Birational finite extensions

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Abstract

The main goal of this note is the following: for any arbitrary finite mapping $f : V \to W$ where $V \subset \mathbf{k}^n$, $W \subset \mathbf{k}^{m-2} \times 0 \subset \mathbf{k}^m$ are algebraic sets there exists a finite mapping $F : \mathbf{k}^n \to \mathbf{k}^m$ birational on its image such that $F|_V = f$.

1 Introduction

Let $\mathbf{k}$ be an algebraicly closed field of characteristic zero, V, W be algebraic subsets of $\mathbf{k}^n$, $\mathbf{k}^m$, respectively, with $n \leq m$. M. Kwieciński proved (in [1]) that any finite mapping $f : V \longrightarrow W$ can be extended to a finite mapping $F : \mathbf{k}^n \longrightarrow \mathbf{k}^m$. We have the following

Theorem 1 ([1]) *Let V, W be algebraic subsets of $\mathbf{k}^n$, $\mathbf{k}^m$, respectively, with $n \leq m$. If $f : V \mapsto W$ is a proper regular mapping then there exists a proper regular mapping $F : \mathbf{k}^n \mapsto \mathbf{k}^m$ such that $F|_V = f$.*

Thus a natural problem follows: to estimate from above the number of points in the *"generic fiber"* of an extension F with the minimal number of points in the *"generic fiber"*. The problem was solved (in [3] and [4]) in a few cases.

All obtained estimations were bigger then gdeg f. It is natural because if $n = m$ then there does not exist finite $F : \mathbf{k}^n \to \mathbf{k}^m$ such that $F|_V = f$ and gdeg $F <$ gdeg f (see the Proposition 7). So in this case there does not exists better estimation than gdeg f.

But if $n < m$ then it may be possible to find a finite mapping $F : \mathbf{k}^n \to \mathbf{k}^m$ with $F|_V = f$ and gdeg $F <$ gdeg f. It is natural to ask if for any such mapping $f : V \to W$ there exists $F : \mathbf{k}^n \to \mathbf{k}^m$ as above. We can go one step further and ask if there exists a finite mapping $F : \mathbf{k}^n \to \mathbf{k}^m$ such that F is birational on its image and $F|_V = f$. In this paper we show that the answer to the second question (and also to the first) is positive if we add some assumptions about the set W. We will prove the following

Theorem 5 *Let $n \leq m - 2$, $V \subset \mathbf{k}^n$, $W \subset \mathbf{k}^{m-2} \times 0 \subset \mathbf{k}^m$ be algebraic sets and $f : V \to W$ finite mapping. If $V \neq \mathbf{k}^n$ then there exists a finite mapping $F : \mathbf{k}^n \to \mathbf{k}^m$ birational on its image such that $F|_V = f$.*

2 Notions and basic facts

The coordinate ring of the algebraic set V will be denoted by $\mathbf{k}[V]$. For any polynomial mapping $f = (f_1, \ldots, f_n) : V \to W$ the homomorphism of the coordinate rings $\mathbf{k}[W] \ni P \mapsto P(f_1, \ldots, f_n) \in \mathbf{k}[V]$ will be denoted by f^*.

Definition 2 *A polynomial mapping $f : V \to W$ is called finite if $\mathbf{k}[V]$ is an integral extension of the ring $f^*(\mathbf{k}[W]) = \mathbf{k}[f_1, \ldots, f_m]$.*

We will need the following

Lemma 3 (see [4]) *If $f : V \to W$ is finite, $g : V \to Z$ and $h : Z \to W$ are polynomial mappings such that $f = h \circ g$ then*

(1) *g is finite,*

(2) *if g is dominating then h is finite.*

In this paper we need the notion of the geometric degree only in the special case when it is equal to the number which provides the following

Theorem 4 (see e.g. [5] Th. 3.17) *Let $V \subset \mathbf{k}^n$ and $W \subset \mathbf{k}^m$ be irreducible algebraic sets of the same dimension. If $f : V \to W$ is dominating then there exists an open and dense subset $U \subset W$ such that*

$$\#f^{-1}(y) = [\mathbf{k}(V) : \mathbf{k}(W)], \qquad \text{for } y \in U.$$

3 Extending to birational finite mapping

Now we are in a position to prove the main result.

Theorem 5 *Let $n \leq m-2$, $V \subset \mathbf{k}^n$, $W \subset \mathbf{k}^{m-2} \times 0 \subset \mathbf{k}^m$ be algebraic sets and $f : V \to W$ finite mapping. If $V \neq \mathbf{k}^n$ then there exists a finite mapping $F : \mathbf{k}^n \to \mathbf{k}^m$ birational on its image such that $F|_V = f$.*

Proof: By Theorem 1, there exists a finite mapping $G = (G_1, \ldots, G_m) : \mathbf{k}^n \to \mathbf{k}^{m-2}$ such that $G|_V = f$. By the primitive element theorem (see e.g. [2] Theorem A.7.1) there exists $l = a_1X_1 + \ldots + a_nX_n$, where $a_1, \ldots, a_n \in \mathbf{k}$, such that $\mathbf{k}(X_1, \ldots, X_n) = \mathbf{k}(\overline{G(\mathbf{k}^n)})(l)$. Let $h \in \mathbf{k}[X_1, \ldots, X_n]$ be any non zero polynomial such that $h|_V = 0$. Now consider the mapping:

$$F : \mathbf{k}^n \ni x \mapsto (G(x), h(x), h(x)l(x)) \in \mathbf{k}^m.$$

If $x \in V$ then $h(x) = 0$ and then

$$F(x) = (G(x), 0, 0) = (f(x), 0, 0).$$

By Lemma 3 the mapping F is finite because $\pi \circ F = G$ is finite, where $\pi : \mathbf{k}^{m-2} \times \mathbf{k}^2 \ni (x, y) \mapsto x \in \mathbf{k}^{m-2}$. Now we show that

$$F^*(\mathbf{k}(\overline{F(\mathbf{k}^n)})) = \mathbf{k}(X_1, \ldots, X_n).$$

Indeed we have that $F^*(\mathbf{k}(\overline{F(\mathbf{k}^n)})) = \mathbf{k}(G_1, \ldots, G_n, h, hl)$ and $l = \frac{1}{h}hl \in \mathbf{k}(G_1, \ldots, G_n, h, hl)$ so $\mathbf{k}(G_1, \ldots, G_n, l) \subset \mathbf{k}(G_1, \ldots, G_n, h, hl)$ but $\mathbf{k}(G_1, \ldots, G_n, l) = \mathbf{k}(X_1, \ldots, X_n)$. Now by theorem 4 gdeg $F = 1$ and so F is a birational mapping. $\square$

To see that if $n = m$ it is impossible to find a finite mapping $F : \mathbf{k}^n \to \mathbf{k}^m$ such that $F|_V = f$ and gdeg $F <$ gdeg f we recall the following

Theorem 6 (see e.g. [6] Th. II 5.6) *If $f : V \to W$ is finite and dominating, V and W are irreducible and W is normal then*

$$\# f^{-1}(y) \leq \text{gdeg } f \qquad \text{for } y \in W.$$

Now it is easy to prove the following

Proposition 7 *Let $V, W \subset \mathbf{k}^n$, be algebraic sets, $f : V \to W$ finite mapping and $F : \mathbf{k}^n \to \mathbf{k}^n$ finite mapping such that $F|_V = f$ then*

$$\text{gdeg } F \geq \text{gdeg } f.$$

References

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