &\le c \sup_n \varepsilon_n^{\frac{1}{6}}|v^{(n)}|_{\mathcal{L}_6} \limsup_{n\to\infty} \varepsilon_n^{\frac{1}{5}-\frac{1}{6}} = 0,\end{aligned}$$

which completes the proof. □

A. Appendix

We precede the proof of Proposition 2.1 by a sequence of lemmas. Recall, see Section 2, that the operator i_ρ given by $i_\rho v = \vartheta_{-\rho/2} v$ is an isometry between $\mathbb{L}^q$ and $\mathbb{L}^q_{q\rho/2}$. In fact we have something more.

Lemma A.1. *Let $\rho \ge 0$. Then for any $r \in \mathbb{R}$ the mapping i_ρ is an isomorphism between $W^r = W^r_0$ and W^r_ρ.*

Proof. First let $r = 2$ and introduce the semigroup $\tilde{S}(t) = i_\rho^{-1} S(t) i_\rho$ on $\mathbb{L}^2$, where S is considered on L^2_ρ. Clearly, the generator of $\tilde{S}$ is equal to $i_\rho^{-1} A i_\rho$. On the other hand, it is a uniformly elliptic operator with coefficients from $C^\infty_{\mathrm{b}}(\mathcal{O})$. Therefore, $i_\rho^{-1}(W^2_\rho) = W^2$. Considering the powers of A we can show in the same way that i_ρ is an isomorphism between W^r and W^r_ρ for even r. Finally, using standard interpolation argument we may extend the result to an arbitrary r. □

Further we will need the following result on Sobolev imbeddings.

Lemma A.2. (i) *For all $0 \le \kappa < \rho$ and $r < s$ the imbedding $W^s_\kappa \hookrightarrow W^r_\rho$ is compact.*
(ii) *Let $\rho \ge 0$, $q \in [2, \infty)$, and let $r \in [0, \frac{d}{2})$ be such that $\frac{d}{2} - r \le \frac{d}{q}$. Then W^r_ρ is continuously imbedded into $\mathbb{L}^q_{q\rho/2}$.*

Proof of (i). Drawing on Lemma A.1 we may assume that $\kappa = 0$. Take a sequence $\{v_n\}$ bounded in W^s. Employing Sobolev's compact imbedding theorem for bounded domains we can extract from $\{(-\nu\Delta + \mathrm{I})^{s/2} v_n\}$ a subsequence convergent almost everywhere. Since each sequence bounded in $\mathbb{L}^2$ is uniformly integrable in $\mathbb{L}^2_\rho$ the proof is complete. □

Proof of (ii). It is well known, see [1, Lemma 5.14], that W^r is continuously imbedded into $\mathbb{L}^q$. The required result follows now from Lemma A.1. □

The lemma below is a version of the result formulated and proved in [16] as Lemma 2, see also [18, Proposition A.1.1].

Lemma A.3. *Let $\rho \in \mathbb{R}$, and let $\alpha \in [0, 1)$ and $p \ge 1$ be such that $\alpha + p^{-1} < 1$. Then for each $\gamma \in [0, 1 - \alpha - p^{-1})$, $\mathcal{J}^\rho_\alpha$ is a bounded linear operator from $L^p(0, T; \mathbb{L}^2_\rho)$ into $C^\gamma([0, T]; \mathbb{L}^2_\rho)$.*

For the convenience of the reader we recall Dubinsky's criterion for compactness. Its proof can be found in [38, Theorem 4.1, p. 132].

Lemma A.4. *Let E_0, E_1 and E be reflexive Banach spaces such that the imbeddings $E_0 \hookrightarrow E \hookrightarrow E_1$ are continuous and the imbedding $E_0 \hookrightarrow E$ is compact. Let $p \in (1,\infty)$ and let M be a bounded set in $L^p(0,T;E_0)$ consisting of equicontinous functions in $C([0,T];E_1)$. Then M is relatively compact in $L^p(0,T;E)$ and $C([0,T];E_1)$.*

Proof of Proposition 2.1.

Proof of (i). Clearly we may assume that $1 < p < 2$. Let $r > 0$ be such that $\frac{1}{p}(r+1) < 1$. Then since the semigroup S is analytic on $\mathbb{L}^2_\rho$, there exists a constant c_1 such that, see [29],

$$\begin{aligned}\int_0^T \|\mathcal{J}_0^\rho(v)(t)\|^2_{r,\rho}\,\mathrm{d}t &= \int_0^T \left\|\int_0^t S(t-s)v(s)\,\mathrm{d}s\right\|^2_{r,\rho}\mathrm{d}t\\ &\le c_1\int_0^T\left(\int_0^t (t-s)^{-\frac{r+1}{2}}\|v(s)\|_{-1,\rho}\,\mathrm{d}s\right)^2\mathrm{d}t.\end{aligned}$$

Now using twice Jensen's inequality we get

$$\begin{aligned}\int_0^T \|\mathcal{J}_0^\rho(v)(t)\|^2_{r,\rho}\,\mathrm{d}t &\le c_2\int_0^T\left(\int_0^t (t-s)^{-\frac{r+1}{2}}\|v(s)\|^p_{-1,\rho}\,\mathrm{d}s\right)^{\frac{2}{p}}\mathrm{d}t\\ &\le c_3\left(\int_0^T \|v(t)\|^p_{-1,\rho}\,\mathrm{d}t\right)^{\frac{2}{p}}.\end{aligned}$$

Hence $\mathcal{J}_0^\rho$ has an extension to a bounded operator acting from $L^p(0,T;W^{-1}_\rho)$ into $L^2(0,T;W^r_\rho)$. By Lemma A.3 for any $\gamma \in [0, 1-p^{-1})$, $\mathcal{J}_0^\rho$ is a bounded linear operator from $L^p(0,T;W^{-1}_\rho)$ into $C^\gamma([0,T];W^{-1}_\rho)$. Now note that both spaces $\mathbb{L}^2_\kappa$ and W^{-1}_ρ are continuously and densely imbedded into $W^{-1}_{2\rho-\kappa}$. Thus Lemma A.2(i) and Dubinsky's criterion applied to the triple $E_0 = W^r_\rho$, $E = \mathbb{L}^2_\kappa$, and $E_1 = W^{-1}_{2\rho-\kappa}$, give the desired conclusion. □

Proof of (ii). By Lemma A.3, $\mathcal{J}_\alpha^\rho$ is a bounded operator from $L^p(0,T;\mathbb{L}^2_\rho)$ into $C^\gamma([0,T];\mathbb{L}^2_\rho)$, where $\gamma < 1-\alpha-p^{-1}$. Proceeding as in the proof of (i) one may easily show that for r sufficiently small but strictly positive, $\mathcal{J}_\alpha^\rho$ is a bounded operator from $L^p(0,T;\mathbb{L}^2_\rho)$ into $L^2(0,T;W^r_\rho)$, and consequently the desired conclusion follows again from Dubinsky's criterion applied to $E_0 = W^r_\rho$, $E = \mathbb{L}^2_\kappa$, and $E_1 = \mathbb{L}^2_\kappa$. □

Proof of (iii). Assume that $\tilde r < -1$. Recall that in the present paper the dimension d is equal to 2 or 3. Thus by Lemma A.2(ii) we have $W^{-\tilde r}_{\rho/3} \hookrightarrow \mathbb{L}^6_\rho$, and consequently by the duality we have $\mathbb{L}^{6/5}_\rho \hookrightarrow W^{\tilde r}_{\tilde\rho}$ where $\tilde\rho = 5\rho/3$. Hence the proof of (iii) will be completed as soon as we find an $\tilde r < -1$ such that $\mathcal{J}_0^\rho$ is has an extension to a compact operator acting from $L^{6/5}(0,T;W^{\tilde r}_{\tilde\rho})$ into $L^2(0,T;\mathbb{L}^2_\kappa)$. By Lemmas A.3, A.2(i) and A.4 it is enough to show that $\mathcal{J}_0^\rho$ is a bounded operator from

$L^{6/5}(0,T;W^{\tilde r}_{\tilde\rho})$ into $L^2(0,T;W^{r}_{\tilde\rho})$ for a certain $r>0$. Take $r>0$ and $\tilde r<-1$ such that $\frac{5}{6}(r-\tilde r)<1$. Then, as S is analytic, there exists a constant c_1 such that

$$\begin{aligned}\int_0^T ||\mathcal{J}_0^\rho(v)(t)||^2_{r,\tilde\rho}\,\mathrm{d}t &= \int_0^T \Big\|\int_0^t S(t-s)v(s)\,\mathrm{d}s\Big\|^2_{r,\tilde\rho}\,\mathrm{d}t\\ &\le c_1\int_0^T\Big(\int_0^t (t-s)^{-\frac{r-\tilde r}{2}}||v(s)||_{\tilde r,\tilde\rho}\,\mathrm{d}s\Big)^2\,\mathrm{d}t.\end{aligned}$$

Applying now Jensen's inequality we get

$$\begin{aligned}&\le c_1\int_0^T\Big(\Big(\int_0^t (t-s)^{-\frac{r-\tilde r}{2}}\,\mathrm{d}s\Big)^{\frac13}\Big(\int_0^t (t-s)^{-\frac{r-\tilde r}{2}}||v(s)||^{\frac65}_{\tilde r,\tilde\rho}\,\mathrm{d}s\Big)^{\frac53}\Big)\,\mathrm{d}t\\ &\le c_2\int_0^T\Big(\int_0^t (t-s)^{-\frac{r-\tilde r}{2}}||v(s)||^{\frac65}_{\tilde r,\tilde\rho}\,\mathrm{d}s\Big)^{\frac53}\,\mathrm{d}t,\end{aligned}$$

and again by Jensen's inequality

$$\begin{aligned}&\le c_2\int_0^T\Big(\int_0^t (t-s)^{-\frac{5(r-\tilde r)}{6}}||v(s)||^{\frac65}_{\tilde r,\tilde\rho}\,\mathrm{d}s\Big)\Big(\int_0^t ||v(s)||^{\frac65}_{\tilde r,\tilde\rho}\,\mathrm{d}s\Big)^{\frac23}\,\mathrm{d}t\\ &\le c_3\Big(\int_0^T ||v(t)||^{\frac65}_{\tilde r,\tilde\rho}\,\mathrm{d}t\Big)^{\frac53},\end{aligned}$$

which completes the proof. □

Proof of Proposition 2.2.

Proof of (i). First note that for each $\rho\in\mathbb{R}$ there exist a constant $c_\rho>0$ such that

$$|\partial_{x_i}\vartheta_\rho(x)|\le c_\rho|\vartheta_\rho(x)|\quad\text{for } x\in\mathbb{R}^d,\ i=1,\dots,d.$$

Now let $v,u\in C_0^\infty(\mathbb{R}^d;\mathbb{R}^d)$. Then integrating by parts and using Hölder's inequality and the estimate above we find a constant C independent of v and u such that

$$|\langle B_0(v),u\rangle_\rho|\le C|v|^2_{\mathbb{L}^4_\rho}||u||_{1,\rho},$$

which easily gives the desired conclusion. □

Proof of (ii). Since $d=2,3$, Lemma A.2(ii) yields that $W^{\frac34}\hookrightarrow\mathbb{L}^4$. Hence the desired conclusion follows from (i) and the following interpolation inequality

$$||v||_{\frac34}\le c|v|^{\frac14}||v||_1^{\frac34}\quad\text{for } v\in W^1.$$

□

Proof of (iii). By Lemma A.2(ii) we have $W^{\frac12}\hookrightarrow\mathbb{L}^4_\rho$. Thus the desired conclusion follows from (i) and the interpolation inequality

$$||v||_{\frac12}\le c|v|^{\frac12}||v||_1^{\frac12}\quad\text{for } v\in W^1.$$

□

B. Appendix

This appendix deals with the existence of an orthonormal basis of H which is a Schauder basis of H^1. This result is needed in the proof of Theorem 1.1.

Theorem B.1. *There exists an orthonormal basis $\{e_k\}$ of H such that:*

(i) $\{e_k\} \subset C_{\mathrm{b}}^{\infty}(\mathcal{O}; \mathbb{R}^d) \cap H^2$,
(ii) *for every $v \in H^1$, the sequence $\{\Pi_n v\}$ converges in H^1 to v,*
(iii) *there exists a constant C such that for every n, $||\Pi_n||_{L(H^1,H^1)} \le C$.*

Proof. Let A be the Stokes operator, that is the generator of S on H, and let $E(\,\mathrm{d}\lambda)$ be the spectral projection corresponding to $\mathrm{I} - A$. Set

$$X_m = \operatorname{Range} E([m, m+1)), \quad m \in \mathbb{N}.$$

Then X_m, $m \in \mathbb{N}$ are closed subspaces of H. For very m let $\{e_{m,l}\}$ be an orthonormal basis of X_m. Put the set $\{e_{m,l} : \ m, l \in \mathbb{N}\}$ into a sequence $\{e_k\}$. We will show that $\{e_k\}$ has the desired properties. First it is an orthonormal basis of H as X_m is orthogonal to $X_{m'}$ if $m \neq m'$, and as

$$X \stackrel{\mathrm{def}}{=} \bigcup_{m \in \mathbb{N}} X_m$$

is dense in H. Then $\{e_k\} \subset C_{\mathrm{b}}^{\infty}(\mathcal{O}; \mathbb{R}^d) \cap H^2$ because

$$X \subset \bigcap_{r \in [0,\infty)} H^r \subset C_{\mathrm{b}}^{\infty}(\mathcal{O}; \mathbb{R}^d).$$

Let Π_n be the projection onto the space $\operatorname{span}\{e_1, \dots, e_n\}$. Then, since on every X_m the norms $|\cdot|$ and $||\cdot||_1$ are equivalent we have

$$||\Pi_n v - v||_1 \to 0 \ \text{ for every } v \in X.$$

Hence, since X is dense, the proof will be complete as soon as we can show that

$$\sup_{n \in \mathbb{N}} ||\Pi_n||_{L(H^1,H^1)} < \infty.$$

Let $v \in X$. Then

$$\begin{aligned}
||\Pi_n||_1^2 = \langle (\mathrm{I} - A)\Pi_n u, \Pi_n u\rangle &= \sum_{m=1}^{\infty} \int_m^{m+1} \lambda \langle E(\,\mathrm{d}\lambda)\Pi_n u, \Pi_n u\rangle \\
&\le \sum_{m=1}^{\infty} (m+1)|E([m, m+1))\Pi_n u|^2 = \sum_{m=1}^{\infty} (m+1)|\Pi_n E([m, m+1))u|^2 \\
&\le \sum_{m=1}^{\infty} \frac{m+1}{m} \int_m^{m+1} \lambda \langle E(\,\mathrm{d}\lambda)u, u\rangle \le 2||u||_1^2,
\end{aligned}$$

which gives the desired conclusion. $\square$

C. Appendix

In the proof of the existence of a solution to (0.2) we fix $1/2 < \rho < 1$, and we project the equation onto the spaces $\operatorname{span}\{e_1, \dots, e_n\}$, $n \in \mathbb{N}$, generated by the elements of a Schauder basis of W^1_ρ, $\mathbb{L}^2_\rho$ and $\mathbb{L}^q_\rho$, $2 \le q \le 6$. Its existence follows from the following theorem.

Recall that $\mathcal{S}^l$ denotes the space of all $v \in C^l(\mathbb{R}^d; \mathbb{R}^d)$ such that

$$\sup_{|\alpha| \le l} \sup_{x \in \mathbb{R}^d} |(1+|x|^2)^n \partial^\alpha v(x)| < \infty \quad \text{for every } n \in \mathbb{N}.$$

Theorem C.1. *For all $l \in \mathbb{N}$ and $\rho \in (1/2, 1)$ there is an orthonormal basis Ψ_ρ of $L^2_\rho(\mathbb{R}^d)$ such that $\Psi_\rho \subset \mathcal{S}^l$ and Ψ_ρ is an unconditional Schauder basis of the spaces $W^1_\rho(\mathbb{R}^d)$ and $L^q_\rho(\mathbb{R}^d)$, $q \in [2, \infty)$.*

Proof. We prove the theorem for $d = 1$. Then taking tensor products one can generalize the result to the case of $d \ge 1$. As in the whole paper we write L^q, L^q_ρ, and W^1_ρ instead of $L^q(\mathbb{R}^d)$, $L^q_\rho(\mathbb{R}^d)$, and $W^1_\rho(\mathbb{R}^d)$. We divide the proof into two steps. First we show the existence of an orthonormal basis Ψ of L^2 such that $\Psi \subset \mathcal{S}^l$ and Ψ is an unconditional Schauder basis of the spaces W^1 and L^q_κ, where

$$1 < q < \infty \quad \text{and} \quad \frac{1}{2} - \frac{q}{2} < \kappa < \frac{1}{2}. \tag{C.1}$$

We construct this basis using the theory of wavelets. Recall, see [28, p. 21 and p. 71] that an *l-regular multiresolution approximation of L^2* is an increasing sequence V_j, $j \in \mathbb{Z}$ of closed linear subspaces of L^2 such that:

(i) $\bigcap_{j=-\infty}^{\infty} V_j = \{0\}$ and $\bigcup_{j=-\infty}^{\infty} V_j$ is dense in L^2,
(ii) for all $v \in L^2$ and $j \in \mathbb{Z}$, $x \mapsto v(x) \in V_j \iff x \mapsto v(2x) \in V_{j+1}$,
(iii) for all $v \in L^2$ and $k \in \mathbb{Z}$, $x \mapsto v(x) \in V_0 \iff x \mapsto v(x-k) \in V_0$,
(iv) there is a function $\varphi \in V_0 \cap \mathcal{S}^l$ such that the sequence $\varphi(\cdot - k)$, $k \in \mathbb{Z}$ is an orthonormal basis of V_0.

Let $\{V_j\}$ be an l-regular multiresolution approximation of L^2. We denote by W_j the orthogonal complement of V_j in V_{j+1}.

According to [28, Theorem 1, p. 72] for any l-regular multiresolution approximation $\{V_j\}$ of L^2 there is a function $\psi \in W_0 \cap \mathcal{S}^l$ such that $\psi(\cdot - k)$, $k \in \mathbb{Z}$, is an orthonormal basis of W_0. For our purposes we take V_0 being the space of l-splines on $\mathbb{R}$, that is the space of all $v \in L^2 \cap C^{l-1}$ such that the restriction of v to each interval $(k, k+1)$, $k \in \mathbb{Z}$, is a polynomial of degree $\le l$.

Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $j \in \mathbb{N}_0$, $k \in \mathbb{Z}$, and $x \in \mathbb{R}$ we write $\psi_{(j,k)}(x) = 2^{j/2}\psi(2^j x - k)$. Note that the family $\{\psi_{(j,k)},\ k \in \mathbb{Z}\}$ is an orthonormal basis of W^j. The following orthonormal basis of L^2,

$$\Psi = \{\varphi(\cdot - k),\ k \in \mathbb{Z}\} \cup \{\psi_{(j,k)},\ j \in \mathbb{N}_0, k \in \mathbb{Z}\}$$

in called a *wavelet basis arising from* $\{V_j\}$.

Our goal is to show that Ψ has the required properties. The fact that it is an unconditional Schauder basis of W^1 follows from [28, Theorem 2, p. 172]. Let q and κ satisfy (C.1). Then, as ϑ_κ is an A_q-basis, see [35, Chapter 5], [22, Theorem 4] yields that Ψ is an unconditional Schauder basis of L^q_κ.

Let $\rho \in (1/2, 1)$, let $i_{\rho/2}v = \vartheta_{-\rho/2}v$, and let $\Psi_\rho = \{j_{\rho/2}v,\ v \in \Psi\}$. Since $i_{\rho/2}$ is an isometry between L^2 and L^2_ρ, and an isomorphism between W^1 and W^1_ρ, see Lemma A.1, we have that Ψ_ρ is an orthonormal basis of L^2_ρ, and Ψ_ρ is an unconditional basis of W^1_ρ. Note that $i_{\rho/2}$ is an isometry between L^q_κ and $L^q_{\kappa+q\rho/2}$. Thus to show that Ψ_ρ is an unconditional basis of the spaces L^q_ρ, $q \in [2, \infty)$, it is enough to observe that for an arbitrary $q \in [2, \infty)$ there is a κ such that (C.1) holds true and $\rho = \kappa + q\rho/2$. $\square$

Lemma C.1. *Let E be equal to either W^1_ρ, or $\mathbb{L}^p_\rho$ for a $q \in [2, \infty)$, and let $\{e_k\}$ be an orthonormal basis of $\mathbb{L}^2_\rho$ and a Schauder basis of E. Let*

$$\Pi_n v = \sum_{k=1}^{n} \langle v, e_k\rangle_\rho e_k \quad \textit{and} \quad \Pi_{n,n+m} v = \sum_{k=n}^{n+m} \langle v, e_k\rangle_\rho e_k.$$

Then:

(i) *For every $v \in E$, $|\Pi_n v - v|_E \to 0$ as $n \to \infty$.*

(ii) $\sup_{n,m} ||\Pi_{n,n+m}||_{L(E,E)} < \infty$.

Proof. By the Banach–Steinhaus theorem (ii) follows from (i). Let $v \in E$, and let $\{v^k\}$ be the coefficients of v in the basis $\{e_k\}$. Write,

$$\tilde{\Pi}_n v = \sum_{k=1}^{n} v^k e_k \quad \text{and} \quad \tilde{\Pi}_{n,n+m} v = \sum_{k=n}^{n+m} v^k e_k.$$

Then $|\tilde{\Pi}_n v - v|_E \to 0$ as $n \to \infty$. Moreover, the uniqueness of the coefficients in an orthonormal basis implies that $\Pi_n v = \tilde{\Pi}_n v$ and $\Pi_{n,n+m} v = \tilde{\Pi}_{n,n+m} v$ for $v \in E$. Thus the proof of (i) will be complete as soon as we show that $\tilde{\Pi}_n$ and $\tilde{\Pi}_{n,n+m}$ are continuous linear operators on E. This follows from the facts that under the identification $(\mathbb{L}^2_\rho)^* = \mathbb{L}^2_\rho$ we have $\{e_k\} \subset E^*$, and $e_k[e_l] = \langle e_k, e_l\rangle_\rho$ is equal to 0 for $k \neq l$ and 1 for $k = l$. $\square$

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Marek Capiński: Institute of Mathematics, Jagiellonian University, Reymonta 4, 30-059 Kraków, Poland

E-mail address: capinski@im.uj.edu.pl

Szymon Peszat: Institute of Mathematics, Polish Academy of Sciences, Św. Tomasza 30/7, Kraków 31-027, Poland

E-mail address: napeszat@cyf-kr.edu.pl