An extension of Markov partitions for a certain toral endomorphism

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IMUJ Preprint 1999/08

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Abstract

We define and construct Markov partition for a certain toral endomorphism and then we use it to obtain a symbolic representation of the semidynamical system induced by the endomorphism.

* Research supported in part by Polish Scientific Grant No.2 P03A 029 12 and the PRODYN programme of the European Science Foundation

Contents

1 Introduction

The increasing interest in the theory of symbolic dynamics is stimulated by the need of a simple tool to investigate complicated dynamical systems. Quite often it is much easier to prove that a given subshift of finite type has some dynamical property, than to show it directly for the considered dynamical system. By finding a semi-conjugacy between an appropriate subshift of finite type and a given dynamical system we often can draw conclusions about dynamical properties of the latter.

Markov partitions provide a way of constructing such semi-conjugaces. The method is based on a division of a space into a finite number of parts. As long as the partition is properly chosen it establishes a correspondence between an orbit and its itinerary, which records the parts visited by the successive points of the orbit. To construct an appropriate partition stable and unstable manifolds are used to indicate the boundaries of these parts.

Markov partitions were first introduced by Adler and Weiss [2] for the hyperbolic automorphisms of the two-dimensional torus and the obtained symbolic model was used to investigate some measure theory problems.

The aim of this paper is to generalize the concept of Markov partitions for the certain toral endomorphism. We search for the partition that gives a symbolic representation of the discrete semi-dynamical system induced by the endomorphism on a torus. It is natural to expect our representation to be a one-sided subshift of finite type, because in the case of the toral automorphism it is a subshift of finite type. Unfortunately the construction of Adler and Weiss does not apply in the non-invertible case. It is impossible to obtain in this way any symbolic representation, which will be explained in Chapter 9.

Our result is a construction of an extended Markov partition for the given toral endomorphism, which raises the expected symbolic representation of the semi-dynamical system of interest.

To reach our aim we generalize the concept of the good intersection property (Definition 3.6), which has been so far described (in a restricted way) only geometrically by pictures (compare Lind, Marcus [16]).

This provides a new method of obtaining the transition rule (Section 7), that is significant for the definition of the extended Markov matrix (Definition 3.9) and the extended Markov partition (Definition 3.10).

One of the most interesting problems which we approach is finding two suitable partitions such that there is a factor map between the one-sided subshifts of finite type associated with them.

In the case of automorphisms we deal with the finite-to-one factor maps. Adler [1], 1998 shows that the existence of Markov partitions is, in some way, equivalent to the existence of these factor maps.

Embeddings are an example of factor maps and the existence of embeddings for the one-sided subshifts of finite type is an open problem. The reader can find more information on this in Kitchens [14], 1998 (Chapter 'Subsystems and infinite-to-one factor maps', Problem 4.4.4). So far there are no known results for the necessary and sufficient conditions for the existence of such embeddings, although the case of the two-sided subshifts of finite type is completed and a procedure constructing such maps is known.

It seems that investigating different partitions of the torus and the associated one-sided subshifts of finite type can contribute to solving the problem. The main benefit is that we can use not only algebraical methods, but also geometrical ones.

Finally we would like to mention other generalizations of Markov partitions. Adler and Weiss' construction was first extended by Sinai [24], [25] for the Anosov diffeomorphisms. Then Bowen [5] created Markov partitions for Axiom A diffeomorphisms, using Smale's Spectral Decomposition Theorem. The detailed description of Bowen's construction is given by Shub [22] under the slightly more general assumptions.

The recent generalizations of Markov condition is the weak local Markov condition, Blank [4], 1997.

Bowen's method, although applying to the broaden class of maps, does not provide any useful information when an explicit dynamical system is considered.

Firstly the rectangles that form the space division are too small and therefore the subshift of finite type, which we obtain as a symbolic model for the considered dynamical system, acts on a very large number of symbols.

Secondly in no boundary point of the rectangles there is a tangent space, so they have a fractal structure, [8] and it causes serious difficulties in constructing these sets in the higher-dimensional spaces (significant problems appear already in dimension three).

More information on this one can find in [1].

In spite of the fact that Markov partitions were created for the hyperbolic systems, similar methods are used for the systems that lack this property, [15]. It is worth mentioning that there are also some attempts to construct Markov partitions with the use of computer methods [11].

2 Acknowledgements

I would like to thank Professor Jerzy Ombach for introducing me to the theory of dynamical systems. Numerous conversations, participation in the seminar and also the pointed literature encouraged me to write this paper, which forms a part of my Master Thesis.

I wish to express my gratitude to Professor Anthony Manning form the University of Warwick, who raised the problem of defining Markov partitions for noninvertible maps and lead the inspiring discussions that contributed significantly to my thesis.

I owe thanks Stefan Batory Foundation in Warsaw, Foreign and Commonwealth Office in London and Stefan Batory Trust in Oxford for founding me

one-term scholarship in the University of Oxford, extended with the consultations in the University of Warwick.

3 Definition of the extended Markov partition

Let us consider the toral endomorphism given by the matrix

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}.$$

All the definitions and results are stated under assumption that A is the above matrix.

Matrix A induces a map on $\mathbf{T}^2$, because it has integer entries, but it is not invertible because $\det A = 2$. We will denote a map induced on the torus also as A.

Matrix A has the following eigenvalues and corresponding eigenvectors

$$\lambda_u = 2 + \sqrt{2},\ \lambda_s = 2 - \sqrt{2}$$

$$x_u = [1 + \sqrt{2}, 1]$$

$$x_s = [1 - \sqrt{2}, 1].$$

It is a hyperbolic matrix, because its eigenvalues are not on the unit circle. Let E^u be a linear subspace of $\mathbf{R}^2$ spanned by the vector x_u and E^s respectively - by the vector x_s.

Let us denote by W^s and W^u projections of E^s and E^u to the torus. By $W^s(x)$ and $W^u(x)$ we mean projections of the subspaces $x + E^s$ and $x + E^u$ on the torus.

Let us introduce some preliminary definitions.

Definition 3.1 *By the interval in $\mathbf{T}^2$ we mean the projection of an interval in $\mathbf{R}^2$.*

Definition 3.2 *By the rectangle in $\mathbf{T}^2$ we mean the projection of a rectangle in $\mathbf{R}^2$.*

We consider only rectangles with the sides parallel to the stable and unstable direction.

Observation 3.3 *Let R be a rectangle on $\mathbf{T}^2$. Then $A(R)$ is also a rectangle and $A^{-1}(R)$ is a union of two congruent rectangles P and Q, such that $P = Q + [\frac{1}{2}, \frac{1}{2}]$.*

Definition 3.4 *An interval J^u is called an unstable section of the rectangle P, if there exists an $x \in \mathbf{T}^2$, such that $J^u = W^u(x) \cap P$.*

Definition 3.5 *By the partition of the torus we mean a finite set of open disjoint rectangles such that the closure of their union is the whole torus.*

Definition 3.6 *Let $\mathcal{R}$ be a partition of the torus. We say that rectangles $P \in \mathcal{R}$ and $A^{-1}(R)$, where $R \in \mathcal{R}$, satisfy the good intersection property if for any unstable section J^u of the rectangle P the set $J^u \cap A^{-1}(R)$ is an interval or for any unstable section J^u of the rectangle P we have $J^u \cap A^{-1}(R) = \emptyset$.*

Definition 3.6 is correctly stated, because of Observation 3.3.

Definition 3.7 *Let $\mathcal{R}$ be a partition of the torus. We say that $\mathcal{R}$ satisfies a good intersection property for the preimages, if for any two rectangles $P, R \in \mathcal{R}$, sets P and $A^{-1}(R)$ satisfy the good intersection property.*

Let us generalize the concept of the transition rule, [1].

Definition 3.8 *Let $\mathcal{R} = \{R_i : 1 \leq i \leq n\}$ be a partition of the torus. We say that $\mathcal{R}$ satisfies the transition rule for A if for any $m \in \mathbf{N}$, $m \geq 3$ the following is satisfied:*
$R_{a_k} \cap A^{-1}(R_{a_{k+1}}) \neq \emptyset,\ 0 \leq k \leq m-1 \ \Rightarrow \bigcap_{k=0}^{m} A^{-k}(R_{a_k}) \neq \emptyset$
for any $R_{a_k} \in \mathcal{R}$, $0 \leq k \leq m$.

By S^T we denote the transpose of a matrix S.

Definition 3.9 *Let $\mathcal{P} = \{P_1, P_2, \ldots, P_n\}$ be a partition of the torus, satisfying a good intersection property for the preimages.*

Let us define a matrix $T = S^T$, where $S = [s_{ij}]_{i,j=1}^{n}$, $s_{ij} = 1$ if $A^{-1}(P_i) \cap P_j \cap J^u \neq \emptyset$ for all J^u unstable sections of the rectangle P_j, $1 \leq i, j \leq n$ and $s_{ij} = 0$ otherwise. The matrix T is called the extended Markov matrix.

Definition 3.10 *An extended Markov partition for the endomorphism A is a pair of the partitions of the torus $(\mathcal{P}, \mathcal{R})$ satisfying the following conditions*
1.) Partition $\mathcal{P}$ satisfies the transition rule.
2.) Partition $\mathcal{R}$ satisfies the following condition

$$(*)\ \forall (a_n)_{n\geq 0} \in \Sigma_{T'}^{+} : \bigcap_{n=0}^{+\infty} A^{-n}(R_{a_n}) = I_{(a_n)} \times \mathcal{C},$$

where $I_{(a_n)} = W^s(x) \cap R_{a_0}$ for some $x \in \mathbf{T}^2$, $\mathcal{C}$ is a topological Cantor set and T' is the extended Markov matrix associated with the partition $\mathcal{R}$.
3.) There exists a surjective, shift commuting map $F : \Sigma_{T'}^{+} \to \Sigma_{T}^{+}$, where T' and T are extended Markov matrices associated respectively with the partitions $\mathcal{R}$ and $\mathcal{P}$.

4 Main results

Let $A : \mathbf{T}^2 \to \mathbf{T}^2$ be an endomorphism of the torus induced by the matrix

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}.$$

Theorem 4.1 *There exists extended Markov partition for the endomorphism A.*

Theorem 4.2 *There exists a one-sided subshift of finite type Σ_T^+ and surjective map $\Phi : \mathbf{T}^2 \setminus B \to \Sigma_T^+$, such that the following diagram commutes.*

$$\begin{tikzcd}
\Sigma_T^+ \arrow[r, "\sigma_T"] & \Sigma_T^+ \\
\mathbf{T}^2 \setminus B \arrow[u, "\Phi"] \arrow[r, "A"'] & \mathbf{T}^2 \setminus B \arrow[u, "\Phi"']
\end{tikzcd}$$

$B = \{x \in \mathbf{T}^2 : A^n x \in \partial\mathcal{R} \ \mathit{for\ some}\ n \geq 0\}$

5 Proof of the existence of the extended Markov partition

CONSTRUCTION:

STEP 1.
We start by repeating the first step of Adler and Weiss' construction, which gives us two rectangles. Let us denote the bigger one by R_1^* and the smaller one by R_2.

STEP 2. (Picture A)
Let us divide rectangle R_1^* into two pieces, named R_1 and R_3, by cutting the original rectangle by the unstable manifold of 0.

STEP 2a. (Picture D)
Let us construct the partition consisting of the preimages of the rectangles from the previous step. By the corollary 3.3 each of the preimages consist of two rectangles. Let us give them respectively the names $R_{1'}$ and $R_{1''}$ for the preimage of R_1, $R_{2'}$ and $R_{2''}$ in a case of R_2 and $R_{3'}$, $R_{3''}$ when R_3 is considered.

They will be used to make some changes in the partition obtained in step 2.

STEP 3. (Pictures A, B)
We want to rebuild the partition obtained in step 2, using partition 2a in order to get a partition satisfying the good intersection property for the preimages.

$R_d = R_{1''} \cap R_1$.

The rectangle R_2 remains unchanged.

$R_e = (R_{1'} \cup R_{2'} \cup R_{3'}) \cap R_3$

$R_f = R_{3''} \cap R_3$

$R_g = (R_{1'} \cup R_{2''} \cup R_{3'} \cup R_{3''}) \cap R_1$

The partition which we obtained satisfies the good intersection property for the preimages and therefore satisfies also the transition rule, by Theorem 7.1.

It is the partition $\mathcal{P}$ required in the definition of the extended Markov partition.

STEP 3a.

Let us make a new partition by taking the preimages of the rectangles R_d, R_2, R_e, R_f and R_g.

Prime or double prime signs are added to the name of the rectangle when the preimage is taken.

It is the partition $\mathcal{R}$ required in the definition of the extended Markov partition.

This is the end of the construction. In the next step we prove required properties of the obtained partitions.

STEP 4.

The extended Markov matrix associated with the partition $\mathcal{P}$, referring to the order of symbols $g, d, 2, e, f$ is as follows

$$T = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

The extended Markov matrix for the partition $\mathcal{R}$, with the given order of symbols $g', g'', d', d'', 2', 2'', e', e'', f', f''$ is as follows

$$T' = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

Let us check if condition 3.) of the definition of the extended Markov partition is satisfied. Let us define a map $F:\Sigma^+_{T'}\to\Sigma^+_T$ by the condition

$$(F(a_n)_{n=0}^{+\infty})_m = b_m :\Longleftrightarrow A(R_{a_m}) = R_{b_m},\ for\ m\geq 0.$$

Let us check if F commutes with the subshift of finite type, that is if the following equality is valid.

$$\sigma_T\circ F = F\circ\sigma_{T'}.$$

From the definition of F we obtain

$$\sigma_T(F(a_n)_{n=0}^{+\infty}) = (c_n)_{n=0}^{+\infty} \Longleftrightarrow$$
$$\Longleftrightarrow (F((a_n)_{n=0}^{+\infty}))_{m+1} = c_m,\ for\ m\geq 0 \Longleftrightarrow$$
$$\Longleftrightarrow A(R_{a_{m+1}}) = R_{c_m}, for\ m\geq 0$$
$$F(\sigma_{T'}((a_n)_{n=0}^{+\infty})) = F((a_{n+1})_{n=0}^{+\infty}) = (d_n)_{n=0}^{+\infty} \Longleftrightarrow$$
$$\Longleftrightarrow (F((a_{n+1})_{n=0}^{+\infty}))_m = d_m,\ for\ m\geq 0 \Longleftrightarrow$$
$$\Longleftrightarrow A(R_{a_{m+1}}) = R_{d_m},\ for\ m\geq 0$$

From above it follows that $c_m = d_m,\ for\ m\geq 0$ and so F commutes with the subshift. Moreover by the definitions of the matrices T and T' the map F is surjective. This proves that condition 3.) of the definition of the extended Markov partition is satisfied.

STEP 5.

Now we will show that the condition $(*)$ is satisfied, that is

Theorem 5.1

$$\forall (a_n)_{n\geq 0}\in\Sigma^+_{T'}:\ \bigcap_{n=0}^{+\infty} A^{-n}(R_{a_n}) = I_{(a_n)}\times\mathcal{C},$$

where $I_{(a_n)} = W^s(x)\cap R_{a_0}$ for some $x\in\mathbf{T}^2$ and $\mathcal{C}$ is a topological Cantor set.

Let us switch to the coordinates system given by the vectors $x_s,\ x_u$. The matrix A in this coordinate system is

$$J_A = \left[\begin{array}{cc} 2+\sqrt{2} & 0\\ 0 & 2-\sqrt{2}\end{array}\right].$$

Let $V=\{v_1,v_2,\ldots,v_k\}$ be a set of vectors in $\mathbf{T}^2$. By $\mathcal{C}(V)$ we mean the set $\{v_{i_1}+v_{i_2}+\ldots+v_{i_s}: v_{i_1},v_{i_2},\ldots,v_{i_s}\in V, 1\leq s\leq k, v_m\neq v_n\ for\ m\neq n\}$.

Let V^k denote the set $\{0,[\frac{1}{4},\frac{1}{4}],J_A^{-1}([\frac{1}{4},\frac{1}{4}]),J_A^{-2}([\frac{1}{4},\frac{1}{4}]),\ldots,J_A^{-(k-1)}([\frac{1}{4},\frac{1}{4}])\}$.

The following lemma describes the structure of the preimages of the given rectangle R_{a_k}.

Lemma 5.2 *$J_A^{-k}R_{a_k} = R_{a_k}^k + \mathcal{C}(V^k)$, where $R_{a_k}^k$ is a rectangle in the torus.*

Proof. Observation 3.3 holds in the coordinate system determined by vectors x_s and x_u, but rectangles in the preimage are moved by the vector $[\frac{1}{4}, \frac{1}{4}]$ towards each other. Then by induction the statement is true for any k. $\square$

Lemma 5.3 *$diam_{x_u} R_{a_k}^k \longrightarrow 0$, when $k \longrightarrow +\infty$.*

Proof. This is the straightforward consequence of the fact that the vector x_u indicates the stable direction for J_A^{-1}. $\square$

Now we can prove Theorem 5.1.

Proof of Theorem 5.1. We will use the coordinates system given by the vectors x_s and x_u. By Lemmas 5.2 and 5.3 for any $k \geq 0$ the set $J_A^{-k}R_{a_k}$ consists of a finite number of rectangles, such that $diam_{x_u} R_{a_k}^k \longrightarrow 0$. Therefore

$$\exists (x_n)_{n=0}^{+\infty},\ x_n \in \mathbf{T}^2 : \bigcap_{k \geq 0} J_A^{-k} R_{a_k} \subset \bigcup_{x_n} W^s(x_n)$$

From Lemma 5.2 and the definition of sets $\mathcal{C}(V^k)$ it follows that (x_n) is a topological Cantor set. Moreover it is obvious that $\bigcap_{k \geq 0} J_A^{-k} R_{a_k} \subset R_{a_0}$.

To complete the proof it is enough to show that the intersection of the sets $W^s(x_n)$ with any rectangle consists of at most one interval, despite the fact that $W^s(x_n)$ is dense in the torus.

This follows from the fact that there is no $m, n \in \mathbf{Z}$ and $a \in \mathbf{R} \setminus \{0\}$ such that

$$R + [m, n] = R + a x_s,$$

where $R \subset \mathbf{R}^2$ is a rectangle on the plane. Indeed, if there were $m, n \in \mathbf{Z}$ and $a \in \mathbf{R} \setminus \{0\}$ such that $[m, n] = a x_s$ then $\frac{m}{n} = 1 - \sqrt{2}$, because $x_s = [1 - \sqrt{2}, 1]$. That is a contradiction. $\square$

6 Proof of the theorem on the correspondence to a one-sided subshift of finite type

Let $(\mathcal{P}, \mathcal{R})$ be the extended Markov partition for A. Moreover let T and T' be the extended Markov matrices associated respectively with the partition $\mathcal{P}$ and $\mathcal{R}$. By F we denote the map from the definition of the extended Markov partition.

We want to show that the following diagram commutes.

$$
\begin{tikzcd}
\Sigma_T^+ \arrow[r, "\sigma_T"] & \Sigma_T^+ \\
\Sigma_{T'}^+ \arrow[u, "F"] & \Sigma_T^+ \arrow[u, swap, "\sigma_T"] \\
\Sigma_{T'}^+ \arrow[u, "\sigma_{T'}"] & \Sigma_{T'}^+ \arrow[u, swap, "F"] \\
\mathbf{T}^2 \setminus B \arrow[u, "\Pi"] \arrow[r, swap, "A"] & \mathbf{T}^2 \setminus B \arrow[u, swap, "\Pi"]
\end{tikzcd}
$$

Then we define the maps $\Phi_1 : \mathbf{T}^2 \setminus B \to \Sigma_T^+$ and $\Phi_2 : \mathbf{T}^2 \setminus B \to \Sigma_T^+$ by

$$\Phi_1 := F \circ \sigma_{T'} \circ \Pi$$

$$\Phi_2 := \sigma_T \circ F \circ \Pi.$$

Let us remind that the map $F : \Sigma_{T'}^+ \to \Sigma_T^+$ was defined in the last chapter by

$$F((a_n)_{n \geq 0}) = (b_n)_{n \geq 0} \Longleftrightarrow A(R_{a_n}) = R_{b_n}, for\ n \geq 0.$$

Let us define $\Pi : \mathbf{T}^2 \setminus B \to \Sigma_{T'}^+$ by

$$\Pi(x) = (a_n)_{n \geq 0} :\Longleftrightarrow A^n(x) \in R_{a_n}, for\ n \geq 0.$$

Maps Φ_1 and Φ_2 are surjective because F is surjective and the set $\bigcap_{n=0}^{+\infty} A^{-n} R_{a_n}$ is not empty, for $(a_n) \in \Sigma_{T'}^+$, which is the consequence of the condition $(*)$.

The maps Φ_1 and Φ_2 are equal (and this map is a map Φ in Theorem 4.2), because F is a shift commuting map.

To show that

$$\sigma_T \circ \Phi = \Phi \circ A$$

it is enough to prove the equality

$$\sigma_T \circ F \circ \sigma_{T'} \circ \Pi(x) = \sigma_T \circ F \circ \Pi \circ A(x)$$

for any $x \in \mathbf{T}^2 \setminus B$.

Let us consider the left hand side of the equation, i.e. $\sigma_T \circ F \circ \sigma_{T'} \circ \Pi(x)$.

$$\Pi(x) = (a_n)_{n \geq 0} \Longleftrightarrow A^n(x) \in R_{a_n}, for\ n \geq 0;$$

$$\sigma_{T'}((a_n)_{n\geq 0}) = (a_{n+1})_{n\geq 0}$$
$$F((a_{n+1})_{n\geq 0}) = (b_n)_{n\geq 0} \Longleftrightarrow A(R_{a_{n+1}}) = R_{b_n}, for\ n \geq 0.$$
$$\sigma_T((b_n)_{n\geq 0}) = (b_{n+1})_{n\geq 0}.$$

Therefore the sequence $(b_n)_{n\geq 0}$ satisfies condition

$$A(R_{a_{n+1}}) = R_{b_n}, for\ n \geq 0$$

$$A^n(x) \in R_{a_n}, for\ n \geq 0,$$

which is the same as

$$A^{n+1}(x) \in A(R_{a_n}) = R_{b_{n-1}}, for\ n \geq 1,$$

hence

$$A^{n+2}(x) \in R_{b_n}, for\ n \geq 0.$$

Consider in turn the right hand side of the equation $\sigma_T \circ F \circ \Pi \circ A(x)$.

$$\Pi(A(x)) = (c_n)_{n\geq 0} \Longleftrightarrow A^{n+1}(x) \in R_{c_n}, for\ n \geq 0;$$
$$F((c_n)_{n\geq 0}) = (d_n)_{n\geq 0} \Longleftrightarrow A(R_{c_n}) = R_{d_n}, for\ n \geq 0.$$
$$\sigma_T((d_n)_{n\geq 0}) = (d_{n+1})_{n\geq 0}.$$

Therefore the sequence $(d_n)_{n\geq 0}$ is such that

$$A(R_{c_n}) = R_{d_n}, for\ n \geq 0$$

$$A^{n+1}(x) \in R_{c_n}, for\ n \geq 0.$$

Consequently

$$A^{n+2}(x) \in A(R_{c_n}) = R_{d_n}, for\ n \geq 0,$$

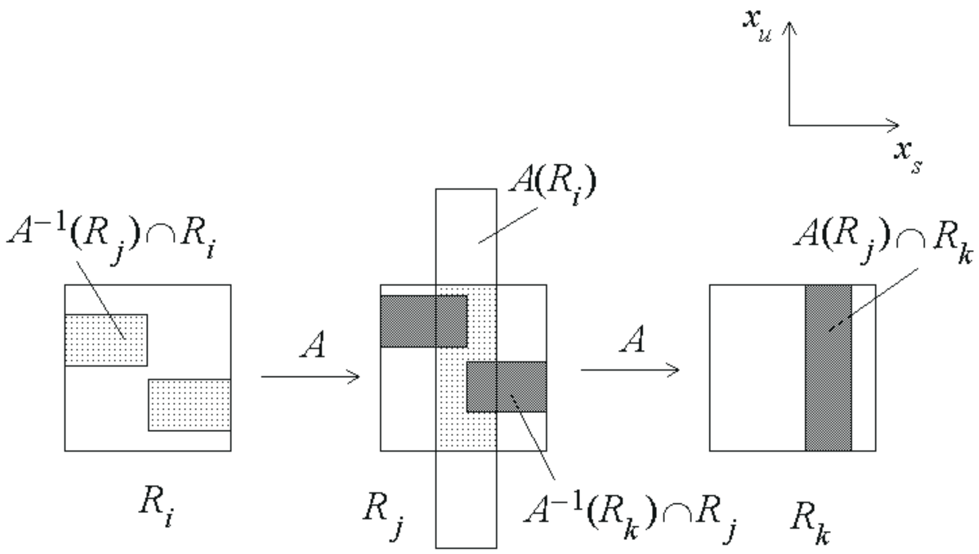
which is equivalent to

$$A^{n+2}(x) \in R_{d_n}, for\ n \geq 0.$$

To finish the proof it is enough to notice that $(b_n)_{n\geq 0} = (d_n)_{n\geq 0}$, because the rectangles in the partition are disjoint. $\Box$

7 A new method of obtaining the transition rule

Theorem 7.1 *Let $\mathcal{R}$ be a partition of the torus satisfying the good intersection property for the preimages. Then $\mathcal{R}$ satisfies the transition rule.*



Picture 1: The transition rule is the consequence of the good intersection property

Proof. The transition rule is the consequence of the following two properties, that are satisfied for any triple of the rectangles R_i, R_j, $R_k \in \mathcal{R}$.

If $A(R_i) \cap R_j \neq \emptyset$ then the intersection is a rectangle that crosses R_j all the way along the unstable direction, because $A(\partial^s \mathcal{R}) \subset \partial^s \mathcal{R}$.

The set $A^{-1}(R_k) \cap R_j$, if non-empty, consists of two rectangles and $A^{-1}(R_k)$ crosses any unstable section of the rectangle R_j, because of the good intersection property.

Therefore if $A(R_i) \cap R_j \neq \emptyset$ and $A^{-1}(R_k) \cap R_j \neq \emptyset$ then $R_i \cap A^{-1}(R_j) \cap A^{-2}(R_k) \neq \emptyset$. The conclusion follows by induction. The idea of the proof is illustrated in Picture 1. $\square$

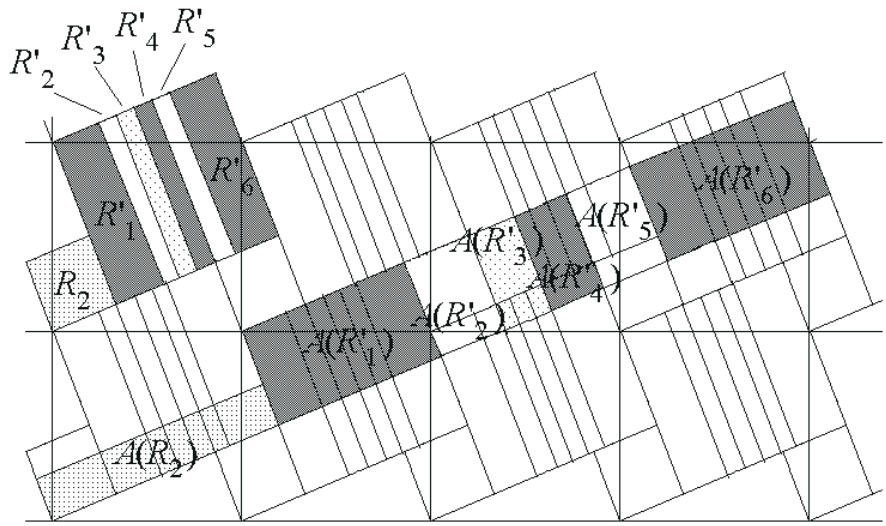
8 Final remarks

The map F defined in Section 5, Step 4 is an infinite-to-one factor map, because among distinguished 2×2 submatrices of the matrix T' there is a matrix of the form

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

An example of such a matrix is a submatrix with rows corresponding to the symbols $g', g"$ and columns $2', 2"$. Therefore in the sequences in which symbol g precedes 2, there are diamonds and that implies that T' induces a infinite-to-one factor map. Compare with the remark 2.5.6, definition 2.5.4 (chapter devoted to automorphisms) and an Introduction .

9 Adler and Weiss' method fails in the case of an endomorphism



Picture 2: Partition obtained by Adler and Weiss' method

Adler and Weiss' method gives a partition that does not satisfy transition rule, which is necessary to obtain a symbolic representation of a dynamical system. More precisely it is described by

Theorem 9.1 *Let $\mathcal{R} = \{R_i \subset \mathbf{T}^2 : 1 \leq i \leq n\}$ be a partition of the torus and let $T = [t_{ij}]_{i,j=1}^n$ be defined as follows*

$$t_{ij} = \begin{cases} 1 & \textit{when } A^{-1}(R_i) \cap R_j \neq \emptyset \\ 0 & \textit{when else} \end{cases}$$

Moreover assume that there is at least one non-zero entry in each row of the matrix T and for any sequence $(a_k)_{k\geq 0}$ admissible by the matrix T the set $\bigcap_{k=0}^{+\infty} A^{-k}(R_{a_k})$ is non-empty.

Then $\mathcal{R}$ satisfies the transition rule.

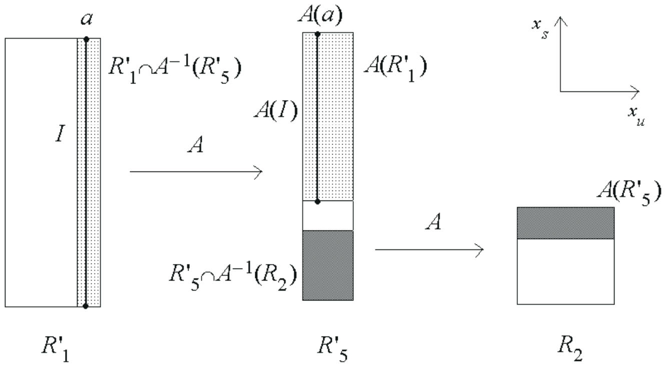
Proof.

Let us assume that $R_{a_k} \cap A^{-1}(R_{a_{k+1}}) \neq \emptyset$, for any k such that $0 \leq k \leq m-1$ and for a given $m \geq 3$. Then the sequence $(a_0, a_1, \ldots, a_m)$ is admissible by the matrix T. It can be extended to the sequence $(a_k)_{k\geq 0}$ admissible by T, because in each row of the matrix T there is at lest one non-zero entry.

Finally the set $\bigcap_{k=0}^{m} A^{-k}(R_{a_k})$ is non-empty, because it includes the set $\bigcap_{k=0}^{+\infty} A^{-k}(R_{a_k})$ which is not empty by the assumption. □

Picture 2 illustrates partition obtained by Adler and Weiss' method and images of the rectangles from this partition.

Analyzing this picture one can notice that rectangles R_1', R_5', R_2 do not satisfy the transition rule, which is clearly visible in Picture 3. To see this one can compare Picture 3 with Picture 1 where the transition rule is satisfied. We omit technical details of the proof.



Picture 3: Transition rule not satisfied

10 Why do we choose such an extension of Markov partitions?

10.1 The aim of the construction of the extended Markov partition

Our aim is to find a partition of the torus, that could be used to obtain a symbolic model for the considered semi-dynamical system. Moreover we expect this model to be a one-sided subshift of finite type, because in the invertible case it was a subshift of finite type. Let us distinguish properties that will explain why we choose partitions constructed in chapter 5 and why we propose such a definition of the extended Markov partition in paragraph 3.

10.2 Properties that are significant for Markov partitions

We already know from Section 9 that the transition rule is the necessary condition to associate a one-sided subshift of finite type with a given toral endomorphism. A sequence $(a_n)_{n\geq 0}$ corresponds to all points in the set $\bigcap_{n=0}^{+\infty} A^{-n}(R_{a_n})$ and this is why we want this set to be non-empty. Therefore we search a partition that satisfies

• PROPERTY A: Transition rule.

In the case of an automorphism the transition rule is the consequence of the good intersection property which follows from the Markov condition. In the non-invertible case $A^{-1}(W^u(0,0)) \not\subset W^u(0,0)$ and Markov condition is not satisfied for any partition. That is why we must search for a new way to obtain the transition rule.

Moreover, we want to obtain the most accurate model, that is we want the set $\bigcap_{n=0}^{+\infty} A^{-n}(R_{a_n})$ to be as small as possible for the sequences $(a_n)_{n\geq 0}$ admissible by the matrix T.

In the invertible case we consider the set $\bigcap_{n=-\infty}^{+\infty} A^{-n}(R_{a_n})$, which is a single point and this is why we obtain a conjugacy. In the non-invertible case we consider the set $\bigcap_{n=0}^{+\infty} A^{-n}(R_{a_n})$ which, at best, can be a union of intervals. Thus we need the following condition.

• PROPERTY B: For any sequence $(a_n)_{n=0}^{+\infty}$ admissible by the extended Markov matrix the set $\bigcap_{n=0}^{+\infty} A^{-n}(R_{a_n}) \subset I \times \mathcal{C}$, where I is an interval parallel to the stable direction and $\mathcal{C}$ is a topological Cantor set.

There is no partition that satisfies both property A and B. That is why we use two partitions, one satisfying property A and an other that satisfies property B. A pair of such partitions can be chosen in many ways, but to obtain Theorem 4.2 these partitions must be compatible, in some way, which is

• PROPERTY C: There exists a shift commuting map from $\Sigma_{T'}^{+}$ to Σ_{T}^{+}, where T and T' are the extended Markov matrices associated respectively with the partitions satisfying property A and B.

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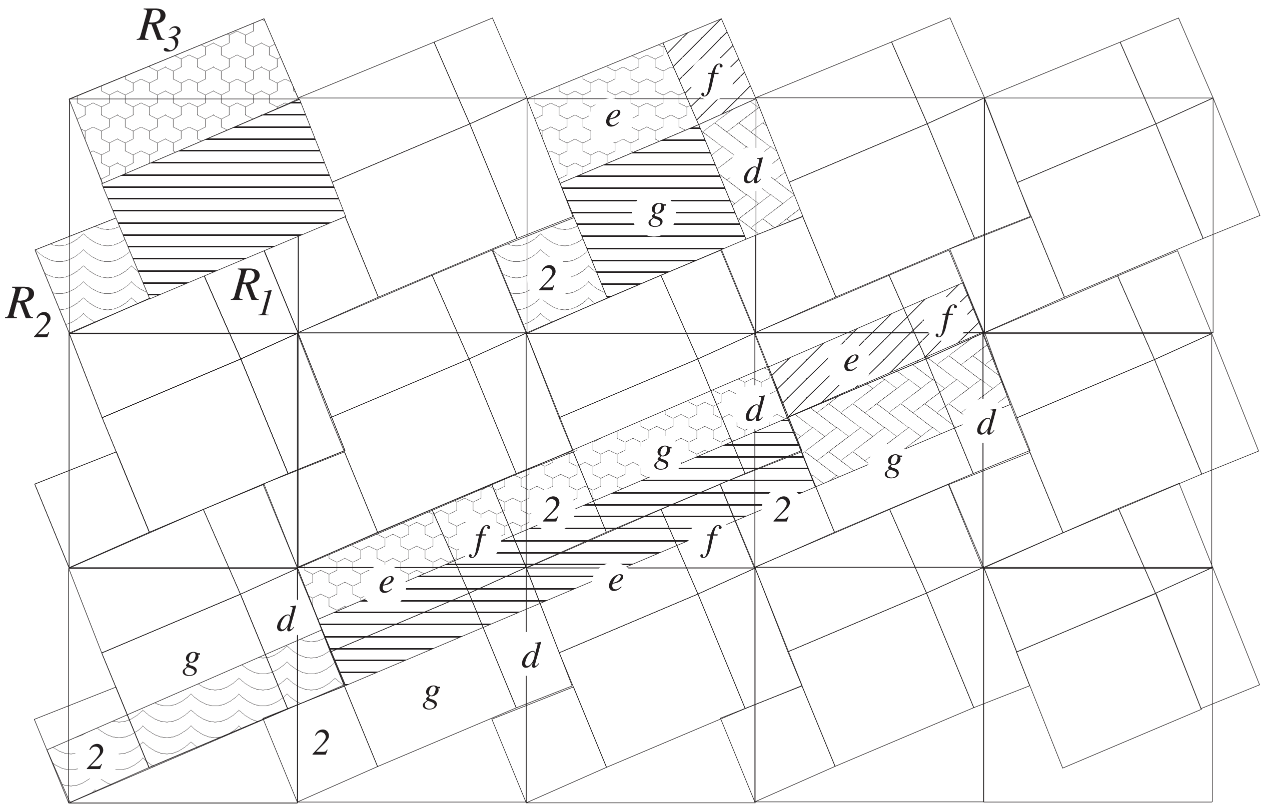


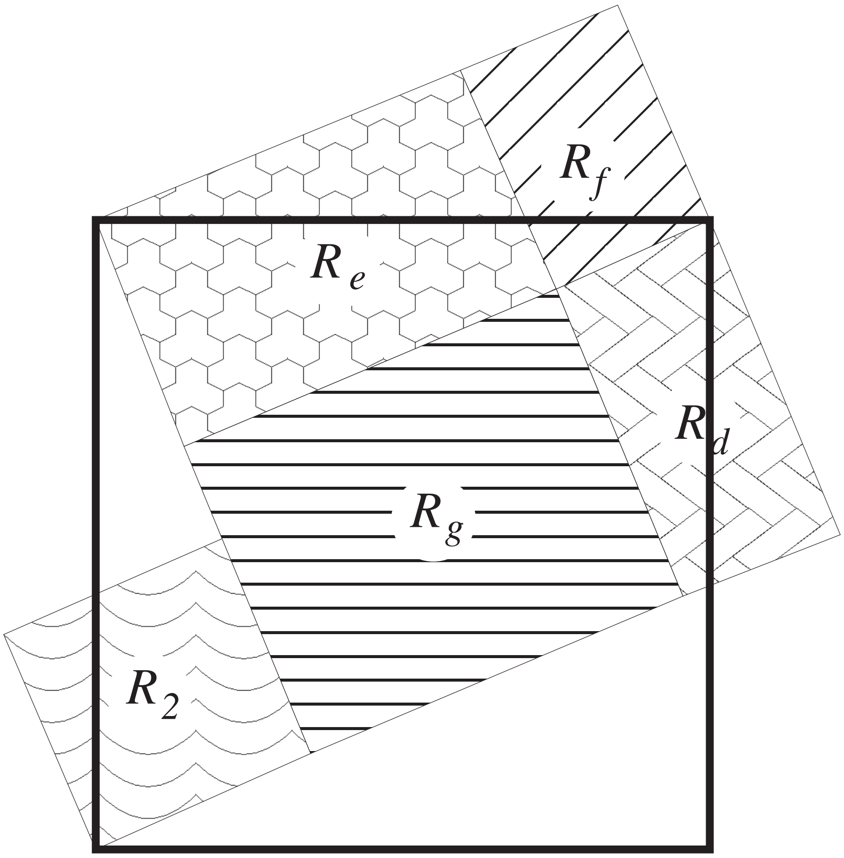
$$R_g = (R_{1'} \cup R_{2''} \cup R_{3'} \cup R_{3''}) \cap R_1$$

$$R_d = R_{1''} \cap R_1$$

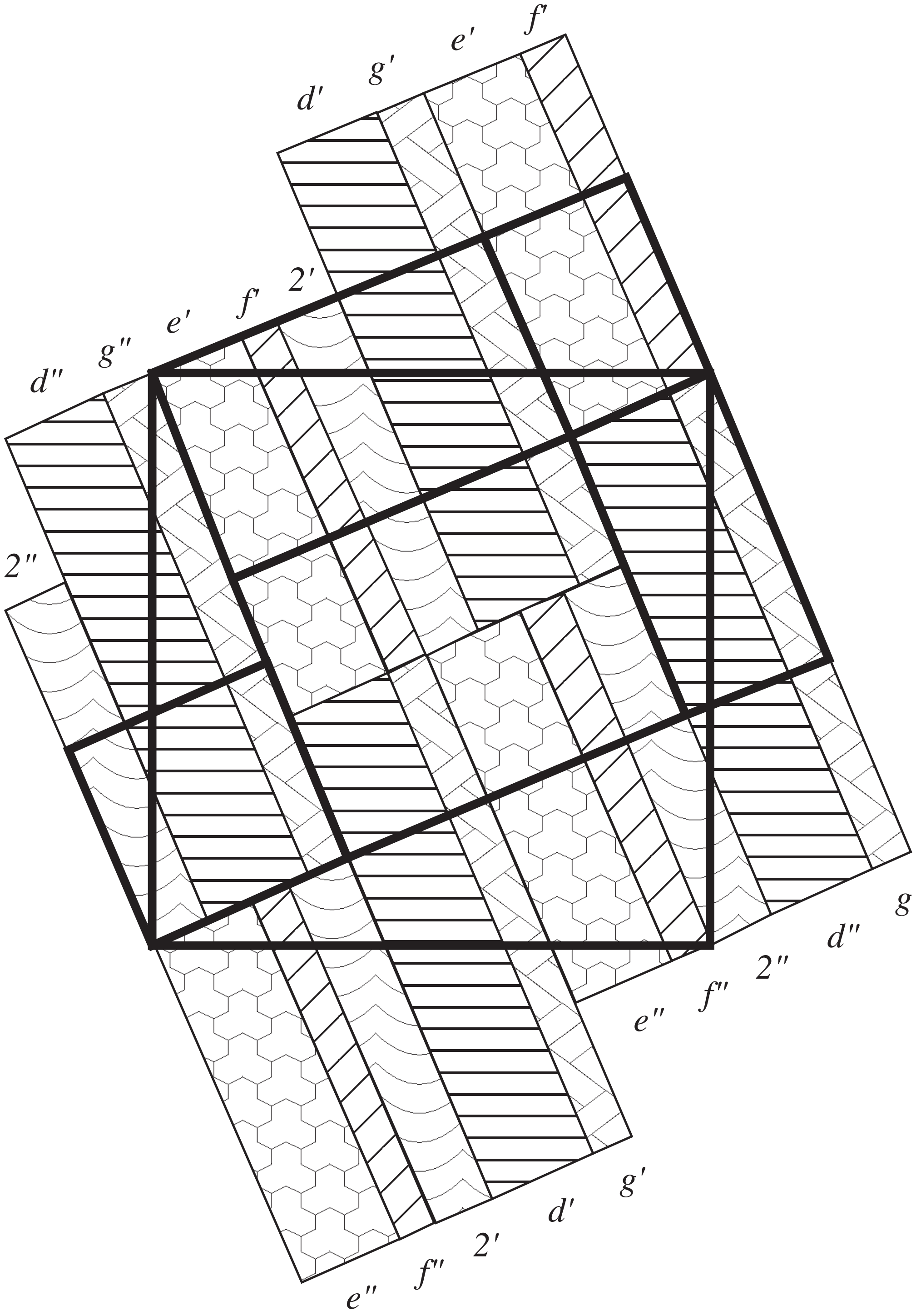
$$R_e = (R_{1'} \cup R_{2'} \cup R_{3'}) \cap R_3$$

$$R_f = R_{3''} \cap R_3$$

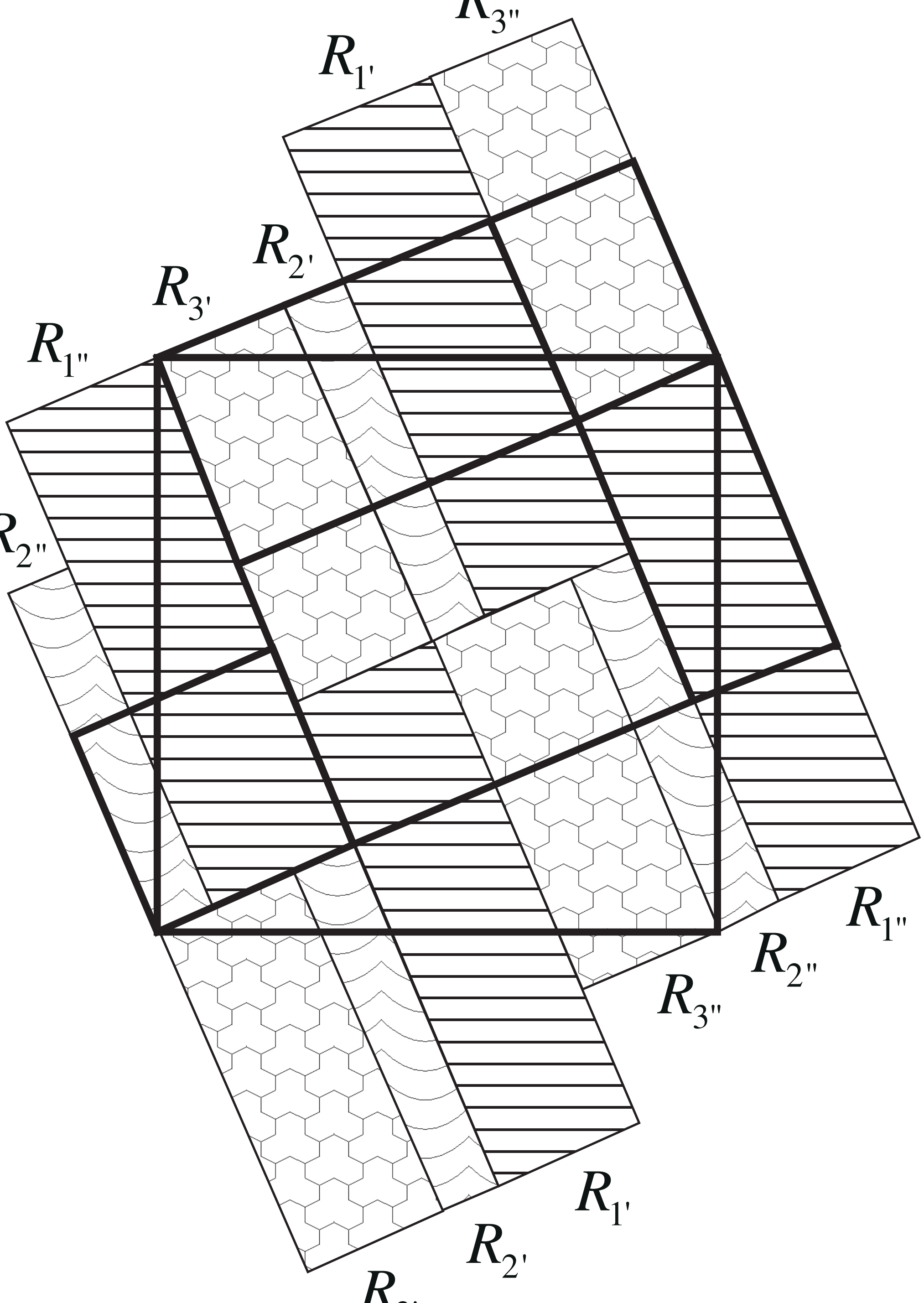
Picture A



Picture B



Picture C



Picture D