Weak solutions of the equation of complex Monge-Ampère type

by

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Abstract. We prove some existence results for the equation of the complex Monge-Ampère type in strictly pseudoconvex domains and on Kähler manifolds.

0. Introduction.

In this paper we extend the results on the existence of bounded (resp. continuous) solutions of the complex Monge-Ampère equation

$$(dd^c u)^n = d\mu \tag{0.1}$$

(with given positive measure $d\mu$, a plurisubharmonic solution u, and the wedge product on the left defined as in [BT1]) to the case of more general equations of Monge-Ampère type

$$(dd^c u)^n = F(u, \cdot)d\mu, \tag{0.2}$$

We shall assume throughout that $F(t, z) \geq 0$, that it is nondecreasing and continuous in the first variable and measurable in the other one.

One can study the Dirichlet problem for the equation (0.1) in a strictly pseudoconvex domain in $\mathbb{C}^n$ imposing a boundary condition

$$\lim_{z \to x} u(z) = \varphi(x) \text{ for } x \in \partial\Omega.$$

with given $\varphi \in C(\partial\Omega)$ (see [BT1], [Ce], [CKNS], [K1], [K2], [K3]). By [K1] this problem has a unique bounded solution, provided a bounded subsolution exists. We shall prove (Theorem 1.1 below) that the same conclusion holds for (0.2) when F is bounded. Thus we generalize the results of Bedford-Taylor [BT2] and Cegrell [Ce]. We refer to [CKNS] for the study of the classical solutions of the equation. Furtheremore, let the measure $d\mu$ be represented as $f\,d\lambda$ with $d\lambda$ denoting the Lebesgue measure and $f \geq 0$ belonging to the Orlicz space

$$L^\chi(\Omega) = \{g \in L^1(\Omega) : \int_\Omega \chi(g)\, d\lambda < \infty\}.$$

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If

$$\chi(t) = |t|(log(1+|t|))^n h(log(1+|t|)), \tag{0.3}$$

where $h : \mathbb{R}_+ \to (1, \infty)$ is an increasing function satisfying

$$\int_1^\infty (yh^{1/n}(y))^{-1}\,dy < \infty,$$

then the Dirichlet problem for (0.1) has a unique continuous solution (see [K2], [K3]). Corollary 1.2 says that it is still true for (0.2) with $d\mu = d\lambda$ and $F(t,z) \leq f(z) \in L^\chi(\Omega)$.

In the next section we study the complex Monge-Ampère equation on a compact Kähler manifold M with the fundamental form ω :

$$(\omega + dd^c u)^n = F(u,\cdot)\omega^n \tag{0.2'}$$

By the Stokes theorem, if u satisfies (0,2') then

$$\int_M F(u,\cdot)\omega^n = \int_M \omega^n.$$

For F positive and smooth, satisfying the normalizing condition, the equation has been solved by Aubin [A1], [A2] and Yau [Y]. It is particulary interesting for $F(t,z) = \exp(ct + f(z))$ when the solution serves to produce Kähler-Einstein metrics (see [A1], [A2], [A3], [S], [Y]). Using a result from [K3], where the case of F not depending on t was treated, we show in Theorem 2.1 that for F only nonnegative and $F \in L^\chi(M)$, χ as above, one can find a continuous solution of (0.2'). This result, applied to $M = \mathbb{P}^n$, leads to solving (0.2) in the family of entire plurisubharmonic functions of minimal growth.

1. The equation of Monge-Ampère type in a strictly pseudoconvex domain.

Theorem 1.1. *Let Ω be a strictly pseudoconvex domain and $\varphi \in C(\partial\Omega)$. Suppose there exists $v \in PSH \cap L^\infty(\Omega)$ such that $(dd^c v)^n = d\mu$ and $\lim_{z\to x} v(z) = \varphi(x)$ for $x \in \partial\Omega$. Furtheremore assume that $F : \mathbb{R} \times \Omega \to \mathbb{R}$ is a bounded nonnegative function which is nondecreasing and continuous in the first variable and $d\mu$ measurable in the second one. Then there exists a unique bounded plurisubharmonic solution of the Dirichlet problem*

$$\begin{aligned} &u \in PSH \cap L^\infty(\Omega),\\ &(dd^c u)^n = F(u,\cdot)d\mu,\\ &\lim_{z\to x} u(z) = \varphi(x) \ \textit{for}\ x \in \partial\Omega. \end{aligned} \tag{1.1}$$

Proof. Without loss of generality we assume $F \leq 1$. As in [Ce] we shall use Schauder's fixed point theorem. Consider $L^2(\Omega,\, d\lambda)$ equipped with the weak topology. Let h be the maximal plurisubharmonic function in Ω with the boundary data equal to φ. Then the set

$$\mathcal{A} = \{u \in PSH(\Omega) : v \leq u \leq h\}$$

is convex and compact in $L^2(\Omega,\, d\lambda)$. We define the mapping

$$\mathcal{F} : \mathcal{A} \to \mathcal{A}$$

taking for $\mathcal{F}(u)$ the solution of

$$\begin{aligned} & w \in \mathcal{A}, \\ & (dd^c w)^n = F(u, \cdot) d\mu. \end{aligned} \tag{1.2}$$

This solution exists by [K1]. We need to show that $\mathcal{F}$ is contiuous. Let $u_j \to u$ in $\mathcal{A}$. Set $w = \mathcal{F}(u)$ and $w_j = \mathcal{F}(u_j)$. By Hartogs' lemma $u = (\limsup u_j)^*$. Consider auxiliary functions $\tilde{u}_k = (\sup_{j \geq k} u_j)^*, \tilde{w}_k = \mathcal{F}(\tilde{u}_k)$. Since $(dd^c \tilde{w}_k)^n$ is decreasing it follows from the comparison principle [BT3] that the sequence $\tilde{w}_k$ is increasing to some $\tilde{w} \in \mathcal{A}$. The convergence theorem [BT3] gives $(dd^c \tilde{w})^n = \lim_k (dd^c \tilde{w}_k)^n$. We also have

$$(dd^c w_k)^n = F(u_k, \cdot) d\mu \leq F(\tilde{u}_k, \cdot) d\mu = (dd^c \tilde{w}_k)^n.$$

Hence, using the comparison principle, $\tilde{w}_k \leq w_k$.

Furtheremore, $\tilde{u}_k \downarrow u$, from the Lebesgue monotone convergence theorem one infers

$$(dd^c \tilde{w})^n = \lim_{k \to \infty} F(\tilde{u}_k, \cdot) d\mu = F(u, \cdot) d\mu.$$

Since the solution to the Dirichlet problem (1.2) is unique we thus get

$$w = \tilde{w} = \lim \uparrow \tilde{w}_k \leq \liminf w_k.$$

It remains to prove that

$$\limsup w_k \leq w.$$

For this consider the sequence $\hat{w}_k$ of functions in $\mathcal{A}$ solving

$$(dd^c \hat{w}_k)^n = F(\hat{u}_k, \cdot) d\mu,$$

where $\hat{u}_k = \inf_{k \leq j} u_j$. Then, by the comparison principle $w_k \leq \hat{w}_k$ and $\hat{w}_k$ decreases to $\hat{w} \in A$. Since $\hat{u}_k \uparrow u$ we obtain, applying the convergence theorem

$$(dd^c \hat{w})^n = \lim \uparrow (dd^c \hat{w}_k)^n = F(u, \cdot) d\mu.$$

This implies $w = \hat{w} = \lim \uparrow \hat{w}_k \geq \limsup w_k$. Thus we have proved the continuity of the mapping $\mathcal{F}$. The Schauder theorem now says that $\mathcal{F}$ has a fixed point which gives the existence part of the statement. Uniqueness follows from the comparison principle and the fact that $F(\cdot, z)$ is nondecreasing.

Remark. *The above theorem remains true for hyperconvex domains provided there exists a maximal plurisubharmonic function with boundary data equal to φ.*

Remark. *The existence part of the theorem still holds when we drop the hypothesis that $F(\cdot, z)$ be monotone, but uniqueness is then lost (see [Ce]).*

Corollary 1.2. *Let Ω, φ and F be as in the above theorem except that instead of assuming that F is bounded we now suppose*

$$F(t,\cdot) \le \psi \in L^{\chi}(\Omega),$$

with χ given by (0.3). *Then there exists a continuous solution to*

$$\begin{aligned} &u \in PSH(\Omega) \cap C(\bar{\Omega}),\\ &(dd^c u)^n = F(u,\cdot)\, d\lambda,\\ &\lim_{z\to x} u(z) = \varphi(x) \text{ for } x \in \partial\Omega, \end{aligned}$$

where $d\lambda$ denotes the Lebesgue measure.

Proof. For the subsolution required in Theorem 1.1 we take $v \in PSH(\Omega) \cap C(\bar{\Omega})$ solving

$$(dd^c v)^n = \psi\, d\lambda,$$

with given boundary data. The existence of such v and continuity of u has been proved in [K2] and [K3].

The equation of Monge-Ampère type on Kähler manifolds.

Our next result is concerned with the equation (0.2').

Theorem 2.1. *Let (M,ω) be a compact Kähler manifold with the fundamental form ω. Assume $F : \mathbb{R} \times M \to \mathbb{R}$, $0 \le F(t,z) \le \psi(z) \in L^{\chi}(M)$ (with χ as in* (0.3)*) is a function such that $F(\cdot,z)$ is continuous and nondecreasing, $F(t,\cdot)$ is ω^n measurable and*

$$\lim_{t\to-\infty} \int_M F(t,z)\omega^n \le \int_M \omega^n \le \lim_{t\to\infty} \int_M F(t,z)\omega^n. \tag{2.1}$$

Then the equation

$$\mathcal{M}u := (dd^c u + \omega)^n = F(u,\cdot)\omega^n$$

has a continuous solution such that $dd^c u + \omega \ge 0$.

We shall need the following lemma.

Lemma 2.2. *Let (M,ω) and F be as in Theorem 2.1 except that we replace the condition* (2.1) *by slightly stronger one*

$$\lim_{t\to-\infty} \int_M F(t,z)\omega^n < \int_M \omega^n < \lim_{t\to\infty} \int_M F(t,z)\omega^n. \tag{2.1'}$$

Then there exists a sequence $F_k \in C^{\infty}(\mathbb{R}\times M)$ with the following properties: $F_k > 0$,

$$\frac{\partial}{\partial t} F_k(t,z) \ge 0,$$

$$\sup_t \sup_k \int_M \chi(F_k(t,\cdot))\omega^n < c < \infty,$$

(2.1') *is fulfilled with* F_k *in place of* F, *and for any bounded function* u *in* M:

$$\lim_{k\to\infty} F_k(u(z), z) = F(u(z), z)$$

almost everywhere in M.

Proof. Using (2.1') we fix $R > 0$ such that

$$\int_M F(-R, z)\omega^n + R^{-1} < \int_M \omega^n < \int_M F(R, z)\omega^n - R^{-1}. \tag{2.2}$$

For any positive integer $k > R$ we shall construct $F_k(t, z)$ as follows. First, note that the assumptions on $F(\cdot, z)$ allow us to find for given $z \in M$ a positive integer N such that

$$F(t_{j+1}, z) - F(t_j, z) < 2^{-k-4} \tag{2.3}$$

where

$$t_j := -k + j2^{-N}, \quad j = 0, 1, ..., k2^{N+1}.$$

We fix N such that (2.3) holds true for all $z \notin E$ with

$$\int_E \omega^n < 2^{-k-2}. \tag{2.4}$$

In the next step we apply Luzin's theorem to choose $g_j \in C(M)$, $g_j \geq 0$, $\int_M \chi(g_j)\omega^n < c$ satisfying

$$\int_{E'_j} \omega^n < 2^{-k-j-3}, \quad E'_j := \{g_j \neq F(t_j, \cdot)\}. \tag{2.5}$$

Set $h_j = \min_{j\leq s} g_j$. Then $\{h_j \neq F(t_j, \cdot)\} \subset E' := \cup_0^N E'_s$. Since $h_j \leq h_{j+1}$ one can approximate h_j by smooth positive f_j such that $f_j \leq f_{j+1}$, $\int_M \chi(f_j)\omega^n < c$ and

$$\{|f_j - F(t_j, \cdot)| > 2^{-k-3}\} \subset E'. \tag{2.6}$$

Fix ψ_j smooth in a neighbourhood of the interval $[t_{j-1}, t_j]$ with $0 \leq \psi_j \leq 1$, $\psi_j = 1$ close to t_j and $\psi_j = 0$ close to t_{j-1}. Set

$$F_k(t, z) = \psi_j(t) f_{j+1}(z) + (1 - \psi_j(t)) f_j(z), \quad t \in [t_{j-1}, t_j].$$

By our choice of ψ_j those functions are smooth and the choice of f_j guarantees that F_k are nondecreasing in t, satisfy the inequalities

$$\int_M F_k(-R, z)\omega^n < \int_M \omega^n < \int_M F_k(R, z)\omega^n, \tag{2.2'}$$

and the condition

$$\sup_t \sup_k \int_M \chi(F_k(t, \cdot))\omega^n < c < \infty.$$

Note that for $z \notin E', t \in [t_{j-1}, t_j]$ we have (see (2.6))

$$F_k(t,z) \geq f_j(z) \geq F(t_j, z) - 2^{-k-3} \geq F(t,z) - 2^{-k-3}. \tag{2.7}$$

If moreover $z \notin E$ then (see (2.3) and (2.6))

$$\begin{aligned} F(t,z) &\geq F(t_{j-1}, z) > F(t_{j+1}, z) - 2^{-k-3} \\ &\geq f_{j+1}(z) - 2^{-k-2} \geq F_k(t,z) - 2^{-k-2}. \end{aligned} \tag{2.8}$$

By (2.4) and (2.5) we have

$$\int_{E\cup E'} \omega^n < 2^{-k-1},$$

and writing $E = E(k), E' = E'(k)$ to indicate the dependece of those sets on k

$$\sum_{j\leq k} \int_{E(k)\cup E'(k)} \omega^n < 2^{-j}.$$

Since, by (2.7) and (2.8), for any $z \notin \cup_{j\leq k}[E(k) \cup E'(k)]$ and any t we have $\lim_{k\to\infty} F_k(t,z) = F(t,z)$ the last part of the statement follows.

Proof of Theorem 2.1. First we prove the statement under extra hypothesis (2.1). We fix a sequence F_k as in the above lemma. Yau's theorem [Y, Theorem 4] provides smooth u_k satisfying $dd^c u_k + \omega > 0$ and

$$\mathcal{M} u_k = F_k(u_k, \cdot).$$

Let us define some auxiliary functions

$$u_{jk} = \max_{j\leq l\leq k} u_l, \quad v_j = (\lim_{k\to\infty} u_{jk})^*, \quad u = (\limsup u_k)^*.$$

One can apply [K3, section 2.3] to conclude that

$$\sup_M u_k - \inf_M u_k < c_0, \tag{2.9}$$

with c_0 depending only on c from the assumptions. By Stokes' theorem

$$\int_M F(u_k, \cdot)\omega^n = \int_M \omega^n.$$

So, in view of (2.2') we obtain

$$\inf_M u_k < R, \quad \sup_M u_k > -R, \qquad k > R.$$

Those inequalities and (2.9) imply that the sequence u_k is uniformly bounded. Passing to a subsequence we assume

$$u_k \to u \quad a.e.$$

Applying [BT1, Proposition 2.8] one gets

$$\mathcal{M}u_{jk} \geq \min_{j \leq l \leq k} F_l(u_l, \cdot).$$

Now, for fixed $\epsilon > 0$ we find an integer j_0 and a set E with

$$u_j(z) \geq u(z) - \epsilon, \ j \geq j_0, \ z \notin E,$$

and $\int_E \omega^n < \epsilon$. Then for j, z as above

$$\mathcal{M}u_{jk}(z) \geq \inf_{j \leq l \leq k} F_l(u(z) - \epsilon, z).$$

Letting k to ∞ and using the convergnce theorem [BT3] we obtain

$$\mathcal{M}v_j(z) \geq \inf_{j \leq l} F_l(u(z) - \epsilon, z), \quad z \notin E.$$

Since v_j decrease to u one can apply the convergence theorem and Lemma 2.2 to get

$$\mathcal{M}u(z) \geq F(u(z) - \epsilon, z) \quad \text{a.e. in } M \setminus E.$$

This is true for any $\epsilon > 0$ and $F(\cdot, z)$ is continuous, so

$$\mathcal{M}u(z) \geq F(u(z), z) \ \ a.e. \tag{2.10}$$

Since by the lemma and the argument above $F_k(u_k(z), z) \to F(u(z), z)$ almost everywhere and, on the other hand,

$$\int_M F_k(u_k, \cdot)\omega^n = \int_M \omega^n$$

for any k, we conclude that the integrals over M of both sides of inequality (2.10) are equal. So the functions are equal. To get the general case, note that if we had equalities in (2.1') then F would be independent of t and the equation would reduce to the one solved in [K3]. If we have one strict inequality we can find a monotone sequence $c_j \to 0$ such that for any j

$$\lim_{t \to -\infty} \int_M (F(t, z) + c_j)\omega^n < \int_M \omega^n < \lim_{t \to \infty} \int_M (F(t, z) + c_j)\omega^n.$$

Then by the preceding part of the proof, there exist u_j satisfying

$$\mathcal{M}u_j = F(u_j(z), z) + c_j.$$

The function $u = (\limsup u_k)^*$ is the desired solution as can be seen by repeating the above reasoning.

Theorem 2.1 when applied for $M = \mathbb{P}^n$, equipped with the Fubini-Study metric, allow us to solve the Monge-Ampère type equation in the class of entire plurisubharmonic functions of minimal growth usually denoted by $\mathcal{L}_+$.

$$\mathcal{L}_+ = \{u \in PSH(\mathbb{C}^n) : |u(z) - \log(1 + |z|^2)| < const.\}.$$

The function

$$v_0(z) = \frac{1}{2}\log(1+|z|^2) \in \mathcal{L}_+$$

is a potential of the Fubini-Study metric in $\mathbb{P}^n$ restricted to $\mathbb{C}^n$ (which is embedded in the standard way). We have

$$\int (dd^c v_0)^n = \int \omega^n = \int \frac{n!}{(1+|z|^2)^{n+1}} d\lambda = (2\pi)^n.$$

Let $F : \mathbb{R} \times \mathbb{C}^n \to \mathbb{R}$, $0 \le F(t,z)$ be continuous and nondecreasing in t, measurable in z, and such that for some t_0

$$\int_{\mathbb{C}^n} F(t_0, z)\, d\lambda = (2\pi)^n.$$

Suppose also that

$$F(t,z) \le f(z)(1+|z|^2)^{-n-1},$$

with $f \in L^\chi(\omega^n)$.

Corollary 2.3. *For F introduced above the following equation*

$$\begin{aligned} &u \in \mathcal{L}_+ \\ &(dd^c u)^n = F(u - v_0, \cdot)\, d\lambda \end{aligned}$$

has a solution.

The uniqueness of those solutions has been shown in [BT4] in the case when F is independent either of t or z.

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