A GEOMETRY OF REAL POLYNOMIAL MAPPINGS OF THE PLANE AND THE PINCHUK EXAMPLE*)

by Janusz Gwoździewicz and Zbigniew Jelonek

IMUJ PREPRINT 1998/16 KRAKÓW

1 Introduction

Let $f : X \to Y$ be a continuous map of locally compact spaces. We say that the mapping f is not proper at a point $y \in Y$, if there is no a neighborhood U of a point y such that the set $f^{-1}(cl(U))$ is compact.

The set S_f of points at which the map f is not proper indicates how the map f differs from a proper map. In particular f is proper if and only if this set is empty, moreover if $f(X)$ is open then S_f contains the border of the set $f(X)$. In the previous paper [3] we described this set in the case of complex polynomial mappings of $\mathbb{C}^n$.

The aim of this paper is to do the same in the case of real polynomial dominant mappings $f : \mathbb{R}^2 \to \mathbb{R}^2$. Our main result is that in this case the set S_f is closed, semi-algebraic and (if non-empty) of pure dimension one. Moreover, the set S_f is a union of finite number (possibly 0) of parametric semi-lines (a parametric semi-line is a semi-algebraic closed curve $\Gamma \subset \mathbb{R}^2$, for which there exists a polynomial surjective mapping $\phi : \mathbb{R} \to \Gamma$). In particular every connected component of S_f is unbounded. Consequently, every connected component Q of the set $\mathbb{R}^2 \setminus S_f$ is simply connected.

In particular these results gives a full geometric picture for polynomial mappings $f : \mathbb{R}^2 \to \mathbb{R}^2$ with a non-zero Jacobian (e.g. for the Pinchuk mapping). Indeed, let $S_1, ..., S_r$ be all connected components of the set $\mathbb{R}^2 \setminus S_f$ with non-empty preimages. Take $f^{-1}(S_i) = \bigcup_{j=1}^{r_i} S_{i,j}$, where $S_{i,j}$ are connected components of the set $f^{-1}(S_i)$. Then mappings $res_{S_{i,j}} f : S_{i,j} \to S_i$ are diffeomorphisms, and consequently the mapping f is a glueing of finite number of diffeomorphisms.

As an illustration we give a detailed description of the Pinchuk mapping. We show that in this case is $r = 2$ and $r_1 = r_2 = 2$. Moreover, we show that the set S_f is an irreducible algebraic (and of course parametric) curve and we give the exact equation of this curve (and we give its picture).

The second author is supported by the grant of KBN number 2 P03A 037 14.

This shows, that the topology of the Pinchuk example is quite different from that which is possible in the complex case. In particular our results suggest that the real counterpart of the complex Jacobian Conjecture has sense only in dimension $n > 2$ and it should be following:

Real Jacobian Conjecture. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a real polynomial mapping with the Jacobian which nowhere vanishes. If* codim $S_f \geq 2$ *then f is one to one.*

2 Preliminaries.

In the beginning we recall some basic facts about polynomial mappings of $\mathbb{C}^2$. We start with the theorem about surfaces which contain $\mathbb{C}^2$ as an open subset ([3]):

Theorem 2.1 *Let X be a complete normal surface. Assume, that X contains a plane $\mathbb{C}^2$ as an open, dense subset. Let $W_1, W_2 \subset X \setminus \mathbb{C}^2$ be two connected, complete curves without common components. Then the intersection $W_1 \cap W_2$ is either the empty set or it is a point.*

Definition 2.2 Let Γ be an affine curve such that there is a surjective polynomial mapping $\phi : \mathbb{C} \to \Gamma$. Then Γ is called to be *an affine parametric line* and the mapping ϕ is called to be *a parametrization of* Γ.

The next result was proved in our papers [2], [3]:

Theorem 2.3 *Let $f = (f_1, f_2) : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ be a dominant polynomial map. Then the set S_f of points at which f is not proper consists of a finite number (possibly 0) of affine parametric lines. Moreover, its degree is not greater than*

$$\frac{\deg f_1 \deg f_2 - \mu(f)}{\min_{i=1,2} \deg\ f_i},$$

where $\mu(f)$ denotes the maximal number of points in (finite) fibers of f.

3 Mappings of the real plane.

In the beginning we compare the real set S_f with the complex one. We have the following simple but basic fact:

Proposition 3.1 *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a dominant polynomial map and let $\mathbb{C}f : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ be its complexification. Then the set S_f of points at which f is not proper is contained in the set $S_{\mathbb{C}f}$ of points at which the mapping $\mathbb{C}f$ is not proper.*

The (easy) purely topological proof is left to the reader.

Definition 3.2 Let $\Gamma \subset \mathbb{R}^n$ be a semi-algebraic closed curve, such that there exists a polynomial surjective mapping $\phi : \mathbb{R} \to \Gamma$. Then Γ will be called *a parametric semi-line* and the mapping ϕ will be called *a parametrization of* Γ. If a curve Γ is algebraic it will be called *a parametric line.*

Theorem 3.3 *Let $f = (f_1, f_2) : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a dominant polynomial mapping. Then the set S_f of points at which f is not proper is a union of finite number (possibly 0) of parametric semi-lines. If $\overline{S_f}$ is an algebraic closure of S_f in $\mathbb{R}^2$, then it is described by a real polynomial of degree not greater than*

$$\frac{\deg f_1 \deg f_2 - \mu(f)}{\min_{i=1,2} \deg\ f_i},$$

where $\mu(f)$ denotes the maximal number of points in (finite) fibers of f. Further, every connected component Q of the set $\mathbb{R}^2 \setminus S_f$ is simply connected. Moreover, the mapping $f : f^{-1}(Q) \to Q$ is proper and consequently the mapping f has a constant topological degree over Q.

Proof. We can treat the mapping f as a rational mapping $f : \mathbb{R}P^2 - \to \mathbb{R}P^2$. Now resolve points of indeterminacy of f by sequence of blowing up's. In this way we obtain a smooth surface X and a regular mapping $F : X \to \mathbb{R}P^2$, such that $res_{\mathbb{R}^2} F = f$. We can also consider the surface $\mathbb{C}X$ and the mapping $\mathbb{C}F$, which are canonical complexifications of X and F (i.e. to obtain X, F we consider complex extension of real blow up's). Hence the mapping $\mathbb{C}F$ is regular in a neighborhood of X. Without loss of generality we can assume that the mapping $\mathbb{C}F$ is regular on the whole of $\mathbb{C}X$ (later we resolve only complex points of indeterminacy).

The set $R := X \setminus \mathbb{R}^2$ is connected and it is a union $\bigcup R_i$ of circles $R_i \cong \mathbb{R}P^1$. Let us denote by l_∞ the line at infinity in $\mathbb{R}^2$ and take $W := F^{-1}(l_\infty)$. Let R_j be a component of R which is not in W. We show that R_j has at most one common point with W. Indeed, the set $W' := \mathbb{C}F^{-1}(\mathbb{C}l_\infty)$ is connected (see [3]). Hence our assertion follows from Theorem 2.1 (applied to the surface $\mathbb{C}X$).

Now let us note that the set S_f is exactly the set $F(R \setminus W)$. In particular the union $S_f \cup l_\infty = F(R) \cup l_\infty$ is a connected set. This means that S_f is either an empty set or it has a pure dimension one. Let Z be an irreducible (non-empty) component of the set S_f. Hence there are indices $j_1, ..., j_k$ such that $Z = \bigcup_{i=1}^k F(R_{j_i} \setminus W)$. On the other hand we know that for every j the curve $R_j \cong \mathbb{R}P^1$ has at most one common point with W. Consequently, the restriction of the mapping F to $R_j \setminus W$ gives a regular mapping $\phi_j : \mathbb{R} \to Z$. Without restriction of generality we can assume that all mappings ϕ_j are non-constant. The mapping ϕ_j must be polynomial.

Indeed, let $Z' := \overline{Z}$ be an algebraic closure of Z. By Proposition 3.1 the curve $\mathbb{C}Z'$ is contained in the set $S_{\mathbb{C}f}$, hence it is an affine parametric curve (over $\mathbb{C}$.). In particular the curve $\mathbb{C}Z'$ has exactly one point at infinity. Since the mapping ϕ_j is regular, we have $\phi_j = (P_1/Q_1, P_2/Q_2)$, where P_1, Q_1 and P_2, Q_2 are relatively prime real polynomials from $\mathbb{R}[t]$ with $Q_1 \cdot Q_2 \neq 0$ (see [1], 3.1.9). If $Q_1 \neq const$ or $Q_2 \neq const$ then $\mathbb{C}Z'$ would have two or more conjugate complex points at infinity (namely images by ϕ_j of zeros of Q_1 or Q_2). Hence, the mapping ϕ_j is indeed a polynomial mapping. This also implies that $R_j \cap W \neq \emptyset$ (i.e. $R_j \setminus W \cong \mathbb{R}$), and consequently $Z = \bigcup_{i=1}^k \phi_{j_i}(\mathbb{R})$.

Further, since every connected component of the set S_f is unbounded (as an union of parametric semi-lines) we have that every connected component Q of the set $\mathbb{R}^2 \setminus S_f$ is simply connected. Since (by definition of the set S_f) the mapping f is proper over Q, we have that the mapping f has a constant topological degree over Q.

Finally, let $\overline{S_f}$ be an algebraic closure of S_f in $\mathbb{R}^2$, which is described by a real reduced polynomial h. If a polynomial $p \in \mathbb{C}[x, y]$ describes the set $S_{\mathbb{C}f}$, then the polynomial h must divide the polynomial p. Consequently, by Theorem 2.3 we have that

$$\deg\ h \leq \deg\ p \leq \frac{\deg f_1 \deg f_2 - \mu(f)}{\min_{i=1,2} \deg\ f_i},$$

where $\mu(f)$ denotes the maximal number of points in (finite) fibers of f.■

Corollary 3.4 *Let f be a polynomial mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$ with a non-zero Jacobian. Let $S_1, ..., S_r$ be all connected components of the set $\mathbb{R}^2 \setminus S_f$ with non-empty preimages and let $f^{-1}(S_i) = \bigcup_{j=1}^{r_i} S_{i,j}$, where $S_{i,j}$ are connected components of the set $f^{-1}(S_i)$.*

Then $res_{S_{i,j}} f : S_{i,j} \to S_i$ are diffeomorphisms and consequently the mapping f is a glueing of finite number of diffeomorphisms.

Proof. Indeed, the mapping $res_{S_{i,j}} f : S_{i,j} \to S_i$ is proper and unramified, hence it is a topological covering. Since the set S_i is simply connected, then the mapping $res_{S_{i,j}} f$ must be a homeomorphism.$\blacksquare$

Remark 3.5 *(see [4]).* Let $S \subset \mathbb{R}^2$ be a parametric semi-line. It can be proved that there exists a polynomial mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$ with finite fibers for which S is exactly the set of points at which f is not proper (i.e., $S = S_f$).$\blacksquare$

Example 3.6 a) There is no polynomial mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that $f(\mathbb{R}^2) = \{(x, y) \in \mathbb{R}^2 : x > 0,\ xy > 1\}$ (or $f(\mathbb{R}^2) = \{(x, y) \in \mathbb{R}^2 :\ x^2 + y^2 > 1\}$). Indeed, borders of latter sets are not unions of parametric semi-lines. More generally, let $U \subset \mathbb{R}^2$ be an open domain. Assume that $bd(U)$ contains as an irreducible component a semi-algebraic curve Γ, such that the algebraic closure $cl(\Gamma)$ does not contain a parametric semi-line. Then there is no polynomial mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that $f(\mathbb{R}^2) = U$.

b) A mapping $f : \mathbb{R}^2 \ni (x, y) \to (1 - 2xy + x^2 + x^2y^2, -y + xy^2) \in \mathbb{R}^2$ satisfies $f(\mathbb{R}^2) = \{(x, y) \in \mathbb{R}^2 : x > 0\}$.

c) (Ruiz) It is an open question whether there is a polynomial mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$, such that $f(\mathbb{R}^2) = \{(x, y) \in \mathbb{R}^2 : x > 0,\ y > 0\}$.

4 The Pinchuk mapping.

In the paper [5] Pinchuk gave an example of a polynomial map $F : \mathbb{R}^2 \to \mathbb{R}^2$ with a nonvanishing Jacobian such that F is not a global diffeomorphism. Let us recall his construction. We define the following auxiliary polynomials in variables x, y:

$$t = xy - 1, \quad h = t(xt + 1), \quad f = (xt + 1)^2(t^2 + y) \tag{1}$$

and two polynomials $v(f,h) = f + f(13+15h)^2 + 12h + 12h^2$, $u(f,h) = 1/4f(75f^3+300f^2h+450fh^2+276f^2+828fh+48h^2+364f+48h)$ satisfying a partial differential equation $\frac{\partial u}{\partial f} - \frac{\partial u}{\partial h} = v$. We put $F=(p,q)$ where p and q are given by

$$p = f+h, \tag{2}$$
$$q = -t^2 - 6th(h+1) + u(f,h). \tag{3}$$

Remark 4.1 *We found two typographical errors in [5]. In the definition of $v(f,h)$ "$12fh$" should be replaced by "$12h^2$" and in the definition of q "$-u(f,h)$" should be replaced by "$+u(f,h)$".*

Let $C = \Phi(\mathbb{R})$ be a parametric line (C is an algebraic curve by Lemma 4.5) given by

$$\Phi : \mathbb{R} \ni t \to (t(t+2), u(t(t+1),t)) \in \mathbb{R}^2 \tag{4}$$

The aim of this section is the following description of the Pinchuk mapping

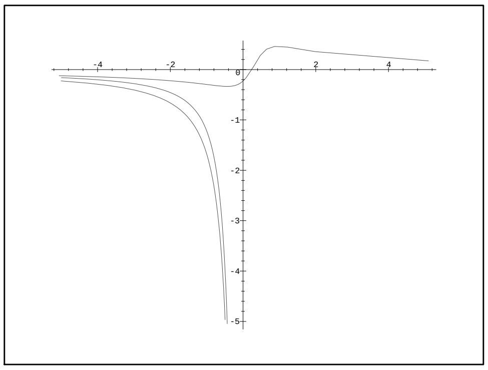
Theorem 4.2 *The set of points at which F is not proper is the parametric line C. The set $F^{-1}(C)$ is a smooth algebraic curve having three connected components. Furthermore*

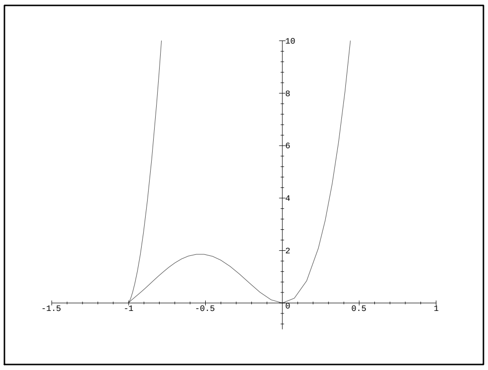
(i) for every $v \in \mathbb{R}^2 \setminus C$ $\quad \#F^{-1}(v) = 2$,

(ii) for every $v \in C \setminus \{(-1,0),(0,0)\}$ $\quad \#F^{-1}(v) = 1$,

(iii) $F^{-1}(-1,0) = F^{-1}(0,0) = \emptyset$.

A behavior of F is illustrated on Figure 1 and Figure 2. A set $\mathbb{R}^2 \setminus F^{-1}(C)$ has 4 connected components. The marked ones are mapped diffeomorphicly to a marked area above the curve C. Indicated connected components of a curve $F^{-1}(C)$ are mapped to indicated parts of the curve C respectively.





For the proof of (4.2) we need a sequence of lemmas.

Lemma 4.3 *The following relations hold for t, h, f, q:*

$$\begin{aligned} (h-t)f &= h^2(h+1) && (5) \\ Q(f,h,q) &= 0 && (6) \end{aligned}$$

where $Q(f,h,q) = f^2(q-u(f,h)) + h^2(f-h(h+1))(f+(6f-h)(h+1))$.

Proof. The formula (5) follows immediately from (1). Multiplying equation (3) by f^2 gives $f^2(q-u(f,h)) + (tf)^2 + 6(tf)h(h+1)f = 0$. By (5) we have $tf = h(f-h(h+1))$. Substituting the right–hand side into above formula we obtain (6). ∎

Let us consider a system of equations

$$\begin{cases} \bar{f}+\bar{h} = a \\ Q(\bar{f},\bar{h},b) = 0 \end{cases} \tag{7}$$

This system is equivalent to $\bar{f}+\bar{h} = a$, $W(\bar{f},a,b) = 0$ where $W(\bar{f},a,b) = Q(\bar{f},a-\bar{f},b)$ is a monic polynomial of degree 6 with respect to $\bar{f}$. Hence, for a given (a,b) a number of solutions of (7) is a number of roots of a polynomial $W(\bar{f},a,b)$.

We see at once that for every $(\bar{f},\bar{h})$ with $\bar{f} \neq 0$ there is a unique pair (a,b) such that $(\bar{f},\bar{h},a,b)$ satisfies (7). Namely

$$\begin{cases} a = \bar{f}+\bar{h} \\ b = u(\bar{f},\bar{h}) - \bar{h}^2(\bar{f}-\bar{h}(\bar{h}+1))(\bar{f}+(6\bar{f}-\bar{h})(\bar{h}+1))/\bar{f}^2 \end{cases} \tag{8}$$

Our basic idea is the observation that, roughly speaking, the Pinchuk mapping is a composition of a polynomial map (f,h) and a map G given by (8). From now we consider all polynomials as complex ones. To distinguish complex and real Pinchuk mappings we shall write $\mathbb{C}F$ for a complex one.

Let $B = \{\,(\bar{f},\bar{h}) \in \mathbb{C}^2 : \bar{f}(\bar{f}-\bar{h}(\bar{h}+1)) = 0\,\}$.

Lemma 4.4 *A mapping $(f,h):\mathbb{C}^2 \to \mathbb{C}^2$ restricted to $\mathbb{C}^2 \setminus f^{-1}(0)$ is a homeomorphism onto $\mathbb{C}^2 \setminus B$. A curve $f^{-1}(0)$ has two algebraic components: $A_1 = (f,h)^{-1}(0,0)$ and $A_2 = (f,h)^{-1}(0,-1)$. They have rational parametrizations $A_1 = \{\,(-1/t, -t(t+1)) : t \neq 0\,\}$, $A_2 = \{\,(-(1+t)/t^2, -t^2) : t \neq 0\,\}$ and inverses of these parametrizations are polynomial functions t.*

Proof. We claim that $(f,h)^{-1}(B) = f^{-1}(0)$. Indeed suppose that $f \neq 0$ and $f - h(h+1) = 0$. Substituting $f = h(h+1)$ into (5) gives $ft = 0$. Hence $t = 0$ and by (1) $h = 0$. But $f = h(h+1) = 0$ – contrary to our assumption.

An easy computation shows that the mapping $\mathbb{C}^2 \setminus B \ni (\bar{f},\bar{h}) \to ((\bar{h}+1)\bar{f}/(\bar{f}-\bar{h}(\bar{h}+1))^2, (\bar{f}-\bar{h}^2)(\bar{f}-\bar{h}(\bar{h}+1))^2/\bar{f}^2) \in \mathbb{C}^2$ composed with (f,g) is an identity on $\mathbb{C}^2 \setminus B$. This together with the claim gives the first part of the lemma.

Now we describe a curve $f^{-1}(0) \subset \mathbb{C}^2$. Assume that $f(x,y) = 0$. Then by (1) $xt + 1 = 0$ or $t^2 + y = 0$. We obtain by (1) in the first case $x = -1/t$, $y = -t(t+1)$, $h = 0$ and in the second case $x = -(t+1)/t^2$, $y = -t^2$, $h = -1$. This finishes the proof. ■

Lemma 4.5 *The complex Pinchuk mapping $\mathbb{C}F$ restricted to $f^{-1}(0)$ is 2:1 between $f^{-1}(0)$ and $\{-1,0\} \times (\mathbb{C} \setminus \{0\})$.*

Proof. By (4.4) $q(x,y) = -t^2 - 6th(h+1) + u(f,h) = -t^2$ for $(x,y) \in f^{-1}(0)$. Successive application of Lemma 4.4 shows that mappings $A_1 \to \{0\} \times (\mathbb{C} \setminus \{0\})$ and $A_2 \to \{-1\} \times (\mathbb{C} \setminus \{0\})$ induced by $\mathbb{C}F$ are 2:1 ■

Lemma 4.6 *The complex Pinchuk mapping $\mathbb{C}F$ restricted to $\mathbb{C}^2 \setminus f^{-1}(0)$ is a composition*

$$\mathbb{C}^2 \setminus f^{-1}(0) \xrightarrow{(f,h)} \mathbb{C}^2 \setminus B \xrightarrow{G} \mathbb{C}^2$$

where $(a,b) = G(\bar{f},\bar{h})$ is given by (8).

Proof. The above lemma follows immediately from 4.3 and 4.4. ■

Lemma 4.7 *Let $(\bar{f},\bar{h},a,b)$ satisfy (7) and let $(\bar{f},\bar{h}) \in B$. Then*

(i) if $\bar{f} = 0$ then $a = -1$ or $a = 0$,

(ii) if $\bar{f} \neq 0$, $\bar{f} = \bar{h}(\bar{h}+1)$ then $(a,b) = \Phi(\bar{h})$.

Proof. Putting $\bar{f} = 0$ in (7) we get $\bar{h} = a$, $Q(0,\bar{h},b) = \bar{h}^4(\bar{h}+1)^2$ and (i) follows. To prove (ii) put $\bar{f} = \bar{h}(\bar{h}+1)$ in (8). We obtain $a = \bar{f} + \bar{h}$, $b = u(\bar{f},\bar{h})$. Thus $(a,b) = \Phi(\bar{h})$. ■

Our aim is a description of the set of points where the complex Pinchuk mapping is not proper.

Lemma 4.8 $S_{\mathbb{C}F} = \Phi(\mathbb{C})$.

Proof. First we show that $S_{\mathbb{C}F} \subset (\{-1\} \times \mathbb{C}) \cup (\{0\} \times \mathbb{C}) \cup \Phi(\mathbb{C})$. Fix $(a, b) \in \mathbb{C}^2$ lying in the complement of above curves and take a neighborhood U of (a, b) such that $cl(U)$ is a compact set disjoint from $(\{-1\} \times \mathbb{C}) \cup (\{0\} \times \mathbb{C}) \cup \Phi(\mathbb{C})$. Let V be the set of all $(\bar{f}, \bar{h}) \in \mathbb{C}^2$ such that $(\bar{f}, \bar{h}, a, b)$ satisfies (7) for some $(a, b) \in cl(U)$. By continuity of roots V is a compact set. Moreover it follows from Lemma 4.7 that $V \cap B = \emptyset$. Hence $V = G^{-1}(cl(U))$. Since $\mathbb{C}F^{-1}(cl(U)) = (f, h)^{-1}(V)$ we conclude by Lemma 4.4 that $\mathbb{C}F^{-1}(cl(U))$ is compact. Therefore the map $\mathbb{C}F$ is proper at the point (a, b).

Now we examine a number of roots of a polynomial $W(\bar{f}, a, b)$ for a generic $(a, b) \in \mathbb{C}^2$. Suppose that $W(\bar{f}, a, b)$ has a multiple factor in a ring $\mathbb{C}[\bar{f}, a, b]$. Then $Q(\bar{f}, \bar{h}, b) = W(\bar{f}, \bar{f} + \bar{h}, b)$ has also a multiple factor. Since a polynomial Q can be rewritten as $Q(\bar{f}, \bar{h}, b) = \bar{f}^2 b + Q_1(\bar{f}, \bar{h})$ the only candidate for a multiple factor is $\bar{f}$. But $\bar{f}$ does not divide $Q(\bar{f}, \bar{h}, b)$ contrary to our claim.

Thus for a generic $(a, b) \in \mathbb{C}^2 \setminus ((\{-1\} \times \mathbb{C}) \cup (\{0\} \times \mathbb{C}) \cup \Phi(\mathbb{C}))$ the polynomial $W(\bar{f}, a, b)$ has 6 single roots. Hence $\#\mathbb{C}F^{-1}(a, b) = 6$.

Our next task is to show that straight lines $\{0\} \times \mathbb{C}$ and $\{-1\} \times \mathbb{C}$ are not contained in $S_{\mathbb{C}F}$. We will use the following property (see [2]): *If $\#\mathbb{C}F^{-1}(a, b) = 6$ then $\mathbb{C}F$ is proper at a point (a, b).*

We have $W(\bar{f}, 0, b) = \bar{f}^2(-197/4\bar{f}^4 + 104\bar{f}^3 - 63\bar{f}^2 + b)$. Thus for a generic $b \in \mathbb{C}$ a system (7) has 4 solutions $(\bar{f}_i, \bar{h}_i)$ $(i = 1, \ldots, 4)$ such that $(\bar{f}_i, \bar{h}_i) \notin B$ and one double solution $(\bar{f}_5, \bar{h}_5) = (0, 0)$. By 4.5 and 4.6 we get $\#\mathbb{C}F^{-1}(0, b) = 6$. Hence by the above mentioned property $\{0\} \times \mathbb{C} \not\subset S_{\mathbb{C}F}$. We show in the same way that $\{-1\} \times \mathbb{C} \not\subset S_{\mathbb{C}F}$. From [3] it follows that the set $S_{\mathbb{C}F}$ is either a complex parametric line $\Phi(\mathbb{C})$ or is empty. The latter is impossible because in this case a real Pinchuk mapping would be proper, so it would be a diffeomorphism. ∎

From now we treat all polynomials under considerations as real polynomials unless otherwise stated.

Lemma 4.9 *The parametrisation Φ of a curve C is injective and C is smooth except a point $(-1, 0)$.*

Proof. Suppose to the contrary that there are $h_1 \neq h_2$ such that $\Phi(h_1) = \Phi(h_2)$ An equation $h_1(h_1 + 2) = h_2(h_2 + 2)$ gives $h_2 = -2 - h_1$. Combining this with equality $u(h_1(h_1+1), h_1) = u(h_2(h_2+1), h_2)$ we get $(75h_1^2+150h_1+104)(h_1+1)^3 = 0$. Hence $h_1 = h_2 = -1$ – a contradiction.

Since $\frac{d}{dh}\Phi(h)$ vanishes for $h = -1$ only, the curve C has a singularity at most at a point $(-1, 0)$. ■

Lemma 4.10 *Let $H : \mathbb{R}^n \to \mathbb{R}^n$ be a local polynomial diffeomorphism. Then H is proper at a point $v_0 \in \mathbb{R}^n$ iff a function $\mathbb{R}^n \ni v \to \#H^{-1}(v) \in \mathbb{Z}$ is locally constant at v_0. Furthermore the function $\mathbb{R}^n \ni v \to \#H^{-1}(v) \in \mathbb{Z}$ is lower semicontinuous.*

The purely topological proof is left to the reader.

Proof of Theorem 4.2. It follows from 4.8 and 3.1 that S_F is a subset of $C = \Phi(\mathbb{R})$. From the topological point of view C is an embedded line. Hence $\mathbb{R}^2 \setminus C$ has 2 connected components. We shall calculate a multiplicity of fibers in each of them.

The curve C cuts a straight line $\{3\} \times \mathbb{R}$ transversely at points $(3, 3142) = \Phi(1)$ and $(3, 8406) = \Phi(-3)$. Hence the points $(3, 0)$ and $(3, 4000)$ lie on opposite sides of C. Using any standard method (e.g. Sturm sequences) one checks that both polynomials $W(\bar{f}, 3, 0)$ and $W(\bar{f}, 3, 4000)$ have 2 real roots. Hence $\#F^{-1}(3, 0) = \#F^{-1}(3, 4000) = 2$ and by Lemma 4.10 for every $(a, b) \in \mathbb{R}^2 \setminus C \quad \#F^{-1}(a, b) = 2$.

Now we will show that $F^{-1}(0, 0) = F^{-1}(-1, 0) = \emptyset$. We have $W(\bar{f}, 0, 0) = -\bar{f}^4(197/4\bar{f}^2 - 104\bar{f} + 63\bar{f})$. The only real root of this polynomial is $\bar{f} = 0$. Since by 4.5 $(0, 0)$ does not belong to the set $F(f^{-1}(0))$ we have $F^{-1}(0, 0) = \emptyset$. We check similarly that $F^{-1}(-1, 0) = \emptyset$.

It remains to compute numbers $\#F^{-1}(v)$ for $v \in C$. Let $D(a, b) = \mathrm{disc}_{\bar{f}} W(\bar{f}, a, b)$. Using a computer one can check that $D(a, b) \neq 0$ for $(a, b) = \Phi(1) \in C$. Since the curve C is irreducible the polynomial D does not vanish on C but a finite number of points. Fix $(a_0, b_0) = \Phi(s) \in C$ for which $D(a_0, b_0) \neq 0$. We have shown before that the polynomial $W(\bar{f}, a, b)$ has two real roots for $(a, b) \in \mathbb{R}^2 \setminus C$. Therefore by continuity of roots system (7) has two real solutions at a point (a_0, b_0). By 4.7 and 4.9 one of these solutions is $(\bar{f}, \bar{h}) = (s(s+1), s) \in B$. Thus $\#F^{-1}(a_0, b_0) = \#(G^{-1}(a_0, b_0) \cap \mathbb{R}^2) = 1$.

Applying Lemma 4.10 we conclude that there is a finite subset K of a curve C such that: $\#F^{-1}(v) = 1$ for $v \in C \setminus K$ and $\#F^{-1}(v) = 0$ for $v \in K$. We have shown before that $\{(-1, 0), (0, 0)\} \subset K$.

A mapping F is a diffeomorphism between an algebraic curve $F^{-1}(C)$ and $C \setminus K$. Hence the curve $F^{-1}(C)$ has $\#K+1$ connected components and its complement $\mathbb{R}^2 \setminus F^{-1}(C)$ has $\#K+2$ connected components. We see by Corollary 3.4 that $\mathbb{R}^2 \setminus F^{-1}(C)$ has 4 connected components. Hence $\#K=2$, so $K=\{(-1,0),(0,0)\}$. This finishes the proof. $\blacksquare$

References

[1] Benedetti, R, Risler, J-J, Real algebraic and semi-algebraic sets, Actualités Mathématiques, Hermann, 1990.

[2] Jelonek, Z, The set of points at which a polynomial map is not proper, **Ann. Polon. Math., 58**, (1993), 259-266.

[3] Jelonek, Z, Geometry of affine polynomial mappings, Preprint PAN, 1997/570.

[4] Jelonek, Z, Geometry of real polynomial mappings, (to appear).

[5] S. Pinchuk, *A Counterexample to the Real Jacobian Conjecture*, Match. Zeitschrift (1994)

[6] Spanier, E.H, Algebraic Topology, Mc Graw-Hill, N.Y., 1966.

[7] R. Walker *Algebraic curves*, Princeton 1953