NOTE ON THE ZEROS OF POLYNOMIALS OF LEAST SQUARE APPROXIMATION

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ABSTRACT. Let the pairs (E, μ) and (F, ν) satisfy Leja's polynomial condition. A version of Siciak's theorem on separately analytic functions due to Nguyen and Zeriahi implies that a Bernstein type theorem on the distribution of the zeros of polynomials of least square approximation with respect to the measure $\mu \otimes \nu$ on the set $E \times F$ holds whenever this theorem holds on the sets E and F for the measures μ and ν, respectively.

0. INTRODUCTION

It was probably S.N. Bernstein [4], who first observed that the distribution of the zeros of the polynomials of least square approximation in a weighted L^2-norm to a function h defined on the interval $I = [-1, 1]$ is tightly associated with the existence of an analytic extension of h in an open neigbourhood of I. Pleśniak [10] extended Bernstein's result to the case of any compact subset $E \subset \mathbb{C}$ and any finite positive measure μ such that the pair (E, μ) satisfies *Leja's polynomial condition.* To the best of our knowledge, in the case of $E \subset \mathbb{C}^N$, $N > 1$, a characterization of analyticity of a function h defined on E in terms

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of the zeros of polynomials of least square approximation to h is known only for balls in a norm of $\mathbb{C}^N$ (see [14], Corollary 5.5).

In this note we introduce a condition (Z) concerning a pair (E, μ), where E is a compact subset of $\mathbb{C}^N$ and μ is a Borel measure on E. Applying a version of Siciak's theorem on separately analytic functions due to Nguyen and Zeriahi [9], we prove that a pair $(E \times F, \mu \otimes \nu)$ satisfies (Z) whenever (E, μ) and (F, ν) do it (Theorem 2.2).

As an aplication of Theorem 2.2 we give a Bernstein type characterization of analyticity of functions in terms of the distribution of zeros of polynomials of least square approximation on a compact set $E \subset \mathbb{C}^N$ with respect to the measure μ such that (E, μ) satisfies (Z) (Proposition 2.9).

1. Definitions and notations

Let $\mathbb{C}^N \ni z \to |z|$ be a fixed norm in $\mathbb{C}^N$. We shall denote by $\mathcal{P}_n(\mathbb{C}^N)$ the set of all polynomials of degree at most n, $n \in \mathbb{N}_0$, and by $\mathcal{P}(\mathbb{C}^N)$ the set of all polynomials.

Following [13], given a set $E \subset \mathbb{C}^N$, we define the *pluricomplex extremal function*

$$V_E(z) = \sup\{u(z),\ u \in \mathcal{L},\ u_{|E} \leq 0\},$$

where $\mathcal{L}$ denotes the *Lelong class* of plurisubharmonic functions on $\mathbb{C}^N$ $(psh(\mathbb{C}^N))$ satisfying

$$\sup_{z \in \mathbb{C}^N} u(z) - \log^+ |z| < \infty,$$

where $\log^+ |z| = \max\{0, \log |z|\}$. We shall denote by $V_E^*(z) = \limsup_{w \to z} V_E(w)$ its upper regularization and for $r > 1$ the set $E_r = \{z \in \mathbb{C}^N : V_E(z) < \log r\}$.

Throughout this note we shall assume that the set $E \subset \mathbb{C}^N$ is compact and *L-regular* i.e. the function V_E is continuouos in $\mathbb{C}^N$.

In this case it is known (see [13], Theorem 4.12) that

$$V_E(z) = \sup_{n \geq 1} \sup\{\frac{1}{n} \log |p(z)| : p \in \mathcal{P}_n(\mathbb{C}^N),\ \|p\|_E \leq 1\}.$$

Then, it follows that for every $b > 1$ there exists an open neighbourhood U of E such that for any $p \in \mathcal{P}(\mathbb{C}^N)$

$$\text{(B-W)} \qquad |p(z)| \leq b^{\deg p} \|p\|_E, \quad \text{as } z \in U,$$

which is refered as to *the Bernstein - Walsh inequality.*

For a set $E \subset \mathbb{C}^N$ and an open set $D \subset \mathbb{C}^N$ we also define the *relative extremal function associated with the pair* (E, D) (see eg. [6]) by

$$\omega(z, E, D) = sup\{u(z) : u \in psh(\mathbb{C}^N), u \leq 1, u_{|E \cap D} \leq 0\},$$

and we denote its upper regularization by $\omega^*(z, E, D) = \limsup_{w \to z} \omega(w, E, D)$.

Definition 1.1. (cf. [8]) Let E be a compact, L-regular subset of $\mathbb{C}^N$ and let μ be a finite Borel measure on E. We say that the pair (E,μ) satisfies *the Bernstein - Markov condition* (BM) with exponent $q \in (0,\infty)$ if for any $\epsilon > 0$ there exists $A_\epsilon > 0$ such that

$$\|p\|_E \leq A_\epsilon (1+\epsilon)^{\deg p} \|p\|_{\mu,q}$$

for any $p \in \mathcal{P}(\mathbb{C}^N)$, where $\|p\|_{\mu,q} = \Big(\int_E |p(z)|^q d\mu(z) \Big)^{1/q}$ and $\|p\|_E = \sup_{z\in E} |p(z)|$.

By Hölder's inequality, the function

$$(0,\infty) \ni q \longrightarrow \Big(\frac{1}{\mu(E)} \int_E |p(z)|^q d\mu(z)\Big)^{1/q}$$

is increasing. Hence, for any q_1, q_2 with $0 < q_1 \leq q_2 < \infty$, if the pair (E,μ) satisfies (BM) with q_1 then it satisfies (BM) with q_2 as well.

We also recall the following

Definition 1.2. Let E be a Borel subset of $\mathbb{C}^N$ and μ be a finite Borel measure on E. We say that the pair (E,μ) satisfies *Leja's polynomial condition* (and we write $(E,\mu) \in \mathcal{L}^*$) if for every family $\mathcal{F} \subset \mathcal{P}(\mathbb{C}^N)$ satisfying

$$\mu(\{z \in E : \sup_{f\in\mathcal{F}} |f(z)| = \infty\}) = 0$$

and for every $b > 1$ there exist $M > 0$ and a neighbourhood U of E such that

$$\|p\|_U \leq M b^{\deg p}, \quad \text{for all } p \in \mathcal{F}.$$

Remark 1.3. (see [8], Théorème 3.1). Let E be a compact, L-regular subset of $\mathbb{C}^N$. If the pair (E,μ) satisfies Leja's polynomial condition then it satisfies the Bernstein - Markov condition with any exponent q, $0 < q < \infty$. In particular, let μ_E be the *complex equilibrium measure* of an L-regular, compact set $E \subset \mathbb{C}^N$, i.e. the Monge-Ampère measure $(dd^c V_E)^N$ associated with the pluricomplex extremal function V_E (see eg. [6], p. 212). Then, by [8], Théorème 2.1 and Théorème 3.1, the pair (E,μ_E) satisfies (BM) with any $q \in (0,\infty)$.

For further examples and properties of (E,μ) satisfying (BM) see [8].

In this paper we shall consider the Hilbert space $L^2_{\mathcal{P}}(E,\mu)$, the closure in the norm $\|\cdot\|_{\mu,2}$ of the algebra $\mathcal{P}(\mathbb{C}^N)$. We shall denote its norm by $\|p\|_\mu := \|p\|_{\mu,2}$.

Let $\alpha : \mathbb{N}_0 \longrightarrow \mathbb{N}_0^N$ be a bijection satisfying $|\alpha(i)| \leq |\alpha(i+1)|$. We denote by (e_k), $k \in \mathbb{N}_0$, the orthonormal polynomial Schauder basis of $L^2_{\mathcal{P}}(E,\mu)$ associated with the monomials $(z^{\alpha(k)})$, $k \in \mathbb{N}_0$. Thus $e_k(z) = \dfrac{t_k(z)}{\|t_k\|_\mu}$, where t_k realizes the infimum

$$\|t_k\|_\mu = \inf\{\|z^{\alpha(k)} + \sum_{i=0}^{k-1} c_i z^{\alpha(i)}\|_\mu,\ c_i \in \mathbb{C}\}.$$

Let $h \in L^2_{\mathcal{P}}(E,\mu)$. We denote by $h_n(z) = \sum_{k=0}^n \sum_{|\alpha(i)|=k} \chi_i e_i(z)$, the orthogonal projection of h onto the space of polynomials of degree at most n, where $\chi_i = \int_E h(z)\overline{e_i(z)} d\mu(z)$ is the ith Fourier coefficient of h.

Remark 1.4. The polynomial h_n does not depend on the choice of the polynomial basis of $L^2_{\mathcal{P}}(E,\mu)$, since $L^2_{\mathcal{P}}(E,\mu)$ is a Hilbert space and h_n is the unique polynomial of best L^2-approximation in $\|\cdot\|_\mu$-norm of degree at most n. In particular, h_n is the orthogonal projection of h onto $\mathcal{P}_n(\mathbb{C}^N)$.

Definition 1.5. Let E be a compact, L-regular subset of $\mathbb{C}^N$ and let μ be a finite Borel measure on E. We say that the pair (E,μ) satisfies *condition* (Z) if there exists a continuous strictly increasing function $\zeta : (1,+\infty) \ni t \to \zeta(t) \in (1,+\infty)$ such that for every continuous function $h \in L^2_{\mathcal{P}}(E,\mu)$ the hypothesis that

$(a1)$ there exists $R > 1$ such that for every $r \in (1,R)$ there exists a complex number A such that $h_n(z) \neq A$, for all $n \in \mathbb{N}$ and all $z \in \overline{E_r}$

ensures that

$(a2)$ h admits an analytic extension onto $E_{\zeta(R)}$.

Remark 1.6. (cf. [14], Remark 3.3) If the pair (E,μ) satisfies (BM) with $q = 2$ then the converse implication $(a2) \implies (a1)$ holds with $\zeta(t) = t$ whenever E is a compact, L-regular subset of $\mathbb{C}^N$. This follows from the Bernstein - Walsh inequality $(B-W)$ and the Bernstein - Walsh - Siciak characterization of analytic functions ([12], Theorem 10.3).

Remark 1.7. (see [10], Theorem 4.1) Let $E \subset \mathbb{C}$ be a compact, L-regular set and μ be a finite Borel measure on E. Then (E,μ) satisfies (Z) with $\zeta(t) = t$ whenever (E,μ) satisfies (BM) with $q = 2$.

To show this it suffices to use the following

Lemma 1.8. *(cf. [10], Lemma 1.4) Let $R > 1$ and let $p(z) = a(z - z_1)\dots(z - z_n)$ be a polynomial. If $z_i \in \mathbb{C} \setminus E_R$, for all $i = 1, 2, \dots, n$, then*

$$|a| \leq \frac{\|p\|_E}{(Rd(E))^n},$$

$d(E)$ being the transfinite diameter of the set E. □

For, let h_n, R, r and A be as in hypothesis $(a1)$ of condition (Z). Applying the above lemma to the polynomial $p(z) := h_n(z) - A = \sum_{k=0}^{n} \chi_k z^k - A$ we have, for all $n \in \mathbb{N}_0$,

$$\frac{|\chi_n|}{\|t_n\|_\mu} \leq \frac{\|h_n\|_E + |A|}{(rd(E))^n}.$$

Fix $\epsilon > 0$. By (BM) there exists A_ϵ such that $\|h_n\|_E \leq A_\epsilon(1+\epsilon)^n\|h_n\|_\mu$, for all $n \in \mathbb{N}_0$.

Since $\lim_{n\to\infty} \|t_n\|_\mu^{1/n} = d(E)$ and since by the Bessel inequality $\|h_n\|_\mu \leq \|h\|_\mu$, we get $\limsup_{n\to\infty} |\chi_n|^{1/n} < (1+\epsilon)/r$. This implies that $\limsup_{n\to\infty} |\chi_n|^{1/n} \leq 1/r$, since $\epsilon > 0$ was chosen arbitrary. Thus, by the Bernstein - Walsh inequality (B-W) and condition (BM), the Fourier series $\sum_{k=0}^{\infty} \chi_k e_k$ converges uniformly on compact subsets of E_r with $r \in (1,R)$. Hence, h admits an analytic extension onto E_R. □

Let us note that in [10] Pleśniak originally assumed that the pair (E,μ) satisfies Leja's polynomial condition. By Remark 1.3, (E,μ) satisfies then (BM).

It is known that if $E \subset \mathbb{C}^N$ $(N \geq 1)$ is a ball in a norm in $\mathbb{C}^N$ and μ is a finite Borel measure, then the pair (E,μ) satisfies (Z) (with $\zeta(t) = t$) if it satisfies (BM) with $q = 2$. (cf. [14], Corollary 5.5).

2. Main result

Let $E \subset \mathbb{C}^N$ and $F \subset \mathbb{C}^M$ be compact, L-regular sets and let μ and ν be finite Borel measures on E and F, respectively. We fix two bijections $\alpha : \mathbb{N}_0 \to \mathbb{N}_0^N$ and $\beta : \mathbb{N}_0 \to \mathbb{N}_0^M$ such that $|\alpha(i)| \le |\alpha(i+1)|$ and $|\beta(j)| \le |\beta(j+1)|$, for all $i, j \in \mathbb{N}_0$. By the Hilbert-Schmidt orthogonalization process we can find two orthonormal polynomial Schauder bases $\{e_i(z)\}$ and $\{f_j(w)\}$ for $L^2_{\mathcal{P}}(E,\mu)$ and $L^2_{\mathcal{P}}(F,\nu)$, respectively, associated with the monomials $\{z^{\alpha(i)}\}$, $i \in \mathbb{N}_0$, and $\{w^{\beta(j)}\}$, $j \in \mathbb{N}_0$.

Remark 2.1. *$\{e_i(z)f_j(w)\}$, $i, j = 0, 1, 2, \dots,$ is an orthonormal polynomial Schauder basis for $L^2_{\mathcal{P}}(E \times F, \mu \otimes \nu)$.*

Indeed, if $h \in L^2_{\mathcal{P}}(E \times F, \mu \otimes \nu)$ then $h = \lim_{n\to\infty} h_n$ in $L^2_{\mathcal{P}}(E \times F, \mu \otimes \nu)$, where $h_n \in \mathcal{P}_n(\mathbb{C}^{N+M})$ is the polynomial of degree at most n of the L^2- approximation to h in the $\|\cdot\|_{\mu\otimes\nu}$ norm (see Remark 1.4). For a fixed $z \in \mathbb{C}^N$, $h_n(z,\cdot)$ is a polynomial of degree at most n in variable $w \in \mathbb{C}^M$. Since $\{f_j\}$ is a Schauder basis of $L^2_{\mathcal{P}}(F,\nu)$, we have $h_n(z,\cdot) = \sum_{|\beta(l)|=0}^{n} \chi_l(z) f_l(\cdot)$, where $\chi_l(z)$ is a polynomial of degree at most $n - |\beta(l)|$ in variable z. Since $\{e_i\}$ is a Schauder basis of $L^2_{\mathcal{P}}(E,\mu)$ there are complex numbers χ_{kl}, where $k \in \mathbb{N}_0$, $|\alpha(k)| \le n - |\beta(l)|$, such that $\chi_l(z) = \displaystyle\sum_{|\alpha(k)|=0}^{n-|\beta(l)|} \chi_{kl} e_k(z)$. Hence $h_n(z,w) = \displaystyle\sum_{|\alpha(k)|+|\beta(l)|\le n} \chi_{kl} e_k(z) f_l(w)$. Since $\int_{E\times F} h_n \overline{e_k f_l} d\mu \otimes \nu = \chi_{kl}$ and $h - h_n$ is orthogonal to $\mathcal{P}_n(\mathbb{C}^{N+M})$ we have $\int_{E\times F}(h - h_n)\overline{e_k f_l} d\mu \otimes \nu = 0$ for k, l such that $|\alpha(k)| + |\beta(l)| \le n$. This implies that

$$\chi_{kl} = \int_{E\times F} (h_n + (h - h_n))\overline{e_k f_l} d\mu \otimes \nu = \int_{E\times F} h\overline{e_k f_l} d\mu \otimes \nu.$$

Thus, $\quad h = \displaystyle\sum_{n=0}^{\infty} \sum_{\substack{k,l \\ |\alpha(k)|+|\beta(l)|=n}} \chi_{kl} e_k f_l, \quad$ where $\chi_{kl} = \displaystyle\int_{E\times F} h\overline{e_k f_l} d\mu \otimes \nu$. This means that $\{e_i(z)f_j(w)\}$, $i, j = 0, 1, 2, ...$ is a polynomial Schauder basis for $L^2_{\mathcal{P}}(E \times F, \mu \otimes \nu)$. By the Fubini theorem, $\int_{E\times F} e_i f_j \overline{e_{i'} f_{j'}} d\mu \otimes \nu = \delta_{ii'}\delta_{jj'}$. Thus, the basis is orthonormal.

Theorem 2.2. *Let $E \subset \mathbb{C}^N$ and $F \subset \mathbb{C}^M$ be compact sets. Assume that the pairs (E,μ) and (F,ν) satisfy Leja's polynomial condition. If the pairs (E,μ) and (F,ν) satisfy (Z), so does the pair $(E \times F, \mu \otimes \nu)$.*

We shall need the following result of Nguyen and Zeriahi ([9], Théorème 1.1)

Theorem 2.3. *Let $E_1 \subset \mathbb{C}^{N_1}$ and $E_2 \subset \mathbb{C}^{N_2}$ be nonpluripolar sets such that at least one of them is $\mathcal{K}$-analytic. Let $D_1 \subset \mathbb{C}^{N_1}$ and $D_2 \subset \mathbb{C}^{N_2}$ be bounded open pseudoconvex sets such that $E_1 \subset D_1$ and $E_2 \subset D_2$. Let f be a complex-valued function defined on $X := D_1 \times E_2 \cup E_1 \times D_2$ that is separately analytic on X i.e. $f(\cdot, z_2)$ is analytic on D_1 for all $z_2 \in E_2$ and $f(z_1, \cdot)$ is analytic on D_2 for all $z_2 \in E_2$. Then there exists a function*

$\tilde{f}$ that is analytic on $\tilde{X} := \{(z_1, z_2) \in D_1 \times D_2 : \omega^*(z_1, E_1, D_1) + \omega^*(z_2, E_2, D_2) < 1\}$ and coincides with f on $X \cap \tilde{X}$.

Proof of Theorem 2.2. We assume that the pairs (E, μ) and (F, ν) satisfy (Z) with continuous increasing functions ζ_E and ζ_F, respectively. We shall prove that the pair $(E \times F, \mu \otimes \nu)$ satisfies (Z) with $\zeta(t) := \exp\left(\left(1/\log \zeta_E(t) + 1/\log \zeta_F(t)\right)^{-1}\right)$. It is seen that $\zeta : (1, +\infty) \to (1, +\infty)$ is a strictly increasing, continuous function whenever ζ_E, $\zeta_F : (1, +\infty) \to (1, +\infty)$ are strictly increasing, continuous functions.

Fix a continuous function $h \in L^2_{\mathcal{P}}(E \times F, \mu \otimes \nu)$ and let

$$h_n(z, w) = \sum_{k=0}^{n} \sum_{|\alpha(i)|+|\beta(j)|=k} \chi_{ij} e_i(z) f_j(w),$$

where $\chi_{ij} = \int_{E \times F} h\overline{e_i f_j} d\mu \otimes \nu$. Assume that $(a1)$ holds. Let

$$R = \sup\{\rho > 1 : \mathbb{C} \setminus \bigcup_{n \in \mathbb{N}_0} h_n\left(\overline{(E \times F)_\rho}\right) \neq \emptyset\}.$$

By [13], Proposition 5.9, we have $V_{E \times F}(z, w) = \max\{V_E(z), V_F(w)\}$. For every $\rho > 1$, the set $(E \times F)_\rho = \{(z, w) \in \mathbb{C}^{N+M} : V_{E \times F}(z, w) < \log \rho\}$ is open since E and F are L-regular. Since $(a1)$ of (Z) holds, we get $R > 1$.

Let $r \in (1, R)$. Chose $A_r \in \mathbb{C} \setminus \bigcup_{n \in \mathbb{N}_0} h_n\left(\overline{(E \times F)_r}\right) \neq \emptyset$. Then $h_n(z, w) - A_r \neq 0$ for any $(z, w) \in \overline{(E \times F)_r}$ and any $n \in \mathbb{N}_0$.

Define

$$E' = \{z \in E : h_n(z, \cdot) \text{ converges to } h(z, \cdot) \text{ in } L^2_{\mathcal{P}}(F, \nu)\},$$

$$F' = \{w \in F : h_n(\cdot, w) \text{ converges to } h(\cdot, w) \text{ in } L^2_{\mathcal{P}}(E, \mu)\}.$$

If we denote by $E_{n,k} = \{z \in E : \int_F |h_n(z, w) - h(z, w)|^2 d\nu(w) \leq \frac{1}{k}\}$ and by $F_{n,k} = \{w \in F : \int_F |h_n(z, w) - h(z, w)|^2 d\mu(z) \leq \frac{1}{k}\}$, we see that

$$E' = \bigcap_{k \in \mathbb{N}_0} \bigcup_{n_0 \in \mathbb{N}_0} \bigcap_{\substack{n \in \mathbb{N}_0 \\ n \geq n_0}} E_{n,k} \quad \text{and} \quad F' = \bigcap_{k \in \mathbb{N}_0} \bigcup_{n_0 \in \mathbb{N}_0} \bigcap_{\substack{n \in \mathbb{N}_0 \\ n \geq n_0}} F_{n,k}.$$

Since h is continuous and the sets E and F are compact, the sets $E_{n,k}$ and $F_{n,k}$ are compact, thus E' and F' are Borel sets. In particular, E' and F' are $\mathcal{K}$-analytic. (For a definition of $\mathcal{K}$-analytic (or Souslin) sets see eg.[5]).

Since h_n converges to h in $L^2_{\mathcal{P}}(E \times F, \mu \otimes \nu)$, we have $\mu(E \setminus E') = \nu(F \setminus F') = 0$.

Fix $z' \in E'$. Then $\overline{\{z'\} \times F_r} \subset \overline{(E \times F)_r}$ and $h_n(z', w) - A_r \neq 0$ for any $w \in F_r$ and any $n \in \mathbb{N}_0$.

Since the pair (F, ν) satisfies (Z) with a continuous increasing function ζ_F, the function $h(z', \cdot)$ admits an analytic extension onto $F_{\zeta_F(r)}$. Repeating this argument for all $r \in (1, R)$

we conclude that $h(z', \cdot)$ admits an analytic extension onto $F_{\zeta_F(R)}$, since F is determining for analytic functions and ζ_F is continuous, increasing function.

Since the pair (E, μ) satisfies (Z) with a continuous inreasing function ζ_E, the same argument shows that $h(\cdot, w')$ admits an analytic extension onto $E_{\zeta_E(R)}$. Thus function h is the restriction of a function separately analytic on $X = E' \times F_{\zeta_F(R)} \cup E_{\zeta_E(R)} \times F'$. Since (E, μ) and (F, ν) satisfy $\mathcal{L}^*$ and $\mu(E \setminus E') = \nu(F \setminus F') = 0$, we conclude that the sets E' and F' are nonpluripolar (see [11]). The sets $E_{\zeta_E(R)}$ and $F_{\zeta_F(R)}$ are polynomialy convex. Thus, by Levi's theorem they are pseudoconvex.

By Theorem 2.3, there exists a function $\tilde{h}$ that is analytic in

$$\tilde{X} = \{(z, w) \in E_R \times F_R : \omega^*(z, E', E_{\zeta_E(R)}) + \omega^*(w, F', F_{\zeta_F(R)}) < 1\}$$

and coincides with h on X.

By [6], Proposition 5.3.3, we have

$$\omega^*(z, E', E_{\zeta_E(R)}) = \frac{V^*_{E'}(z)}{\log \zeta_E(R)}, \quad \text{for } z \in E_{\zeta_E(R)},$$

$$\omega^*(z, F', F_{\zeta_F(R)}) = \frac{V^*_{F'}(z)}{\log \zeta_F(R)}, \quad \text{for } z \in F_{\zeta_F(R)}.$$

Since the pairs (E, μ) and (F, ν) satisfy $\mathcal{L}^*$ and $\mu(E \setminus E') = \nu(F \setminus F') = 0$, by Levenberg's theorem ([7], Theorem 3.1, see also [11]), we get $V^*_{E'} = V^*_E$ and $V^*_{F'} = V^*_F$. This implies that

$$\tilde{X} = \{(z, w) \in \mathbb{C}^{N+M} : \frac{V_E(z)}{\log \zeta_E(R)} + \frac{V_F(w)}{\log \zeta_F(R)} < 1\}.$$

Let $(z, w) \in (E \times F)_{\zeta(R)}$. Then $V_E(z) < \log \zeta(R)$ and $V_F(w) < \log \zeta(R)$ and we get

$$\frac{V_E(z)}{\log \zeta_E(R)} + \frac{V_F(w)}{\log \zeta_F(R)} < \log \zeta(R) \left(\frac{1}{\log \zeta_E(R)} + \frac{1}{\log \zeta_F(R)} \right) = 1,$$

i.e. $(E \times F)_{\zeta(R)} \subset \tilde{X}$ and h admits an analytic extension onto $(E \times F)_{\zeta(R)}$.

This means that the pair $(E \times F, \mu \otimes \nu)$ satisfies (Z). □

Remark 2.4. In the proof of Theorem 2.2 we can put $\zeta_1(t) := \min\{\sqrt{\zeta_E(t)}, \sqrt{\zeta_F(t)}\}$ instead of $\zeta(t) := \exp\left((1/\log \zeta_E(t) + 1/\log \zeta_F(t))^{-1} \right)$. There arises a natural question of whether the pair (E, μ) satisfies (Z) with $\zeta_1(t) = t$ if it satisfies (Z) with a continuous increasing function ζ.

If $B := E \times F$ is a ball in a norm of $\mathbb{C}^{N+M}$, using a different technique (see [14], Corollary 5.5) one may prove that if (B, μ) satisfies (BM) then it satisfies (Z) with $\zeta(t) = t$.

Let μ_E be the *complex equilibrium measure* of the compact set $E \subset \mathbb{C}^N$. It is known that μ_E is supported on the boundary of the polynomial hull of E and posseses the following product property.

Proposition 2.5. *(cf. [3], Proposition 2.2) Let E_1, ..., E_N be compact, L-regular subsets of $\mathbb{C}$. Then $\mu_{E_1\times\cdots\times E_N} = \mu_{E_1}\otimes\cdots\otimes\mu_{E_N}$.* □

Question 2.6. Does the above property hold for compact sets $E_i \subset \mathbb{C}^{N_i}$ for any dimension $N_i \geq 1$?

Recently Baran proved that the answer to the question is affirmative whenever the sets $E_i \subset \mathbb{R}^{N_i} \subset \mathbb{C}^{N_i}$, $i = 1, 2, ...$, are balls in a norm of R^{N_i} (see [2], Theorem 3.12, see also [1], where several explicite formulae for μ_E are given).

Hence by Remark 1.3, Remark 1.7, Proposition 2.5 and by Theorem 2.2 we get

Example 2.7. Let E_1, ..., E_N be compact, L-regular subsets of $\mathbb{C}$. Then the pair

$$(E_1\times\cdots\times E_N, \mu_{E_1\times\cdots\times E_N})$$

satisfies condition (Z).

In [14], Theorem 5.3, Corollary 5.5, we have proved that if $E \subset \mathbb{C}^N$ is a ball in a norm of $\mathbb{C}^N$ and the pair (E,μ) satisfies (BM) then it satisfies (Z) (with $\zeta(t) = t$). This gives another

Example 2.8. Let $E_i \subset \mathbb{C}^{N_i}$ be a compact set and let μ_i be a finite Borel measure such that the pair (E_i, μ_i) satisfy $\mathcal{L}^*$, $i = 1, \ldots, k$. If for every $i = 1, \ldots, k$ we have $N_i = 1$ or E_i is a ball in a norm of $\mathbb{C}^{N_i}$ then the pair $(E_1\times\cdots\times E_k, \mu_1\otimes\cdots\otimes\mu_k)$ satisfies condition (Z).

The above example includes such pairs (E,μ) satisfying (Z) that E is neither a ball in $\mathbb{C}^N$ nor a Cartesian product of one-dimensional compact, L-regular sets, e.g. $E := E_1\times E_2 \subset \mathbb{C}^3$, where $E_1 := \{(u,v) \in \mathbb{C}^2 : |u|^2 + |v|^2 \leq 1\}$, $E_2 = \{z \in \mathbb{C} : |\Re z| \leq 1, \Im z = 0\}$ with the measure $\mu = \mu_{E_1}\otimes\mu_{E_2}$. (If the answer to Question 2.6 were affirmative then $\mu = \mu_E$).

As an application of Theorem 2.2 we present the following Bernstein type characterization of analyticity of functions defined on a compact set $E \subset \mathbb{C}^N$ that we proved in [14], Corollary 5.5, in the particular case of $E \subset \mathbb{C}^N$ being a ball in a norm of $\mathbb{C}^N$ (cf. [10], Theorem 4.1, for N=1).

Proposition 2.9. *Let E be a compact, L-regular subset of $\mathbb{C}^N$ and let μ be a finite Borel measure such that the pair (E,μ) satisfies (BM). Let $h \in L^2_{\mathcal{P}}(E,\mu)$ be a continuous function and let $R > 1$. Consider the following conditions*

1) $\limsup_{i\to\infty} |\chi_i|^{1/|\alpha(i)|} < \frac{1}{R}$;

2) function h admits an analytic extension onto E_R;

3) for every $r \in (1,R)$ there exists $A_r \in \mathbb{C}$ such that $h_n(z) - A \neq 0$ for any $n \in \mathbb{N}_0$ and any $z \in \overline{E_r}$;

4) function h admits an analytic extension onto an open neighbourhood of E;

5) $\limsup_{i\to\infty} |\chi_i|^{1/|\alpha(i)|} < 1$.

If the pair (E,μ) satisfies (Z) then (3) $\Longrightarrow$ (4)$\Longleftrightarrow$(5).

If the pair (E,μ) satisfies (Z) with $\zeta(t) = t$ then (1)$\Longleftrightarrow$(2)$\Longleftrightarrow$(3).

Here $h_n(z) = \sum_{k=0}^{n} \sum_{|\alpha(i)|=k} \chi_i e_i(z)$, $\chi_i = \int_E f(z)\overline{e_i(z)} d\mu(z)$ *and* (e_i), $i \in \mathbb{N}_0$, *is the orthonormal polynomial Schauder basis for* $L^2_{\mathcal{P}}(E, \mu)$ *associated with the monomials* $\{z^{\alpha(i)}\}$, $i \in \mathbb{N}_0$, *where* $\alpha : \mathbb{N}_0 \to \mathbb{N}_0$ *is a bijection satisfying* $|\alpha(n)| \leq |\alpha(n+1)|$ *for all* $n \in \mathbb{N}_0$. □

Proof. The implications (2) $\Longrightarrow$ (1) and (4) $\Longrightarrow$ (5) easily follow from the Bernstein-Walsh-Siciak characterization of analytic functions ((BWS), [12], Theorem 10.3). The implications (1) $\Longrightarrow$ (2) and (5) $\Longrightarrow$ (4) follow from (BWS) and from condition (BM). The implication (2) $\Longrightarrow$ (3) easily follows from the Bernstein - Walsh inequality (B-W) and from (BWS) (see eg. [14], Remark 3.3).

Since the pair (E, μ) satisfies (Z) by Theorem 2.2, we get the implication (3) $\Longrightarrow$ (4) and (3) $\Longrightarrow$ (2).

□

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