Complicated dynamics in nonautonomous ODE's.

Leszek Pieniążek and Klaudiusz Wójcik *

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Abstract

We present a topological method for detecting complicated dynamics in nonautonomous ordinary differential equations (not necesserily periodic with respect to the time variable). Our main result gives a sufficient condition for existence of a class of solutions, whose presence displays some chaotic features of the dynamics. The method is based on the Ważewski Retract Theorem and the Lefschetz Fixed Point Theorem. Some applications to the nonautonomous systems on the plane are considered.

Keywords: dynamical systems, local process, nonautonomous ODE's, multibump solutions, periodic solutions, Lefschetz number, fixed point index, Ważewski sets.

1 Introduction.

In this note we study a topological method for detecting complicated dynamics in the local processes generated by the nonautonomous ordinary differential equations (not necesserily periodic with respect to the time variable). We show the existence of a class of solutions, whose presence displays some chaotic features of the dynamics. The results presented here, in the same spirit as in [S1],

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[SW], [W1] are inspired by a lot of papers on the existence of the multibump orbits ([AM], [BM], [CZN], [CZR], [CZMN], [MN]) starting with the novel minimax method in [Se]. In the context of the Lagrangian systems the multibump orbits mean the orbits which are close to chains of homoclinics of the limit system (see [AM], [BM], [CZR]). The results of [BM] are based on the variational version of the Birkhoff-Smale-Shilnikov theory. In contrast to the above method, we do not need any global information on particular solutions of the considered equation. Our method for detecting chaotic dynamics is based on the existence of some sets, called admissible proper sets and the knowledge of the values of some topological invariants of these sets. The notion of the proper pair is based on the concept of the Ważewski set ([C]) and is an inessential modification of the periodic block in [S1]. In the case when the right-hand side of the considered equation is periodic with respect to the time variable, our main result (up to slightly different notation) improves Th.2 in [SW] (comp. [W1]). In fact, the method presented in this paper, is addapted to the general nonautonomous systems from the periodic case in [SW], [W1], [W4]. In order to apply the topological method introduced in [SW], we have to prove the existence of some sets, called periodic isolating segments, in the extended phase space (see [SW]). Basic property of the segment is that in any point on its boundary the vector field is directed outward or inward with respect to the segment (compare the notion of the isolating block in the Conley index theory). Notice that an admissible proper pair (in our sense) can be easily obtained by gluing translated copies of a periodic isolating segment. It was observed by Roman Srzednicki in [S1], that the fixed point index of the Poincaré map inside the segment is equal to the Lefschetz number of the monodromy homeomorphism given by the segment (see Th.7.1 in [S1]). Our Proposition 1 is a non-periodic version of this result. Theorem 2 in [SW] gives a sufficient condition for the chaotic dynamics in the periodic systems in the sense that the Poincaré map is semiconjugated to the shift on two symbols and the counterimage (by the semiconjugacy) of any periodic point in the shift contains a periodic point of the Poincaré map. It follows by our Theorem 1, that any small perturbation of the T-periodic system for which the results in [SW] show chaos has also complicated dynamics. In non-periodic case we are not able to prove the existence of periodic solutions but the map after time T is still semiconjugated to the Bernoulli shift on some compact set. In particular, the topological entropy is positive.

2 Proper pairs.

Assume that X is a metric space and $\varphi : D \to X$ is a continuous mapping, $D \subset \mathbb{R} \times X \times \mathbb{R}$ is an open set. We will denote by $\varphi_{(\sigma,t)}$ the function $\varphi(\sigma, \cdot, t)$.

φ is called a local process if the following conditions are satisfied

(1) $\forall \sigma \in \mathbb{R},\, x \in X : \{t \in \mathbb{R} : (\sigma, x, t) \in D\}$ is an interval,

(2) $\forall \sigma \in \mathbb{R} : \varphi_{(\sigma,0)} = \mathrm{id}_X$

(3) $\forall \sigma \in \mathbb{R} : \varphi_{(\sigma,s+t)} = \varphi_{(\sigma+s,t)} \circ \varphi_{(\sigma,s)}$, If $D = \mathbb{R} \times X \times \mathbb{R}$, we call φ a (global) process. For $(\sigma, x) \in \mathbb{R} \times X$ the set

$$\{(\sigma + t, \varphi_{(\sigma,t)}(x)) \in \mathbb{R} \times X : (\sigma, x, t) \in D\}$$

is called the trajectory of (σ, x) in φ. If T is a positive number such that

(4) $\forall \sigma, t \in \mathbb{R} : \varphi_{(\sigma+T,t)} = \varphi_{(\sigma,t)}$

we call φ a T-periodic local process. It follows, that the interval in (1) is open and, by (2), it contains 0. Since the domains of the both maps in (3) are equal, $(\sigma, x, s + t) \in D$ if and only if $(\sigma, x, s) \in D$ and $(\sigma + s, \varphi_{(\sigma,s)}(x), t) \in D$.

A local process φ on X determines a local flow Φ on $\mathbb{R} \times X$ by the formula

$$\Phi_t(\sigma, x) = (\sigma + t, \varphi_{(\sigma,t)}(x)).$$

Remark 1 The differential equation

$$(*) \quad \dot{x} = f(t, x)$$

such that f is regular enough to guarantee the uniqueness of solutions of Cauchy problems associated to $(*)$ generates a local process as follows. For $x(t_0, x_0; \cdot)$ the solution of $(*)$ such that $x(t_0, x_0; t_0) = x_0$ we put

$$\varphi_{(t_0,\tau)}(x_0) = x(t_0, x_0; t_0 + \tau).$$

If f is T-periodic with respect to t then φ is a T-periodic local process and in order to determine all T-periodic solutions of the equation $(*)$ it suffices to look for fixed points of $\varphi_{(0,T)}$ (called the Poincaré map).

We will use the following notation: by $\pi_1 : \mathbb{R} \times X \to \mathbb{R}$ and $\pi_2 : \mathbb{R} \times X \to X$ we denote the projections and for every $Z \subset \mathbb{R} \times X$ and $t \in \mathbb{R}$ we put

$$Z_t = \{x \in X : (t, x) \in Z\}.$$

Let (U, U^-) be a pair of subsets of $\mathbb{R} \times X$ (i.e. $U^- \subset U \subset \mathbb{R} \times X$). We call (U, U^-) a proper pair (for the process φ) and U^- the exit set of U if:

(i) U and U^- are closed ENR's, U_0, U_0^- are compact,

(ii) there exists a homeomorphism

$$h : \mathbb{R} \times (U_0, U_0^-) \longrightarrow (U, U^-)$$

such that $\pi_1 = \pi_1 \circ h$,

(iii) for every $\sigma \in \mathbb{R}$ and $x \in \partial U_\sigma$ there exists a $t \in \mathbb{R}$ such that $\varphi_{(\sigma,t)}(x) \notin U_{\sigma+t}$,

(iv) $U^- = \{(\sigma, x) \in U : \ \exists s_n > 0 \ s_n \to 0 : \varphi_{(\sigma,s_n)}(x) \notin U_{\sigma+s_n}\}$.

Define a map

$$\tau_U : U_0 \ni x \longrightarrow \sup\{t \geq 0 : \forall s \in [0,t] : \Phi_s(0,x) \in U\} \in [0,\infty].$$

τ_U is continuous (by the argument in a proof of the Ważewski Theorem, [C], [S1]).

Let (U, U^-) be a proper pair, $U_a = U_b$, $U_a^- = U_b^-$ for some a, $b \in \mathbb{R}$, $a < b$. Define a homeomorphism

$$h_{a,b} : (U_a, U_a^-) \longrightarrow (U_b, U_b^-) = (U_a, U_a^-)$$

by $h_{a,b}(x) = \pi_2(h(b, \pi_2 h^{-1}(a,x)))$ for $x \in U_a$.

Geometrically, $h_{a,b}$ moves a point $x \in U_a$ to $U_b = U_a$ along the arc $h([a,b] \times \{\pi_2 h^{-1}(a,x)\})$. Consider the automorphism

$$\mu_{U_{a,b}} : H(U_a, U_a^-) \longrightarrow H(U_a, U_a^-)$$

induced by $h_{a,b}$ in the singular homology with rational coefficients. Recall that its Lefschetz number is defined as

$$\mathrm{Lef}(\mu_{U_{a,b}}) = \sum_{n=0}^{\infty} (-1)^n \mathrm{tr}\ H_n(h_{a,b}).$$

In particular, if $\mu_{U_{a,b}} = \mathrm{id}_{H(U_a, U_a^-)}$ then $\mathrm{Lef}(\mu_{U_{a,b}})$ is equal to the Euler characteristic $\chi(U_a, U_a^-)$.

In the sequel we will use the following non-periodic version of the Theorem 7.1 in [S1]

Proposition 1 *If (U, U^-) is a proper pair, $U_a = U_b$, $U_a^- = U_b^-$ for some a, $b \in \mathbb{R}$, $a < b$ then the set*

$$F_{U_{a,b}} = \{x \in X : \varphi_{(a,b-a)}(x) = x,\ \forall t \in [0, b-a] : \varphi_{(a,t)}(x) \in U_{a+t}\}$$

is compact and open in the set of fixed points of $\varphi_{(a,b-a)}$ and the fixed point index of $\varphi_{(a,b-a)}$ in $F_{U_{a,b}}$ is given by

$$\mathrm{ind}(\varphi_{(a,b-a)}, F_{U_{a,b}}) = \mathrm{Lef}(\mu_{U_{a,b}}).$$

Proof: Our proof of Proposition 1 is simpler but similar in the spirit to the proof of Theorem 7.1 in [S1]. By Lemma 2.3.1 in [S2] $F_{U_{a,b}}$ is compact and open in the set of fixed points of the $\varphi_{(a,b-a)}$. Let $\tau = \tau_U : U \to [0,\infty]$. For $s \in [a,b]$ we define a homeomorphism

$$h_{s,b} : (U_s, U_s^-) \to (U_b, U_b^-),$$

by $h_{s,b}(x) = \pi_2 h(b, \pi_2 h^{-1}(s,x))$. Consider a homotopy $H : (U_a, U_a^-) \times [0,1] \to (U_b, U_b^-) = (U_a, U_a^-)$ given by

$$H(x,t) = \begin{cases} h_{a+\tau(x),b}(\varphi_{(a,\tau(x))}(x)), & \tau(x) \leq (1-t)(b-a) \\ h_{a+(1-t)(b-a),b}(\varphi_{(a,(1-t)(b-a))})(x)), & \tau(x) \geq (1-t)(b-a) \end{cases}$$

Put $H_t(x) = H(x,t)$. It is easy to check that $H_t(x) = h_{a,b}(x)$ for $x \in U_a^-$, $t \in [0,1]$ and $H_1 = h_{a,b}$. By the homotopy property of the Lefschetz number

$$\mathrm{Lef}(h_{a,b}) = \mathrm{Lef}(H_0).$$

Moreover

$$\mathrm{Lef}(H_0) = \mathrm{ind}(\varphi_{(a,b-a)}, F_{U_{a,b}}) + \mathrm{ind}(h_{a,b}, \mathrm{Fix}(h_{a,b}|_{U_a^-})),$$

and by the commutativity property of the fixed point index

$$\mathrm{ind}(h_{a,b}, \mathrm{Fix}(h_{a,b}|_{U_a^-})) = \mathrm{Lef}(h_{a,b}|_{U_a^-}),$$

so the proof is complete, because

$$\mathrm{Lef}(\mu_{U_{a,b}}) = \mathrm{Lef}(h_{a,b}) - \mathrm{Lef}(h_{a,b}|_{U_a^-}).$$

□

3 Main result.

Let (U, U^-) be a proper pair for the process φ. We call U admissible iff there is a sequence $\{t_n\}_{n\in\mathbb{Z}} \subset \mathbb{R}$ such that $t_n < t_{n+1}$, $U_{t_n} = U_{t_0}$, $U_{t_n}^- = U_{t_0}^-$ for all $n \in \mathbb{Z}$.

Suppose that (U, U^-), (Z, Z^-) are proper pairs, $U \subset Z$ are admissible with the same sequence $\{t_n\}_{n\in\mathbb{Z}}$, $U_{t_0} = Z_{t_0}$, $U_{t_0}^- = Z_{t_0}^-$ and the set $P = \{t \in \mathbb{R} : U_t = Z_t, U_t^- = Z_t^-\}$ is unbounded both from below and above. Assume that for any $n \in \mathbb{Z}$ there is an $s \in (t_n, t_{n+1})$ such that $U_s \neq Z_s$.
Consider the following conditions

(a) for all $n \in \mathbb{Z}$ $\mu_{U_{t_n,t_{n+1}}} = \mathrm{id}_{H(U_{t_0}, U_{t_0}^-)}$,

(b) there is an $n_0 \in \mathbb{N} \setminus \{1\}$ and the authomorphism $G : H(U_{t_0}, U_{t_0}^-) \to H(U_{t_0}, U_{t_0}^-)$ such that $G^{n_0} = \mathrm{id}_{H(U_{t_0}, U_{t_0}^-)}$, for all $n \in \{1, \ldots, n_0 - 1\}$ $\mathrm{Lef}(G^n) = \mathrm{Lef}(G)$ and $\mu_{Z_{t_n,t_{n+1}}} = G$ $(n \in \mathbb{N})$,

(c) $\mathrm{Lef}(G) \neq \chi(U_{t_0}, U_{t_0}^-) \neq 0$.

Remark 2 Let us consider the planar differential equation

$$\dot{z} = \mathrm{e}^{it} \bar{z}^n.$$

From the phase portrait one can deduce (comp. [S1]) the existence of the admissible proper pair Z with the sequence $t_k = 2\pi k$ such that Z_0 is a regular $2(n+1)$-gon, Z_0^- consist of $n+1$ disjoint segments and both the sets Z_0 and Z_0^- are invariant with respect to the rotation by the angle $\frac{2\pi}{n+1}$. It is easy to see that

$$\mathrm{Lef}(\mu_{Z_{t_k,t_{k+1}}}) = \ldots = \mathrm{Lef}(\mu^n_{Z_{t_k,t_{k+1}}}) = 1,$$

$$\mu^{n+1}_{Z_{t_k,t_{k+1}}} = \mathrm{id}_{H(Z_0, Z_0^-)},$$

so the condition (b) holds with $n_0 = n + 1$.

We prove the following

Theorem 1 *Under the assumptions (a), (b), (c) for any subset $S \subset \mathbb{Z}$ there is $x_0 \in U_{t_0}$ such that*
(1) for all $t \in \mathbb{R}$ $\varphi_{(t_0,t)}(x_0) \in Z_{t_0+t}$
(2) if $n \in S$ then there exists $t \in (t_n - t_0, t_{n+1} - t_0)$ such that $\varphi_{(t_0,t)}(x_0) \notin U_{t_0+t}$,
(3) if $n \notin S$ then for all $t \in [t_n - t_0, t_{n+1} - t_0]$ $\varphi_{(t_0,t)}(x_0) \in U_{t_0+t}$.

For $S \subset \mathbb{Z}$, by U^S we denote the proper pair such that if $n \in S$ then

$$U^S \cap ([t_n, t_{n+1}] \times X) = Z \cap ([t_n, t_{n+1}] \times X),$$

and if $n \notin S$ then

$$U^S \cap ([t_n, t_{n+1}] \times X) = U \cap ([t_n, t_{n+1}] \times X).$$

Assume that S is a finite set and $\mathrm{card}(S) = k$ for some $k \in \mathbb{N}$. Let $S = \{n_1, \ldots, n_k\}$, a, $b \in \{t_n : n \in \mathbb{Z}\}$ and $a < t_{n_1} < \ldots < t_{n_k} < b$. Recall that the set

$$F_{U^S_{a,b}} = \{x \in X : \varphi_{(a,b-a)}(x) = x, \forall t \in [0, b-a] : \varphi_{(a,t)}(x) \in U^S_{a+t}\}$$

is compact and open in the set of fixed points of $\varphi_{(a,b-a)}$. Observe that if $n \in S$, $x \in F_{U^S_{a,b}}$ then $\tau_U(\varphi_{(a,t_n-a)}(x)) \neq t_{n+1} - t_n$. For any subset $L \subset S$ by F^L we denote the set of points $x \in F_{U^S_{a,b}}$ such that if $n \in L$ then $\tau_U(\varphi_{(a,t_n-a)}(x)) < t_{n+1} - t_n$ and if $n \in S \setminus L$ then $\tau_U(\varphi_{(a,t_n-a)}(x)) > t_{n+1} - t_n$. The sets F^L over all subsets $L \subset S$ form a compact and disjoint covering of $F_{U^S_{a,b}}$. The proof of Theorem 1 is based on the following

Lemma 1 *If $\mathrm{card}(S) = k$ then*

$$\mathrm{ind}(\varphi_{(a,b-a)}, F^S) = \sum_{l=0}^{k} (-1)^{k-l} \binom{k}{l} \mathrm{Lef}(G^l).$$

Proof: The case $k = 0$ follows immediately from Proposition 1. For $k \geq 1$ we use the induction with respect to k. Let $k = 1$. Since F^S and $F^{\emptyset}$ form a compact and disjoint covering of $F_{U^S_{a,b}}$, by the additivity property of the fixed point index,

$$\mathrm{ind}(\varphi_{(a,b-a)}, F_{U^S_{a,b}}) = \mathrm{ind}(\varphi_{(a,b-a)}, F^S) + \mathrm{ind}(\varphi_{(a,b-a)}, F^{\emptyset}),$$

so by Proposition 1

$$\mathrm{ind}(\varphi_{(a,b-a)}, F^S) = \mathrm{Lef}(G) - \chi(U_{t_0}, U^-_{t_0}).$$

Assume now that the lemma holds for $p \leq k$. We prove it for $k+1$. Again by the additivity of the fixed point index,

$$\mathrm{ind}(\varphi_{(a,b-a)}, F_{U^S_{a,b}}) =$$

$$\mathrm{ind}(\varphi_{(a,b-a)}, F^S) + \sum_{L\subset S, L\neq S} \mathrm{ind}(\varphi_{(a,b-a)}, F^L).$$

By Proposition 1 and the inductive step,

$$\mathrm{ind}(\varphi_{(a,b-a)}, F^S) =$$

$$\mathrm{Lef}(G^{k+1}) - \sum_{s=0}^{k} \binom{k+1}{s} \sum_{l=0}^{s} (-1)^{s-l} \binom{s}{l} \mathrm{Lef}(G^l) =$$

$$\mathrm{Lef}(G^{k+1}) - \sum_{s=0}^{k} \sum_{l=0}^{s} (-1)^{s-l} \binom{k+1}{s} \binom{s}{l} \mathrm{Lef}(G^l) =$$

$$\mathrm{Lef}(G^{k+1}) - \sum_{s=0}^{k} \sum_{l=0}^{s} (-1)^{s-l} \binom{k+1}{l} \binom{k+1-l}{k+1-s} \mathrm{Lef}(G^l).$$

Let $s_0 \in \{0, \dots, k\}$ be fixed. We show that the coefficient of $\mathrm{Lef}(G^{s_0})$ equals $(-1)^{k+1-s_0}\binom{k+1}{s_0}$. It is easy to see that this coefficient equals

$$\left[- \sum_{r=s_0}^{k} (-1)^{r-s_0} \binom{k+1-s_0}{k+1-r} \right] \binom{k+1}{s_0}.$$

We put $m = k+1-s_0$. Then

$$\sum_{r=s_0}^{k} (-1)^{r-s_0} \binom{k+1-s_0}{k+1-r} =$$

$$\sum_{w=0}^{m-1} (-1)^w \binom{m}{m-w} = \sum_{w=0}^{m-1} (-1)^w \binom{m}{w}.$$

By

$$\sum_{w=0}^{m} (-1)^w \binom{m}{w} = 0,$$

we have that if $m = k+1-s_0$ is even then $\sum_{w=0}^{m-1}(-1)^w \binom{m}{w} = -1$ and if $m = k+1-s_0$ is odd then $\sum_{w=0}^{m-1} \binom{m}{w} = 1$. This finishes the proof.

□

Corollary 1 (1) *Under the assumptions of Theorem 1, if* $\mathrm{card}(S) = k$ *then*

$$\mathrm{ind}(\varphi_{(a,b-a)}, F^S) = (\sum_{n_0|s} (-1)^{k-s} \binom{k}{s})(\chi(U_0, U_0^-) - \mathrm{Lef}(G)).$$

(2) *In particular, if* n_0 *is even then* $\mathrm{ind}(\varphi_{(a,b-a)}, F^S) \neq 0$ *and if* n_0 *is odd then* $\mathrm{ind}(\varphi_{(a,b-a)}, F^S) \neq 0$ *iff* k *is not an odd multiplicity of* n_0.

Proof: (1) It follows from Lemma 1 because $\sum_{s=0}^{k}(-1)^{k-s}\binom{k}{s}=0$ and

$$\mathrm{Lef}(G^l)=\begin{cases}\chi(U_0,U_0^-) & ,n_0\mid l\\ \mathrm{Lef}(G) & ,otherwise.\end{cases}$$

(2) If n_0 is even then $(-1)^{k-s}=(-1)^k$ for all s such that $n_0 \mid s$, so the conclusion follows by (1). The case of n_0 odd will be proved in Corollary 2 in the appendix.

□

Proof of the Theorem 1: Assume first that S is a finite set. Let $a_n, b_n \in P$, $a_n \to -\infty$, $b_n \to +\infty$. If a_n is sufficiently small and b_n sufficiently large then by Lemma 1 there is $y_n \in F^S \subset F_{U^S_{a_n,b_n}}$. By the compactness of U_{t_0} there is a subsequence of the sequence $\{\varphi_{(a_n,t_0-a_n)}(y_n)\} \subset U_{t_0}$ which converges to some $x_0 \in U_{t_0}$. The standard arguments show that the trajectory of x_0 is defined on the whole real line. The conditions (1) and (3) are easy to verify so we claim only (2). Let $t_{n_0} \in S$ be fixed. Because

$$\varphi_{(a,t_{n_0}-a)}(y_n)=\varphi_{(t_0,t_{n_0}-t_0)}(\varphi_{(a,t_0-a)}(y_n)),$$

so by the continuity of τ_U

$$\tau_U(\varphi_{(a,t_{n_0}-a)}(y_n)) \to \tau_U(\varphi_{(t_0,t_{n_0}-t_0)}(x_0)).$$

On the other hand, by the definition of the sequence $\{y_n\}$ we have

$$0<\tau_U(\varphi_{(a,t_{n_0}-a)}(y_n))<t_{n_0+1}-t_{n_0},$$

so

$$0\le\tau_U(\varphi_{(t_0,t_{n_0}-t_0)}(x_0))\le t_{n_0+1}-t_{n_0}.$$

Because $\tau_Z(x_0)=+\infty$, $Z^-_{t_{n_0}}=U^-_{t_{n_0}}$ and $Z^-_{t_{n_0+1}}=U^-_{t_{n_0+1}}$, we obtain that in fact

$$0<\tau_U(\varphi_{(t_0,t_{n_0}-t_0)}(x_0)<t_{n_0+1}-t_{n_0},$$

so (2) holds for S finite. If S is an infinite set then we use the proved finite case and similar arguments based on the compactness of U_{t_0}.

□

4 Periodic case.

Assume that the vector field f (on manifold M) is T-periodic ($T>0$) with respect to the time variable. Let U and Z be two proper sets such that the assumptions (a), (b), (c) hold with the sequence $t_n = nT$ ($n \in \mathbb{Z}$). As a corollary we obtain

Theorem 2 (1) *There are a compact set $I \subset M$ invariant with respect to the Poincaré map $\varphi_{(0,T)}$ and a continuous surjective map $g : I \to \Sigma_2$ such that $\varphi_{(0,T)}$ is semiconjugated to the shift $\sigma : \Sigma_2 \to \Sigma_2$ by the map g in I.*
(2) *If n_0 is even then for every n-periodic sequence $s \in \Sigma_2$ its counterimage by*

g contains at least one n-periodic point of the Poincaré map.
(3) *If n_0 is odd and $s \in \Sigma_2$ is n-periodic sequence in which the symbol 1 appears k-times, k is not an odd multiplicity of n_0 then the counterimage of s by g contains n-periodic point of $\varphi_{(0,T)}$.*

Proof: The set I and map g are defined in the same way like in [SW]. Surjectivity of g follows by density of periodic orbits in Σ_2 and Corollary 1.

□

Remark 3 Let U and W be two periodic isolating segments over $[0,T]$ for the equation $(*)$ (see [SW] for the definition). Suppose that U and W fulfil the assumptions of the Theorem 2 in [SW] (see also [W1], [W2], [W3], [W4] for examples of concrete differential equations). We define two admissible proper pairs $\tilde{U}$, $\tilde{W}$ by the conditions

$$\tilde{U}_t = U_{t \bmod T},$$

$$\tilde{W}_t = W_{t \bmod T},$$

for $t \in \mathbb{R}$. It follows that for $\tilde{U}$, $\tilde{W}$ hold all assumptions of our Theorem 1 with $t_n = nT$, so for any sufficiently small perturbation (not necesserily T-periodic) of the system $(*)$ there is a compact set Λ which is invariant with respect to $\varphi_{(0,T)}$ and $\varphi_{(0,T)}$ restricted to Λ is semiconjugated to the Bernoulli shift with two symbols. In particular, the topological entropy of $\varphi_{(0,T)}$ is positive.

5 Applications.

In the present section we consider the planar nonautonomous equation of the variable $z \in \mathbb{C}$

$$\dot{z} = (1 + (\cos(t^2) + 2)e^{i\phi t}|z|^2)\bar{z}, \tag{1}$$

for some $\phi \in \mathbb{R}$.

Theorem 3 *The equation (1) fulfils assumptions of theorem 1 for sufficiently small $\phi > 0$.*

Although a proof is similar to the proof of Th.2 in [SW], we give it in a detailed way for the sake of completeness. Our proof of Theorem 3 consists of the construction of two proper pairs U and Z (admissible with $t_n = \frac{2\pi}{\phi}n$) satisfying the conditions (a), (b), and (c) in Theorem 1. Z will be a twisted prism with a square base centered at the origin. Its cross-sections Z_t will be obtained by rotating the base with the angle velocity $\phi/2$ over the t-interval $[0, 2\pi/\phi]$. The set U will be a regular square-based prism with broadening ends. Its cross-sections U_t corresponding to t near the centre of the interval will have the small side and they will broaden when t approaches the ends of the interval (because one should get the same cross-sections of U at times 0 and $2\pi/\phi$ as Z).

The reminder of this section will be devoted to presentation of the proof of the above theorem. The equation (1) coincides with the system of two planar equations:

$$\begin{cases} \dot{x} = x + (\cos(t^2)+2)(x^2+y^2)(x\cos(\phi t)+y\sin(\phi t)), \\ \dot{y} = -y + (\cos(t^2)+2)(x^2+y^2)(x\sin(\phi t)-y\cos(\phi t)). \end{cases} \tag{2}$$

By F we denote the vector-field in the extended phase space $\mathbb{R}^3$ generated by the right-hand side of the system (2), i.e.

$$F(t,x,y) = \begin{pmatrix} 1 \\ x + (\cos(t^2)+2)(x^2+y^2)(x\cos(\phi t)+y\sin(\phi t)) \\ -y + (\cos(t^2)+2)(x^2+y^2)(x\sin(\phi t)-y\cos(\phi t)) \end{pmatrix}$$

In the sequel we assume that $\phi > 0$. In order to construct U and Z we will introduce several auxiliary functions and sets. Let $R > 0$. Put

$$\Lambda_R^1(t,x,y) = \frac{1}{R^2}(x\cos(\frac{\phi}{2}t)+y\sin(\frac{\phi}{2}t))^2 - 1,$$
$$\Lambda_R^2(t,x,y) = \frac{1}{R^2}(x\sin(\frac{\phi}{2}t)-y\cos(\frac{\phi}{2}t))^2 - 1,$$

and

$$L_R = \{(t,x,y) \in \mathbb{R}^3 : \Lambda_R^i(t,x,y) \le 0,\ i=1,2\},$$
$$L_R^- = \{(t,x,y) \in \mathbb{R}^3 : \Lambda_R^1(t,x,y) = 0,\ \Lambda_R^2(t,x,y) \le 0\},$$
$$L_R^+ = \{(t,x,y) \in \mathbb{R}^3 : \Lambda_R^1(t,x,y) \le 0,\ \Lambda_R^2(t,x,y) = 0\}.$$

Lemma 2 *If $\phi \le 1$ and $R \ge 3$ then*

$$F(t,x,y)\cdot\nabla\Lambda_R^1(t,x,y) > 0 \qquad ((t,x,y) \in L_R^-), \tag{3}$$
$$F(t,x,y)\cdot\nabla\Lambda_R^2(t,x,y) < 0 \qquad ((t,x,y) \in L_R^+). \tag{4}$$

Proof: A direct calculation shows that

$$F(t,x,y)\cdot\nabla\Lambda_R^1(t,x,y) =$$
$$\frac{2}{R^2}\Big((x\cos(\frac{\phi}{2}t)+y\sin(\frac{\phi}{2}t))(x\cos(\frac{\phi}{2}t)-\ y\sin(\frac{\phi}{2}t)$$
$$-\phi x\sin(\frac{\phi}{2}t)+\phi y\cos(\frac{\phi}{2}t))+$$
$$(\cos(t^2)+2)(x^2+y^2)(x\cos(\frac{\phi}{2}t)+y\sin(\frac{\phi}{2}t))^2\Big)$$

$$F(t,x,y)\cdot\nabla\Lambda_R^2(t,x,y) =$$
$$\frac{2}{R^2}\Big((x\sin(\frac{\phi}{2}t)-y\cos(\frac{\phi}{2}t))(y\cos(\frac{\phi}{2}t)+x\sin(\frac{\phi}{2}t)$$
$$+\phi y\sin(\frac{\phi}{2}t)+\phi x\cos(\frac{\phi}{2}t))-$$

$$(\cos(t^2)+2)(x^2+y^2)(x\sin(\frac{\phi}{2}t)-y\cos(\frac{\phi}{2}t))^2\Big)$$

It is easy to verify that $x^2+y^2 \geq R^2$, $|x| \leq R\sqrt{2}$, and $|y| \leq R\sqrt{2}$ provided $(t,x,y) \in \partial L_R$ for some t. If $(t,x,y) \in L_R^-$ then

$$|x\cos(\frac{\phi}{2}t)+y\sin(\frac{\phi}{2}t)| = R$$

and

$$F(t,x,y)\cdot\nabla\Lambda_R^1(t,x,y) \geq \frac{2}{R^2}(R^4-R(2R\sqrt{2}+2R\phi\sqrt{2})) = 2(R^2-2\sqrt{2}(1+\phi)),$$

hence (3) follows. In the same manner we see that if $(t,x,y) \in L_R^+$ then

$$|x\sin(\frac{\phi}{2}t)-y\cos(\frac{\phi}{2}t)| = R$$

and

$$F(t,x,y)\cdot\nabla\Lambda_R^2(t,x,y) \leq \frac{2}{R^2}(-R^4+R(2R\sqrt{2}+2R\phi\sqrt{2})) = 2(-R^2+2\sqrt{2}(1+\phi)),$$

hence (4) is also satisfied and Lemma 2 is proved.

$\square$

Similarly to the previous construction, let $r > 0$ and put

$$\begin{aligned}\Xi_r^1(t,x,y) &= \frac{1}{r^2}x^2-1,\\ \Xi_r^2(t,x,y) &= \frac{1}{r^2}y^2-1,\end{aligned}$$

and

$$\begin{aligned}K_r &= \{(t,x,y)\in\mathbb{R}^3 : \Xi_r^i(t,x,y) \leq 0,\ i=1,2\},\\ K_r^- &= \{(t,x,y)\in\mathbb{R}^3 : \Xi_r^1(t,x,y) = 0,\ \Xi_r^2(t,x,y) \leq 0\},\\ K_r^+ &= \{(t,x,y)\in\mathbb{R}^3 : \Xi_r^1(t,x,y) \leq 0,\ \Xi_r^2(t,x,y) = 0\}.\end{aligned}$$

Lemma 3 *For an arbitrary ϕ and $r \leq \frac{1}{4}$,*

$$F(t,x,y)\cdot\nabla\Xi_r^1(t,x,y) > 0 \quad ((t,x,y)\in K_r^-), \tag{5}$$
$$F(t,x,y)\cdot\nabla\Xi_r^2(t,x,y) < 0 \quad ((t,x,y)\in K_r^+). \tag{6}$$

Proof: We have

$$F(t,x,y)\cdot\nabla\Xi_r^1(t,x,y) = \frac{2}{r^2}\left(x^2+(\cos(t^2)+2)(x^2+y^2)(x^2\cos(\phi t)+xy\sin(\phi t)\right) \tag{7}$$
$$F(t,x,y)\cdot\nabla\Xi_r^2(t,x,y) = \frac{2}{r^2}\left(-y^2+(\cos(t^2)+2)(x^2+y^2)(xy\sin(\phi t)-y^2\cos(\phi t)\right) \tag{8}$$

If $(t,x,y)\in K_r^-$ then $|x|=r$ and $|y|\le r$, hence, by (7),

$$F(t,x,y)\cdot\nabla\Xi_r^1(t,x,y)\ge\frac{2}{r^2}(r^2-6r^2\cdot 2r^2)=2-24r^2,$$

hence (5) is satisfied. By (8), it follows on an analogous way that if $(t,x,y)\in K_r^+$ then

$$F(t,x,y)\cdot\nabla\Xi_r^2(t,x,y)\le -2+24r^2,$$

hence (6) follows. Lemma 3 is proved.

$\square$

Let $\omega>0$. For $k\in\mathbb{Z}$ and $t\in[\frac{2\pi}{\phi}k,\frac{2\pi}{\phi}k+R/\omega]$ put

$$\Pi^1_{R,\omega}(t,x,y)=\frac{1}{(R-\omega(t-\frac{2\pi}{\phi}k))^2}x^2-1,$$
$$\Pi^2_{R,\omega}(t,x,y)=\frac{1}{(R-\omega(t-\frac{2\pi}{\phi}k))^2}y^2-1.$$

and

$$\begin{aligned}
P_{r,R,\omega}&=\{(t,x,y)\in[\tfrac{2\pi}{\phi}k,\tfrac{2\pi}{\phi}k+\tfrac{R-r}{\omega}]\times\mathbb{R}^2:\Pi^i_{R,\omega}(t,x,y)\le 0,\ i=1,2\},\\
P^-_{r,R,\omega}&=\{(t,x,y)\in[\tfrac{2\pi}{\phi}k,\tfrac{2\pi}{\phi}k+\tfrac{R-r}{\omega}]\times\mathbb{R}^2:\Pi^1_{R,\omega}(t,x,y)=0,\ \Pi^2_{R,\omega}(t,x,y)\le 0\},\\
P^+_{r,R,\omega}&=\{(t,x,y)\in[\tfrac{2\pi}{\phi},\tfrac{2\pi}{\phi}k+\tfrac{R-r}{\omega}]\times\mathbb{R}^2:\Pi^1_{R,\omega}(t,x,y)\le 0,\ \Pi^2_{R,\omega}(t,x,y)=0\}.
\end{aligned}$$

Lemma 4 *For $\phi>0$ sufficiently small and $\omega\le\frac{r}{2}$*

$$F(t,x,y)\cdot\nabla\Pi^1_{R,\omega}(t,x,y)>0\qquad((t,x,y)\in P^-_{r,R,\omega}),\tag{9}$$
$$F(t,x,y)\cdot\nabla\Pi^2_{R,\omega}(t,x,y)<0\qquad((t,x,y)\in P^+_{r,R,\omega}).\tag{10}$$

Proof:

$$F(t,x,y)\cdot\nabla\Pi^1_{R,\omega}(t,x,y)=\frac{2\omega x^2}{(R-\omega(t-\frac{2\pi}{\phi}k))^3}+$$
$$\frac{2}{(R-\omega(t-\frac{2\pi}{\phi}k))^2}\left(x^2+(\cos(t^2)+2)(x^2+y^2)(x^2\cos(\phi t)+xy\sin(\phi t))\right),$$
$$F(t,x,y)\cdot\nabla\Pi^2_{R,\omega}(t,x,y)=\frac{2\omega y^2}{(R-\omega(t-\frac{2\pi}{\phi}))^3}+$$
$$\frac{2}{(R-\omega(t-\frac{2\pi}{\phi}))^2}\left(-y^2+(\cos(t^2)+2)(x^2+y^2)(xy\sin(\phi t)-y^2\cos(\phi t))\right).$$

Let $\phi>0$ be small enough that

$$\cos(\phi t)>0\qquad(t\in[\frac{2\pi}{\phi}k,\frac{2\pi}{\phi}k+\frac{R-r}{\omega}]).\tag{11}$$

Let $(t,x,y) \in P^-_{R,\omega}$, hence $|x| = R - \omega(t - \frac{2\pi}{\phi}k)$ and $|y| \leq R - \omega(t - \frac{2\pi}{\phi}k)$. For $\phi > 0$ sufficiently small we obtain:

$$\begin{aligned} F(t,x,y) \cdot \nabla \Pi^1_{R,\omega}(t,x,y) &> \tfrac{2\omega}{r} + 2 - 2(R - \omega(t - \tfrac{2\pi}{\phi}k))^2 \phi (t - \tfrac{2\pi}{\phi}k) \\ &\geq 2 - 6R^2 \phi \tfrac{R-r}{\omega} \geq 1. \end{aligned}$$

By a similar argument we conclude that if $(t,x,y) \in P^+_{R,\omega}$ then for $\phi > 0$ sufficiently small

$$F(t,x,y) \cdot \nabla \Pi^2_{R,\omega}(t,x,y) < -2 + \frac{2\omega}{r} + 6R^2\phi \frac{R-r}{\omega} \leq -1 + \frac{2\omega}{r}.$$

Since $\omega \leq r/2$ is assumed, the proof of Lemma 4 is finished.

□

For $t \in [\frac{2\pi}{\phi}k - \frac{R-r}{\omega}, \frac{2\pi}{\phi}k]$ we put

$$\Sigma^1_{R,\omega}(t,x,y) = \Pi^1_{R,\omega}(\frac{2\pi}{\phi}k - t, x, y),$$
$$\Sigma^2_{R,\omega}(t,x,y) = \Pi^2_{R,\omega}(\frac{2\pi}{\phi}k - t, x, y),$$

$$\begin{aligned} S_{r,R,\omega} &= \{(t,x,y) \in [\tfrac{2\pi}{\phi}k - \tfrac{R-r}{\omega}, \tfrac{2\pi}{\phi}k] \times \mathbb{R}^2 : \Sigma^i_{R,\omega}(t,x,y) \leq 0,\ i = 1,2\}, \\ S^-_{r,R,\omega} &= \{(t,x,y) \in [\tfrac{2\pi}{\phi}k - \tfrac{R-r}{\omega}, \tfrac{2\pi}{\phi}k] \times \mathbb{R}^2 : \Sigma^1_{R,\omega}(t,x,y) = 0,\ \Sigma^2_{R,\omega}(t,x,y) \leq 0\}, \\ S^+_{r,R,\omega} &= \{(t,x,y) \in [\tfrac{2\pi}{\phi}k - \tfrac{R-r}{\omega}, \tfrac{2\pi}{\phi}k] \times \mathbb{R}^2 : \Sigma^1_{R,\omega}(t,x,y) \leq 0,\ \Sigma^2_{R,\omega}(t,x,y) = 0\}. \end{aligned}$$

One can check that

Lemma 5 *For $\phi > 0$ sufficiently small*

$$F(t,x,y) \cdot \nabla \Sigma^1_{R,\omega}(t,x,y) > 0 \qquad ((t,x,y) \in S^-_{r,R,\omega}) \tag{12}$$
$$F(t,x,y) \cdot \nabla \Sigma^2_{R,\omega}(t,x,y) < 0 \qquad ((t,x,y) \in S^+_{r,R,\omega}) \tag{13}$$

□

Lemma 6 *If $\phi < \frac{2\omega}{R}$ then $P_{r,R,\omega} \subset L_R$ and $S_{r,R,\omega} \subset L_R$.*

Proof: Essentially it is lemma 6 in [SW].

□

Proof of Theorem 2: All the estimates in the above lemmas are satisfied for $R = 3$, $r = 1/4$, $\omega = 1/8$, and $\phi > 0$ sufficiently small. Define

$$U = P_{\frac{1}{4},3,\frac{1}{8}} \cup K_{\frac{1}{4}} \cup S_{\frac{1}{4},3,\frac{1}{8}},$$
$$Z = L_3.$$

It follows by Lemmas 2, 3, 4, and 5 that U and Z are admissible proper pairs with the sequence $t_n = \frac{2\pi}{\phi}n$, and

$$U_0 = Z_0 = \{(x,y) \in \mathbb{R}^2 : |x| \leq 3, |y| \leq 3\},$$

$$U_0^- = Z_0^- = \{(x,y) \in \mathbb{R}^2 : |x| = 3, |y| \leq 3\}.$$

The set Z is a twisted prism with a square base, its successive cross-sections Z_t are obtained by the rotation of Z_0 by the angle $\phi t/2$ ($t \in [0, 2\pi]$), hence we can take the map $(x,y) \to (-x,-y)$ as a homeomorphism $\tilde{h}$ corresponding to Z. Thus

$$\mu_{Z_{t_n,t_n+1}} \circ \mu_{Z_{t_n,t_n+1}} = \mathrm{id}_{H(U_0,U_0^-)}, \qquad \mathrm{Lef}(\mu_W) = 1.$$

U is a regular square-based prism, broadening at t_n, hence

$$\mu_{U_{t_n,t_n+1}} = \mathrm{id}_{H(U_0,U_0^-)}.$$

Moreover, $U \subset Z$ because $K_{\frac{1}{6}} \subset L_3$ and Lemma 6 is valid. Since

$$\chi(U_0, U_0^-) = -1,$$

all conditions (a), (b), and (c) are satisfied.

□

6 Appendix.

Let n be an odd natural number. For a $p \in \mathbb{N}$ and an $r \in \mathbb{Z}$ define:

$$S_n(p,r) = \sum_{\substack{0 \leq s \leq p \\ n \mid s-r}} (-1)^{s-r} \binom{p}{s}. \tag{14}$$

It is easy to check that

$$S_n(0,r) = \begin{cases} (-1)^r & \text{if } r = k \cdot n; \\ 0 & \text{otherwise.} \end{cases}$$

Two following lemmas gather some useful properties of $S_n(p,r)$

Lemma 7 $S_n(p,r) = S_n(p-1,r) + S_n(p-1,r-1)$.

Proof:

$$\begin{aligned} S_n(p,r) &= \sum_{\substack{0 \leq s \leq p \\ n \mid s-r}} (-1)^{s-r} \binom{p}{s} \\ &= \sum_{\substack{0 \leq s \leq p \\ n \mid s-r}} (-1)^{s-r} \binom{p-1}{s-1} + \sum_{\substack{0 \leq s \leq p \\ n \mid s-r}} (-1)^{s-r} \binom{p-1}{s} \\ &= \sum_{\substack{1 \leq s \leq p \\ n \mid s-r}} (-1)^{s-r} \binom{p-1}{s-1} + \sum_{\substack{0 \leq s \leq p-1 \\ n \mid s-r}} (-1)^{s-r} \binom{p-1}{s} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{\substack{0\le s\le p-1\\ n\,|\,s+1-r}} (-1)^{s+1-r}\binom{p-1}{s} + \sum_{\substack{0\le s\le p-1\\ n\,|\,s-r}} (-1)^{s-r}\binom{p-1}{s}\\
&= \sum_{\substack{0\le s\le p-1\\ n\,|\,s-(r-1)}} (-1)^{s-(r-1)}\binom{p-1}{s} + \sum_{\substack{0\le s\le p-1\\ n\,|\,s-r}} (-1)^{s-r}\binom{p-1}{s}\\
&= S_n(p-1,r-1)+S_n(p-1,r).
\end{aligned}
$$

□

Lemma 8 *The 2–directional sequence $\{S_n(p,r)\}_{r\in\mathbb{Z}}$ satisfies following properties:*

1. $S_n(p,r) = -S_n(p,r+n)$*;*
2. $S_n(p,p/2+\alpha) = S_n(p,p/2-\alpha)$ *for such α that $p/2+\alpha\in\mathbb{Z}$.*

Proof: The lemma is obvious for $p=0$.

Now suppose, that it is satisfied for $p-1$.

1.

$$
\begin{aligned}
S_n(p,r) &= S_n(p-1,r)+S_n(p-1,r-1)\\
&= -S_n(p-1,r+n)-S_n(p-1,r-1+n)\\
&= -(S_n(p-1,r+n)+S_n(p-1,r+n-1))\\
&= -S_n(p,r+n);
\end{aligned}
$$

2.

$$
\begin{aligned}
S_n(p,p/2+\alpha) &= S_n(p-1,p/2+\alpha-1)+S_n(p-1,p/2+\alpha)\\
&= S_n(p-1,\frac{(p-1)}{2}+\alpha-\frac{1}{2})+S_n(p-1,\frac{(p-1)}{2}+\alpha+\frac{1}{2})\\
&= S_n(p-1,\frac{(p-1)}{2}-\alpha+\frac{1}{2})+S_n(p-1,\frac{(p-1)}{2}-\alpha-\frac{1}{2})\\
&= S_n(p-1,p/2-\alpha)+S_n(p-1,p/2-\alpha-1)\\
&= S_n(p,p/2-\alpha).
\end{aligned}
$$

□

Now we are ready to state an important property of numbers $S_n(p,r)$:

Theorem 4

$$
S_n(p,r)\begin{cases} =0 & \text{if } \frac{p-2r}{n}\in 2\mathbb{N}+1\\ >0 & \text{if } \frac{p-2r}{n}\in(4k-1,4k+1)\\ <0 & \text{if } \frac{p-2r}{n}\in(4k+1,4k+3)\end{cases}
$$

for odd $n\in\mathbb{N}$ and all $p\ge n-1$.

Proof: Let $p = n-1$. Note, that $\frac{p-2r}{n}$ cannot be an odd number.

From the definition,

$$S_n(n-1,r) = (-1)^{s-r}\binom{n-1}{s}$$

for such $s \in \{0,1,\ldots,n-1\}$, that there exists $k \in \mathbb{N}$ that $s - r = kn$. But $(-1)^{s-r} = (-1)^{kn} = (-1)^k = (-1)^{\frac{s-r}{n}} = (-1)^{\frac{r-s}{n}}$, so

$$S_n(n-1,r) > 0 \Leftrightarrow \frac{r-s}{n} \in 2\mathbb{N} \Leftrightarrow [\frac{r}{n}] \in 2\mathbb{N}$$

$$\Leftrightarrow \frac{r}{n} \in [2m, 2m+1) \Leftrightarrow \frac{p-2r}{n} \in (-4m-1-\frac{1}{n}, -4m+1-\frac{1}{n}].$$

Since p is even number, no number of the form $\frac{p-2r}{n}$ lies in intervals of the form $(2l+1-\frac{1}{n}, 2l+1+\frac{1}{n})$. Hence

$$\frac{p-2r}{n} \in (-4m-1-\frac{1}{n}, -4m+1-\frac{1}{n}] \Leftrightarrow \frac{p-2r}{n} \in (-4m-1, -4m+1).$$

Similarly,

$$S_n(n-1,r) < 0 \Leftrightarrow \frac{p-2r}{n} \in (-4m+1-\frac{1}{n}, -4m+3-\frac{1}{n}]$$

$$\Leftrightarrow \frac{p-2r}{n} \in (-4m+1, -4m+3),$$

and these equivalences complete the proof for $p = n-1$.

Now suppose, that the theorem is fulfilled for $p-1$.

1. $\frac{p-2r}{n} = 2k+1$

   From the Lemma 8 we have $S_n(p-1,r) = S_n(p-1,p-1-r)$ so:

   $$S_n(p,r) = S_n(p-1,r) + S_n(p-1,r-1) = S_n(p-1,p-1-r) + S_n(p-1,r-1)$$

   $$= S_n(p-1,p-1-r) + S_n(p-1,p-1-r+(2r-p))$$

   $$= S_n(p-1,p-1-r) + S_n(p-1,p-1-r-(2k+1)n) = 0$$

   because $(2k+1)n$ is an odd multiplicity of n.

2. $\frac{p-2r}{n} \in (4k-1, 4k+1)$

   The numbers $\frac{(p-1)-2(r-1)}{n} = \frac{p-2r}{n} + \frac{1}{n}$ and $\frac{(p-1)-2r}{n} = \frac{p-2r}{n} - \frac{1}{n}$ are both of the form $\frac{q}{n}$ and differ from $\frac{p-2r}{n}$ by $\frac{1}{n}$. Since both ends of the interval $[4k-1, 4k+1]$ are of the form $\frac{q}{n}$ and $\frac{p-2r}{n} \in (4k-1, 4k+1)$, numbers $\frac{(p-1)-2(r-1)}{n}, \frac{(p-1)-2r}{n}$ must lie in $[4k-1, 4k+1]$ and at least one of them is in the interior of $[4k-1, 4k+1]$. Thus we have following inequalities:

   $$S_n(p-1,r) \geq 0, \quad S_n(p-1,r-1) \geq 0$$

   and at most one of them is an equality. Hence $S_n(p,r) = S_n(p-1,r) + S_n(p-1,r-1) > 0$.

3. $\frac{p-2r}{n} \in (4k+1, 4k+3)$
   Analogically as above we get: $S_n(p,r) = S_n(p-1,r) + S_n(p-1,r-1) < 0$.

$\square$

As a consequence of the theorem, we obtain:

Corollary 2 $\sum_{\substack{0 \le s \le p \\ n \mid s}} (-1)^{p-s} \binom{p}{s} = 0 \iff p/n$ *is an odd number.*

Proof:

$$\sum_{\substack{0 \le s \le p \\ n \mid s}} (-1)^{p-s} \binom{p}{s} = (-1)^p \cdot \sum_{\substack{0 \le s \le p \\ n \mid s}} (-1)^s \binom{p}{s} = (-1)^p \cdot S_n(p,0).$$

$\square$

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