Existence and regularity of global solutions to the complex Monge-Ampère equation.

by

Sławomir Kołodziej

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ABSTRACT. We prove some existence and regularity results for the complex Monge-Ampère equation in $\mathbb{C}^n$.

0. Introduction.

In this paper we address the question: for which nonnegative Borel measures $d\mu$ in $\mathbb{C}^n$ there exist entire plurisubharmonic functions solving the complex Monge-Ampère equation

$$(dd^c u)^n = d\mu \ ?$$

Furthermore, what additional assumptions guarantee that the solution is unique or smooth? Recall that if u is C^2 smooth then

$$(dd^c u)^n = c_n det \left(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} \right) \ d\lambda,$$

where $c_n = 4^n n!$ and $d\lambda$ denotes the Lebesgue measure. Moreover the positive current $(dd^c u)^n$ is well defined for any u which is plurisubharmonic and locally bounded as it was first shown in [BT1]. Fairly much is known about solvability of the Dirichlet problem for the complex Monge-Ampère equation in a strictly

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pseudoconvex domain in $\mathbb{C}^n$ (see [BT1], [CKNS], [C], [K1], [K2], [K3]) but the problem of global solutions has been treated only in some special cases. There are rather sharp sufficient conditions which guarantee that a measure μ having finite mass in $\mathbb{C}^n$ admits a solution of $(dd^c u)^n = d\mu$, which (after suitable normalization) has logarithmic growth ([K2], see also [CK]). The result is based on a generalized version of Yau's theorem [Y]. Uniqueness, up to an additive constant, of those solutions had been earlier shown by Bedford and Taylor [BT3]. In [J] Jeune proved that perturbing (the density of) the Lebesgue measure in $\mathbb{C}^n$ by a smooth function which, together with all its derivatives, tends to 0 fast enough at infinity one can solve the Monge-Ampère equation for this new measure. The solution is smooth and differs from the potential of the Euclidean metric by a bounded function. Tian and Yau [TY] solved the equation in a more general setting on some special non-compact complete Kähler manifolds. In this case, as well as in [J] and [Y], the attempt to solve the Monge-Ampère equation is motivated by geometrical problem of finding a Kähler metric with preassigned Ricci curvature.

In the present paper we develop an approach based on methods of the pluripotential theory. One can apply the above mentioned results on the Dirichlet problem for the Monge-Ampère equation to show the following existence theorem. Given two entire locally bounded plurisubharmonic functions $w \le v$ with $(dd^c v)^n \le (dd^c w)^n$ and $\lim_{|z|\to\infty}(v-w)=0$ we can solve the Monge-Ampère equation for any measure which is between the given ones:

$$(dd^c v)^n \le d\mu \le (dd^c w)^n,$$

and the solution u is unique among functions satisfying $w \le u \le v$. In particular, if one perturbs a rotation invariant (or briefely: *radial*) measure $d\nu$ in a suitable way then the equation is still solvable and the solution is close at infinity to the one of $(dd^c v)^n = d\nu$.

A simple observation that two functions: $u_1(z_1,z_2) = |z_1|^2 + |z_2|^2$ and $u_2(z_1,z_2) = 2|z_1|^2 + 2^{-1}|z_2|^2$ have the same Monge-Ampère mass (and their difference is not pluriharmonic) shows that to have uniqueness one has to specify the behaviour of the solution at infinity. Such extra conditions are also necessary when we require that the solution be smooth even when the right hand side has constant density with respect to the Lebesgue measure (see the example at the end of the paper).

In the second section we prove a general regularity result which is then tested on solutions of $(dd^c u)^n = f\ d\lambda$ which are close (in the sense of the above existence theorem) to $v(z) = |z|^{2\alpha},\ \alpha > 0$. In this case if $f > 0,\ f \in C^\infty$ and both $|\nabla f|$ and $|\Delta f|$ are bounded by $const.f$ then the solution is C^∞ smooth. For $\alpha = 1$ we thus get a result more general than the one proven in [J].

1. Existence.

First we prove the existence statement mentioned in the Introduction.

Theorem 1.1. *Suppose we have two locally bounded, entire plurisubharmonic functions v and w satisfying $w \le v$, $(dd^c v)^n \le (dd^c w)^n$ and $\lim_{|z|\to\infty}(v-w) = 0$. Then for any nonnegative Borel measure $d\mu$ satisfying*

$$(dd^c v)^n \le d\mu \le (dd^c w)^n$$

there exists a unique $u \in PSH(\mathbb{C}^n)$ *such that*

$$(dd^c u)^n = d\mu$$

and $w \leq u \leq v$. *Furthermore, if* $d\mu = f\, d\lambda$ *with* $f \in L^p_{loc}(\mathbb{C}^n), p > 1$, *then* u *is continuous.*

Proof. We begin with a lemma which is essentially Theorem A from [K4].

Lemma 1.2. *Let* Ω *be a strictly pseudoconvex domain and* w *a bounded plurisubharmonic function in* Ω. *Then for any nonnegative Borel measure* $d\mu$ *in* Ω *satisfying* $d\mu \leq (dd^c w)^n$ *one can find* $u \in PSH \cap L^\infty(\Omega)$ *such that*

$$(dd^c u)^n = d\mu,$$

$u \geq w$ *and* $\limsup_{\eta \to z} u(\eta) \limsup_{\eta \to z} w(\eta)$ *for every* $z \in \partial\Omega$.

Proof of the lemma. Fix a sequence φ_k of continuous functions on $\partial\Omega$ decreasing to (the restriction of) w and a sequence χ_k of functions defined in Ω such that $0 \leq \chi_k \leq 1$, $E_k := supp \chi_k \subset\subset \Omega$ and $\chi_k \uparrow 1$. For every k choose $\rho_k \in PSH(\Omega) \cap C(\bar{\Omega})$ with $\rho_k = \varphi_k$ on $\partial\Omega$ and $\rho_k \leq w$ on E_k.

If we denote by w_k the plurisubharmonic function $\max(w, \rho_k)$ then $(dd^c w_k)^n \geq \chi_k d\mu$ and thus by [K4] one can solve the following Dirichlet problem:

$$\begin{aligned} &u_k \in PSH(\Omega) \cap L^\infty(\Omega) \\ &(dd^c u_k)^n = \chi_k d\mu \\ &\lim_{\eta \to z} u_k(\eta) = \varphi_k(z) \text{ on } \partial\Omega, \end{aligned}$$

The sequence u_k is decreasing due to the comparison principle [BT2]. Thus the convergence theorem [BT2] implies that $u = \lim u_k$ satisfies $(dd^c u)^n = d\mu$. Since by the comparison principle $u_k \geq w_k = w$ on E_k we get $u \geq w$. On the other hand $\limsup_{\eta \to z} u(\eta) \leq \limsup_{\eta \to z} u_k(\eta) = \varphi_k(z)$ for every k which proves the boundary condition is fulfilled. The proof of the lemma is thus finished.

Continuing the proof of Theorem 1.1 we use the lemma to solve the following Dirichlet problem:

$$\begin{aligned} &u_j \in PSH(B(j)) \cap L^\infty(B(j)) \\ &(dd^c u_j)^n = d\mu \\ &\limsup_{\eta \to z} u_j(\eta) = w(z) \text{ on } \partial B(j), \end{aligned} \tag{1.1}$$

where $B(r)$ denotes the ball $\{|z| < r\}$.

From the comparison principle it follows that $w \leq u_j \leq v$ thereby $\{u_j\}$ is locally bounded. Since $u_{j+1} \geq w = u_j$ on $\partial B(j)$ the inequality between u_{j+1} and u_j extends to $B(j)$ by the comparison principle again. Thus the sequence is also locally increasing. Using convergence theorem [BT2] we conclude that $u = \lim u_j$ is a plurisubharmonic solution of the equation

$$(dd^c u)^n = d\mu$$

with $\lim_{|z|\to\infty}(u-v)=0$ and $w\le u\le v$.

Uniqueness follows easily from the comparison principle since if u' is another such solution then for every j

$$\sup_{B(j)}|u-u'| = \sup_{\partial B(j)}|u-u'|,$$

and the left hand side tends to 0 as $j\to\infty$.

It remains to show that u is continuous under the extra assumption. For this we need a lemma from [K2, Lemma 2.3.1]. Let us recall the definition of the relative capacity introduced in [BT2]:

$$cap(K,\Omega)=\sup\{\int_K (dd^c u)^n; u\in PSH(\Omega), -1<u<0\},$$

for compact K in Ω. This set function extends to open sets as inner capacity.

Lemma 1.3. *Let Ω be a strictly pseudoconvex subset of $\mathbb{C}^n$ and let w, v belong to PSH(Ω). Assume that $\emptyset\ne\mathcal{U}_c := \{w<v-c\}\subset\Omega'\subset\Omega$ and*

$$\int_{\mathcal{U}_c}(dd^c w)^n \le F(cap(\mathcal{U}_c,\Omega)), \tag{1.2}$$

for $F(x) = Ax(\log(1+x^{-1}))^{-(n+\epsilon)}$, with $A,\epsilon>0$, and any $c\in(0,C)$. Then $C<\kappa=\kappa(A,||v||,\epsilon,\Omega',\Omega)$ (the estimate is independent of w and v as long as $||v||$ is under control). Moreover, $\kappa\to 0$ as $cap(\mathcal{U}_0,B(R))\to 0$ when $A,\epsilon,\Omega',\Omega$ are fixed.

If $(dd^c w)^n$ belongs to $L^p(\Omega), p>1$, then the inequality (1.2) is satisfied (see [K1]).

Now, suppose u_j were not locally uniformly convergent. Then for a bounded sequence z_j of points in $\mathbb{C}^n$ and a positive constant C we would have

$$u_j(z_j) > u(z_j) + C.$$

Having $\lim_{|z|\to\infty}(v-u)(z)=0$ one can fix $R>\sup_j|z_j|$ such that $v-u<C/2$ on $\mathbb{C}^n\setminus B(R)$. Then the sets

$$\mathcal{U}_j(c) := \{u_j > u+c\},\quad c\in(C/2,C)$$

are non-empty and relatively compact in $B(R)$. Since the sequence u_j is decreasing one can apply [BT2] (see [K2, Corollary 1.2.10]) to conclude that

$$cap((\mathcal{U}_j(C/2),B(R))\to 0$$

as $j\to\infty$, which contradicts the lemma above. The proof of Theorem 1.1 is thus complete.

Suppose we have a radial entire plurisubharmonic function v with $(dd^c v)^n = g\ d\lambda,\ g\in L^1_{loc}$ and a radial nonnegative function h. Set

$$G(r)=\int_{B(r)} g\ d\lambda \text{ and } H(r)=\int_{B(r)} h\ d\lambda.$$

Using this notation we can state the following corollary to Theorem 1.1.

Corollary 1.4. *If $f \in L^p_{loc}(\mathbb{C}^n)$ satisfies $g \le f \le g+h$ and*

$$\int_1^\infty \frac{H(t)}{tG^{(n-1)/n}(t)}\, dt < \infty.$$

then the equation

$$(dd^c u)^n = f\ d\lambda$$

has a unique solution such that $\lim_{|z|\to\infty}(u-v) = 0$.

Proof. In view of Theorem 1.1 it is enough to show that there exists a radial plurisubharmonic solution w of

$$(dd^c w)^n = (g+h)\ d\lambda$$

with $\lim_{|z|\to\infty}(v-w) = 0$.

Following [M], one can compute w when $(dd^c w)^n$ is given. For a radial function g we shall use the notation: $\tilde{g}(t) := g(t,0,0,...,0)$.

$$\frac{d}{dt}\tilde{w}(t) = \frac{2}{t}[2nc_n^{-1}\int_0^t x^{2n-1}(\tilde{g}(x)+\tilde{h}(x))\,dx]^{1/n}.$$

Use this formula to estimate $\tilde{v}(t) - \tilde{w}(t)$.

$$\begin{aligned}\frac{d}{dt}[(\tilde{w}-\tilde{v})(t)] &= \frac{2}{t}(2nc_n^{-1})^{1/n}\{[\int_0^t x^{2n-1}(\tilde{g}(x)+\tilde{h}(x))\,dx]^{1/n}\\ &-[\int_0^t x^{2n-1}\tilde{g}(x)\,dx]^{1/n}\} \le \frac{const.}{t}[(G(t)+H(t))^{1/n} - G^{1/n}(t)]\\ &\le \frac{const.}{t}[\frac{H(t)}{nG^{(n-1)/n}(t)}].\end{aligned}$$

To get the last inequality write $(a-b)/(a^{1/n}-b^{1/n})$ as $\sum a^{k/n}b^{(n-k-1)/n}$.

Integrating both sides of the above formula one gets the result.

Example 1.5. *Let us consider the family of radial functions*

$$g_\alpha(z) := |z|^{2n\alpha},\ \alpha \ge 0.$$

A simple computation yields that the corresponding solution of the Monge-Ampère equation is

$$v_\alpha(z) = c_n(\alpha)|z|^{2(\alpha+1)}$$

with $c_n(\alpha) = c_n^{-1}(\alpha+1)^{-1-1/n}$. Then

$$h_\alpha(z) = (c_0 + c_1|z|^2)^{n\alpha-\alpha-1-\epsilon}, \quad \epsilon > 0,$$

fulfills the hypothesis of Corollary 1.4 and so for f satisfying $g_\alpha \le f \le g_\alpha + h_\alpha$ we can solve the equation

$$(dd^c u)^n = f\ d\lambda$$

with $\lim_{|z|\to\infty}(v_\alpha - u) = 0$. In particular, if $\alpha = 0$ one can take $h_0(z) = const.(1+|z|)^{-2-\epsilon}$, $\epsilon > 0$.

2. Regularity.

In this section we address the question of regularity of solutions obtained in Theorem 1.1 when the right hand side of the equation is sufficiently smooth. We shall employ a method similar to the one applied in [K4] to show the regularity of entire solutions having logarithmic growth. This approach, however, requires variuos adjustments as the growth of functions under consideration changes.

We shall stick to the notation of previous sections. Furtheremore, for any real function f we shall denote throughout, by f_ϵ the convolution of f with the characteristic function of the ball $B(\epsilon)$ multiplied by $\sigma_n := (\int_{B(\epsilon)} d\lambda)^{-1}$.

$$f_\epsilon(z) = \sigma_n \int_{B(\epsilon)} f(z+w)\, d\lambda(w).$$

It is well known (see [BT1]) that the operator

$$T_\epsilon f := 4(n+1)\epsilon^{-2}(f_\epsilon - f) \tag{2.1}$$

tends to the Laplacian of f when $\epsilon \to 0$ in the sense of distributions, and for $f \in C^2$ we have locally uniform convergence.

We are going to prove second order a priori estimates, letting the existing theory of elliptic fully nonlinear equations to take care of higher order regularity. Let us begin with the statement of three propositions.

Proposition 2.1. *If $u, v \in PSH \cap C^2(\Omega)$ and $(dd^c u)^n = f\, d\lambda, (dd^c v)^n = g\, d\lambda$, then*

$$(dd^c u)^k \wedge (dd^c v)^{n-k} \geq f^{k/n} g^{(n-k)/n}$$

at any point of Ω.

The inequalities are obtained by simultanuous diagonalization of both forms (which is possible due to the fact that we are dealing with positive forms) and the application of the inequality between geometric and arithmetic means. For a more general version and details see [D].

For the proof of the next proposition we refer to [K4].

Proposition 2.2. *With the notation of Proposition 2.1 one can find a constant $c(n)$ depending only on the dimension of the space and such that*

$$(dd^c u)^{n-1} \wedge dd^c v \geq c(n) \left(\frac{\Delta u}{\Delta v}\right)^{1/(n-1)} \left(\frac{g}{f}\right)^{1/(n-1)} f d\lambda.$$

Proposition 2.3. *If $(dd^c u_\epsilon)^n = f_{(\epsilon)}\, d\lambda$, then $f_{(\epsilon)} \geq [(f^{1/n})_\epsilon]^n$. Similary, if for a fixed unit vector γ we define $u_{[\epsilon]}(x) := (1/2)[u(x+\epsilon\gamma) + u(x-\epsilon\gamma)]$, and $(dd^c u_{[\epsilon]})^n = f_{[\epsilon]}\, d\lambda$ then $f_{[\epsilon]} \geq [(f^{1/n})_{[\epsilon]}]^n$.*

The first statement comes from [BT1, Theorem 5.7] and the other follows directly from Proposition 2.1 above. It is enough to write $(dd^c u_{[\epsilon]})^n$ as

$$(1/2^n) \sum \binom{n}{k} (dd^c u(x+\epsilon\gamma))^k \wedge (dd^c(u(x-\epsilon\gamma))^{n-k}$$

and apply the proposition to every term of this sum.

Next, we shall prove some a priori estimates for plurisubharmonic functions which are close to $v(z) = |z|^2$ at infinity.

Lemma 2.4. *Let $v, w \in PSH \cap C^2(\mathbb{C}^n)$ and $(dd^c v)^n = g\ d\lambda$, $(dd^c w)^n = h\ d\lambda$, with $w \leq v, \lim_{|z|\to\infty}(v - w) = 0$ and let $f \in C(\mathbb{C}^n)$ be a function such that $g \leq f \leq h$. Suppose $\Delta v \leq 4(n+1)C_0$, and assume that there exists a nonnegative bounded function ψ such that for any positive $\epsilon < \epsilon_0$ we have*

$$T_\epsilon f^{1/n} \leq K\psi f^{1/n}. \tag{2.2}$$

and

$$g(1 + K_1\psi)^n \leq h \tag{2.3}$$

for some positive K, K_1. Then the solution u of the equation

$$(dd^c u)^n = f\ d\lambda,$$

given by Theorem 1.1, satisfies the inequality

$$\Delta u \leq 4(n+1)[\frac{2}{K}(u - w) + C_0].$$

Furthermore, if

$$\frac{f^{1/n} - (f^{1/n})_{[\epsilon]}}{\epsilon^2 f^{1/n}} \leq \psi, \tag{2.4}$$

where $f_{[\epsilon]}$ is defined as in Proposition 2.3 then

$$\frac{\partial^2 u}{\partial \gamma^2} \leq 4(n+1)[\frac{2}{K}(u - w) + C_0].$$

In particular, if $f > 0$, $f \in C^\infty$ satisfies (2.3) - (2.4) then $u \in C^\infty$.

Proof. We shall prove the estimate for the Laplacian, the proof of the other one being analogous. In order to show that

$$\Delta u \leq 4(n+1)[c(u - w) + C_0],$$

it is enough to prove that for sufficiently small ϵ the set

$$E(\epsilon, c) := \{u < u_\epsilon - \epsilon^2[c(u - w) + C_0]\}$$

is empty (see (2.1)). In any case, it is relatively compact in $\mathbb{C}^n$ for $c > 0$ since by Theorem 1.1 the function $u - w$ tends to 0 at infinity, and, by the assumption on Δv, we have $u_\epsilon \leq v_\epsilon \leq v + C_0\epsilon^2$. Applying the comparison principle one gets

$$\begin{aligned} \int_{E(\epsilon,c)} \sum_0^n \binom{n}{k} c^k \epsilon^{2k} (dd^c w)^k \wedge (dd^c u_\epsilon)^{n-k} &= \int_{E(\epsilon,c)} [dd^c(u_\epsilon + c\epsilon^2 w)]^n \\ \leq \int_{E(\epsilon,c)} (1 + c\epsilon^2)^n (dd^c u)^n &= \int_{E(\epsilon,c)} \sum_0^n \binom{n}{k} c^k \epsilon^{2k} f\ d\lambda. \end{aligned} \tag{2.5}$$

Using the assumptions (2.2), (2.3), Proposition 2.1 and Proposition 2.3 we can estimate for $k>0$ and $\epsilon<\epsilon_1<\epsilon_0$, with ϵ_1 small enough:

$$
\begin{aligned}
(dd^c w)^k\wedge(dd^c u_\epsilon)^{n-k} &\geq h^{k/n} f_{(\epsilon)}^{(n-k)/n} \geq (1+K_1\psi)^k(1-\epsilon^2K\psi)f \\
&\geq [1+(3K/4)\psi]g.
\end{aligned} \tag{2.6}
$$

Let us plug this into (2.5) and reshuffle the terms, putting those "ϵ-free" on the right hand side and discarding those containing $\epsilon^k, k>2$. We thus obtain

$$
nc\epsilon^2\int_{E(\epsilon,c)}(K/2)\psi f\ d\lambda \leq \int_{E(\epsilon,c)}(f-f_{(\epsilon)})\ d\lambda.
$$

Since by (2.2) and Proposition 2.3 the right hand side of this inequality is dominated by the integral of $n\epsilon^2\psi f$ a contradiction would follow for $c>2/K$ unless the set $E(\epsilon,c)$ was empty. Thus the estimate for the Laplacian follows.

If the hypothesis (2.4) is fulfilled one can repeat the above argument with $u_{[\epsilon]}$ in place of u_ϵ and apply the other part of Proposition 2.3.

It remains to show the last statement of the lemma. By the preceding part of the lemma pure (real) second order derivatives of u are locally bounded. Hence, applying the argument from [BT1] (see also [K]) one obtains $u\in C^{1,1}$. By Trudinger's theorem [T, Theorem 3.1] we thus have $u\in C^{2,\alpha}$ for some $\alpha>0$. The Schauder estimates complete the proof.

Lemma 2.4 can be applied to show that a solution of the Monge-Ampère equation is smooth provided (among other things) that it is bounded from above by $const.(|z|^2+1)$. The next result works also for functions growing faster at infinity, but in the course of its proof we shall use a priori estimates for solutions which are known to be C^2 smooth and those will be provided by an application of Lemma 2.4. So, first we need to modify our data to meet the hypothesis of Lemma 2.4 and prove a priori estimates for a sequence of smooth functions which, hopefully, approximate the solution we are interested in.

We shall say that the function u is *well approximated* if the following is true:

There exists a sequence of smooth functions f_j such that f_j-c_n have compact support, $f_j\to f$, locally uniformly for f satisfying $(dd^cu)^n=f\ d\lambda$. Moreover, the sequence of solutions of

$$
(dd^cu_j)^n=f_j\,d\lambda,
$$

provided by Theorem 1.1 (in which we take $v(z)=|z|^2$) tends to u locally uniformly.

Note that every radial function is well approximated. This follows from the formula for the solution when the Monge-Ampère mass is given which we used in the proof of Corollary 1.4. For f_j one can take any continuous radial function equal to f on $B(j)$ and equal to c_n outside $B(2j)$. Then normalizing u_j so that $u_j(0)=u(0)$ we get $u_j=u$ on $B(j)$.

Theorem 2.5. *Let u be a well approximated function satisfying*

$$
(dd^cu)^n=f\ d\lambda,
$$

where $f > 0, f \in C(\mathbb{C}^n)$. We define f_j (resp. u_j) as in the above definition and assume that

$$-\Delta f_j^{1/n} \leq K f_j^{1/n}. \tag{2.7}$$

Finally we assume that there exists a sequence of C^2 smooth entire plurisubharmonic functions φ_j converging locally uniformly to φ and satisfying the following properties

$$\varphi_j \leq u_j, \tag{2.8}$$

$$\Delta \varphi_j \leq K_1(u_j - \varphi_j), \tag{2.9}$$

$$\lim_{|z|\to\infty} \sup \ \Delta \varphi_j(z) \leq 4n, \tag{2.10}$$

$$dd^c \varphi_j \geq K_2 |dd^c T_\epsilon \varphi_j|, \quad 0 < \epsilon < \epsilon_1, \tag{2.11}$$

$$(dd^c \varphi_j)^n \geq K_3 f_j \ d\lambda, \tag{2.12}$$

where $\epsilon_1, K, K_1, K_2, K_3$ are independent of j.

Then for some positive constants c, c_1:

$$\Delta u \leq c(u - \varphi) + c_1.$$

Proof. We fix an integer j and write, for the sake of brevity, U, W, Φ instead of u_j, w_j, φ_j respectively. Set

$$\Psi_\epsilon := \Phi - \frac{1}{c} T_\epsilon \Phi$$

for $\epsilon \in (0, \epsilon_1)$ and $c > C > 1$ with C so large that $\Psi_\epsilon \in PSH(\mathbb{C}^n)$ and

$$dd^c \Phi \geq n! |\frac{1}{c} dd^c T_\epsilon \Phi|, \quad dd^c \Psi_\epsilon > 0, \tag{2.13}$$

(see (2.11)).

We shall use the comparison principle for the pair of functions

$$F_1 := (1 + c\epsilon^2)U, \quad F_2 := U_\epsilon + c\epsilon^2 \Psi_\epsilon - c_0,$$

$c_0 > 4n$, and the set

$$\Omega = \Omega_\epsilon = \{F_1 < F_2\} = \{c(U - \Phi) + c_0 < T_\epsilon(U - \Phi)\}.$$

Let us first observe that Ω is bounded. Indeed, by Lemma 2.4

$$\lim_{|z|\to\infty} \Delta U(z) \leq 4n < c_0$$

and so, via Jensen formula (see e.g. [BT1]),

$$T_\epsilon U < c_0$$

away from a ball, say $B(R)$. Hence $\Omega_\epsilon \subset B(R)$ for $\epsilon \in (0, \epsilon_1)$. Since both ΔU an $\Delta\Phi$ are locally bounded one can use (2.7) to find $\epsilon_0 \in (0, \frac{1}{cn!}), \epsilon_0 < \epsilon_1$ such that for all $\epsilon < \epsilon_0$ the following inequalities hold on $B(R)$:

$$|\Delta(U_\epsilon - U)| < 1, \tag{2.14}$$

$$|T_\epsilon U - \Delta U| < 1, \tag{2.15}$$

$$|T_\epsilon \Phi - \Delta\Phi| < 1, \tag{2.16}$$

$$-T_\epsilon f_j^{1/n} \leq 2K f_j^{1/n}. \tag{2.17}$$

Suppose

$$\Omega' := \{c(U - \Phi) + c_0 + 2 < \Delta(U - \Phi)\} \neq \emptyset.$$

Then also $\Omega \neq \emptyset$ (see (2.15), (2.16)). By the comparison principle

$$\int_\Omega (dd^c F_2)^n \leq \int_\Omega (dd^c F_1)^n.$$

Note that $(dd^c F_2)^n \geq c^n \epsilon^{2n} (dd^c \Psi_\epsilon)^n > 0$, so if $(dd^c U_\epsilon)^n \geq 2(dd^c U)^n$ then $(dd^c F_2)^n > (dd^c F_1)^n$. Hence

$$\int_{\Omega_1} (dd^c F_2)^n \leq \int_{\Omega_1} (dd^c F_1)^n$$

where

$$\Omega_1 := \Omega \cap \{(dd^c U_\epsilon)^n < 2(dd^c U)^n\} \neq \emptyset. \tag{2.18}$$

Furthermore, since we deal with positive forms and $\sum_2^n \binom{n}{k} \epsilon^{2k} < \epsilon^2$ for $\epsilon < \epsilon_0$, we get from the last inequality

$$\int_{\Omega_1} (dd^c U_\epsilon)^n + nc\epsilon^2 dd^c \Psi_\epsilon \wedge (dd^c u_\epsilon)^{n-1} \leq (1 + 2nc\epsilon^2) \int_{\Omega_1} (dd^c U)^n.$$

Hence, by Proposition 2.3 and (2.17) one obtains

$$\begin{aligned} nc\epsilon^2 \int_{\Omega_1} [dd^c \Psi_\epsilon \wedge (dd^c u_\epsilon)^{n-1} - 2f\, d\lambda] &\leq \int_{\Omega_1} (f - f_{(\epsilon)})\, d\lambda \\ = \int_{\Omega_1} (f^{1/n} - f_{(\epsilon)}^{1/n}) (\sum_{k=0}^{n-1} f^{\frac{n-1-k}{n}} f_{(\epsilon)}^{\frac{k}{n}}) &\leq 2K\epsilon^2 \int_{\Omega_1} f\, d\lambda, \end{aligned}$$

and further, after regrouping the terms

$$\int_{\Omega_1} dd^c\Psi_\epsilon \wedge (dd^c u_\epsilon)^{n-1} \leq 2\left((\frac{K}{nc}+1\right) \int_{\Omega_1} f\ d\lambda.$$

To estimate the left hand side of this inequality we make use of Proposition 2.2 and get

$$c(n)\int_{\Omega_1} \left(\frac{\Delta U_\epsilon}{\Delta \Psi_\epsilon} \frac{(dd^c\Psi_\epsilon)^n}{(dd^c U_\epsilon)^n} \right)^{\frac{1}{n-1}} f\ d\lambda \leq 2\left(\frac{K}{nc}+1\right) \int_{\Omega_1} f\ d\lambda. \tag{2.19}$$

Since $(dd^c U_\epsilon)^n < 2(dd^c U)^n$ on Ω_1 we get using (2.12) and (2.13),

$$(dd^c\Psi_\epsilon)^n \geq \frac{1}{2}(dd^c\Phi)^n \geq \frac{K_3}{2}(dd^c U)^n \geq \frac{K_3}{4}(dd^c U_\epsilon)^n.$$

From (2.9) and (2.13) one infers

$$\Delta\Psi_\epsilon \leq 2\Delta\Phi \leq 2K_1(U-\Phi).$$

Finally (2.14) and (2.15) imply that in Ω we have

$$\Delta U_\epsilon \geq \Delta U - 1 \geq T_\epsilon U - 2 \geq T_\epsilon \Phi + c(U-\Phi) + c_0 - 2 \geq c(U-\Phi).$$

Combining the last three chains of inequalities one can estimate from below the integrand on the left hand side of (2.19) by

$$\left(\frac{cK_3}{8K_1}\right)^{\frac{1}{n-1}} f$$

and conclude, in view of (2.18), that for

$$c > \frac{8K_1}{K_3}\left[\frac{2}{c(n)}(K+1)\right]^{n-1}$$

the sets $\Omega_1, \Omega, \Omega'$ must be empty. From locally uniform convergence of plurisubharmonic functions $u_j \to u$ it follows that $\Delta u_j \to \Delta u$ in the weak* topology. Thus $\Omega' = \emptyset$ implies the estimate from the statement. The proof is completed.

Corollary 2.6. *If f in Theorem 2.5 is C^∞ smooth then so is the solution u.*

Proof. Locally, a bound for the Laplacian of a plurisubharmonic function gives also a bound of mixed second order derivatives. This allows us to apply [RS, Theorem 1] and get locally bounded mixed third order derivatives. Thus, in particular, $\Delta(\partial u/\partial z_j)$ is bounded for every j. Therefore, by the regularity theory of the Poisson equation (see [GT, Theorem 8.34]) $\partial u/\partial z_j \in C^{1,\alpha}, \alpha \in (0,1)$, and so $u \in C^{2,\alpha}$. Finally, we use Schauder estimates to complete the proof.

To see what Theorem 2.5 is worth let us return to Example 1.5.

Example 2.7. *Let f, g_α and h_α be as in Example 1.5. Suppose further that $f > 0, f \in C^\infty$ fulfills*

$$|\nabla f| \leq const. f^{1/n}, \quad -\Delta f^{1/n} \leq const. f^{1/n}.$$

Then the solution u of $(dd^c u)^n = f\ d\lambda$ is also C^∞ smooth.

Proof. We need to verify the hypothesis of Theorem 2.5. We shall write *const.* or *const.'* for a constant independent of $z \in \mathbb{C}^n$ and indices. It may be different in different places. Furthermore, $t = |z|$ whenever t and z appear in the same formula.

Choose smooth functions $\chi_j : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\begin{aligned} &\chi_j(t) = (\alpha+1)^{-1} t^{\alpha+1} \text{ for } t < j, \quad \chi_j(t) = c_n^{1/n} t \text{ for } t > t_j, \\ &|t^k \chi_j^{(k+1)}(t)| \leq const. \chi_j'(t) \leq const. (1+t)^{-1} \chi_j(t), \ k = 1, 2; \\ &\chi_j'(t) + t\chi_j''(t) \geq const. \chi_j'(t). \end{aligned} \tag{2.20}$$

Such a choice is possible provided t_j are large enough. Set

$$\begin{aligned} &\psi_j(t) = \chi_j'(t^{1/\alpha}) \\ &g_j^{1/n} = \psi_j(g_\alpha^{1/n}), \quad f_j^{1/n} = \psi_j(f^{1/n}), \quad \eta_j^{1/n} = \psi_j((g_\alpha + h_\alpha)^{1/n}). \end{aligned}$$

The radial functions g_α and $g_\alpha + h_\alpha$ are well approximated, and so is f, since $g_j \leq f_j \leq \eta_j$. The auxiliary functions φ_j from Theorem 2.5 are defined by

$$\varphi_j(z) = c\tilde{\varphi}_j(t) - c_0, \quad \text{where} \quad \tilde{\varphi}_j(t) = \chi_j(1+t^2),$$

with $c, c_0 > 0$ to be chosen. For a radial function $v(z) = \tilde{v}(|z|)$ its Laplacian and the Monge-Ampère mass are given by the formulae

$$\begin{aligned} \Delta v(z) &= t^{1-2n} \frac{d}{dt}(t^{2n-1} \tilde{v}'(t)), \\ Mv(z) &= \frac{c_n}{4} \left[\frac{\tilde{v}'(t)}{2t}\right]^{n-1} \left[\tilde{v}''(t) + \frac{\tilde{v}'(t)}{t}\right] \text{ with } (dd^c v)^n(z) = Mv(z)\ d\lambda. \end{aligned}$$

Applying this to φ_j one gets

$$\begin{aligned} \Delta\varphi_j(z) &= const. [2n\chi_j'(1+t^2) + t^2\chi_j''(1+t^2)], \\ M\varphi_j(z) &= const. [\chi_j'(1+t^2)]^{n-1} [\chi_j'(1+t^2) + t^2\chi_j''(1+t^2)]. \end{aligned} \tag{2.21}$$

Hence in view of (2.20)

$$const. c(\chi_j'(1+t^2))^n \leq M\varphi_j(z) \leq const.' c(\chi_j'(1+t^2))^n. \tag{2.22}$$

Now, since

$$Mv_j(z) = (\psi_j(g^{1/n}))^n = (\chi_j'(g^{1/n\alpha}))^n = (\chi_j'(t^2))^n.$$

by chosing c small enough we get

$$2^n M\varphi_j(z) \le Mv_j(z) \le Mw_j(z), \quad \text{for} \quad |z| \ge const. > 0.$$

All functions are radial, so taking c_0 large enough we end up with

$$2\varphi_j \le w_j \le u_j, \tag{2.23}$$

which verifies (2.8). The inequality 2.12 also follows from (2.22) since

$$const. f_j(z) \le g_j(z) = (\chi_j'(t^2))^n \le const.' M\varphi_j(z).$$

To obtain (2.9) we use (2.20), (2.21) and (2.23):

$$\begin{aligned}\Delta\varphi_j(z) &\le const.\chi_j'(1+t^2) \le const.(1+t^2)^{-1}\chi_j(1+t^2)\\ &\le const.\varphi_j(z) \le const.(u_j - \varphi_j)(z).\end{aligned}$$

Then, (2.10) follows from Lemma 2.4. Next we are going to verify (2.11). Having $\Delta\varphi_j$ given by (2.21) one can use Jensen's formula to compute $T_\epsilon\varphi_j$ and then calculate the second order derivatives of the latter function (we omit this tedious calculation). These derivatives can be estimated by applying (2.20):

$$|dd^c T_\epsilon \varphi_j(z)| \le const.(1+t^2)^{-1}\chi_j'(1+t^2). \tag{2.24}$$

On the other hand, by (2.20) and (2.21) we obtain

$$M\varphi_j(z) \ge const.[\chi_j'(1+t^2)]^n \ge const.'(\Delta\varphi_j(z))^n.$$

Since $M\varphi_j(z)$ is, up to a constant, the product of eigenvalues of the complex Hessian of φ_j at z, the last inequality says that the ratio of the largest and the smallest eigenvalues is bounded from above by a constant not depending on z nor j. Thus

$$dd^c\varphi_j(z) \ge const.\chi_j'(1+t^2)dd^c|z|^2,$$

which combined with (2.24) gives (2.11). In order to apply Theorem 2.5 we need yet to check (2.7). Let us compute

$$\Delta f_j^{1/n} = \psi_j''(f^{1/n})|\nabla f^{1/n}|^2 + \psi_j'(f^{1/n})\Delta f^{1/n}.$$

Since $\psi_j(t) = \chi_j'(t^{1/\alpha})$ we can estimate using (2.20)

$$|\psi_j''(f^{1/n})| \le const.\chi_j'(f^{1/n\alpha})f^{-2/n}, \ \psi_j'(f^{1/n}) \le const.\chi_j'(f^{1/n\alpha})f^{-1/n}.$$

From our assumptions on f we get

$$|\nabla f^{1/n}| \le const.f^{1/n}, \ \ -\Delta f^{1/n} \le const.f^{1/n}.$$

Combining the last four inequalities we obtain (2.7). Thus, we may conclude that our solution is indeed smooth.

Remark. *Example 1.5 and Example 2.7 (for $\alpha = 0$) generalize the result of Jeune [J], where the derivatives are required to satisfy*

$$|D^k f(z)| \le (1+|z|)^{-2-k-\epsilon}, \quad k = 0, 1, 2, \dots, \epsilon > 0.$$

Recently Chou and Wang [CW] solved the real Monge-Ampère equation with any function on the right hand side which is bounded from above and from below by two positive constants. So far we have no complex analogue of this existence result but using Theorem 2.5 one can show a relevent regularity statement. Its proof is just a special case ($\alpha = 0$) of the proof from Example 2.7.

Example 2.8. *Suppose $f \in C^\infty$, for some positive c we have*

$$0 < c^{-1} < f < c, \quad |\nabla f| < c, \quad |\Delta f| < c,$$

and there exists a solution of

$$(dd^c u)^n = f \ d\lambda$$

which satisfies

$$c|z|^2 \le u(z) \le c'|z|^2, \quad c, c' > 0. \tag{2.25}$$

Then $u \in C^\infty$.

It is worth noting that if one drops the assumption (2.25) this statement is no longer true even for $f = const. > 0$. A counterexample, borrowed from [BL] is the following function

$$u(z_1, z_2) = |z_1|(1+|z_2|^2)$$

which has Monge-Ampère mass equal to $const.\, d\lambda$ but is not C^1 smooth.

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postal address:
Jagiellonian University, Institute of Mathematics
Reymonta 4, 30-059 Kraków, Poland
e-mail: kolodzie@ im.uj.edu.pl