A GEOMETRY OF REAL POLYNOMIAL MAPPINGS*)

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Abstract

In this paper we study the set of points at which a real polynomial mapping is not proper.

1 Introduction

Let $f : X \to Y$ be a continuous map of locally compact spaces. We say that the mapping f is not proper at a point $y \in Y$, if there is no a neighborhood U of a point y such that the set $f^{-1}(cl(U))$ is compact.

The set S_f of points at which the map f is not proper indicates how the map f differs from a proper map. In particular f is proper if and only if this set is empty, moreover if $f(X)$ is open then S_f contains the border of the set $f(X)$. The set S_f is a minimal set S with a property that the mapping $f : X \setminus f^{-1}(S) \to Y \setminus S$ is proper.

Key words: real polynomial mapping, quasi-finite mapping, the set of points over which a polynomial mapping is not proper.

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Further, if $f : \mathbb{R}^n \to \mathbb{R}^n$ is a generically finite polynomial mapping, then the mapping f has a constant topological degree over every connected component of the set $\mathbb{R}^n \setminus S_f$.

In our previous paper [6] we described the set S_f in the case of complex polynomial mappings $f : \mathbb{C}^n \to \mathbb{C}^m$ and in the paper [7] we described this set in the case of real polynomial mappings $f : \mathbb{R}^2 \to \mathbb{R}^2$.

The aim of this paper is to do the same in the general case of real polynomial generically finite mappings $f : \mathbb{R}^n \to \mathbb{R}^m$. Our main result is the following

Theorem. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a polynomial generically-finite mapping. Then the set S_f is closed, semi-algebraic and for every non-empty connected component $S \subset S_f$ we have $1 \leq \dim S \leq n-1$.*

Moreover, the set S_f is $\mathbb{R}$-uniruled. It means, that for every point $a \in S_f$ there is a non-constant polynomial mapping $\phi : \mathbb{R} \to S_f$ such that $\phi(0) = a$. Further, there is a real polynomial $P \in \mathbb{R}[x_1, ..., x_m]$ of degree at most $(\deg f)^{n-1} - 1$, such that $S_f \subset P^{-1}(0)$.

Corollary. *If $f : \mathbb{R}^n \to \mathbb{R}^m$ is a polynomial generically-finite mapping, then every non-empty connected component of S_f is unbounded.*

In this paper we use methods from our recent papers [4] and [5].

2 Terminology.

Let $X \subset \mathbb{R}^n(\mathbb{P}^n(\mathbb{R}))$ be a semi-algebraic set. The Zariski closure of X will be denoted by $cl_Z(X)$. By $\mathbb{C}X$ we denote a complexification of X, i.e., $\mathbb{C}X$ is a Zariski closure of the set X in $\mathbb{C}^n$ $(\mathbb{P}^n(\mathbb{C}))$.

More generally, if $X \subset \mathbb{P}^n(\mathbb{R}) \times \mathbb{P}^n(\mathbb{R})$ is an algebraic set, then by its complexification $\mathbb{C}X$ we mean a Zariski closure of the set X in $\mathbb{P}^n(\mathbb{C}) \times \mathbb{P}^n(\mathbb{C})$ etc.

Let $X \subset \mathbb{R}^n$ be an algebraic set. A mapping $f : X \to \mathbb{R}$ is called polynomial if it is a restriction of a polynomial $F : \mathbb{R}^n \to \mathbb{R}$, $F \in \mathbb{R}[x_1, ..., x_n]$. More generally, a mapping $f : X \to \mathbb{R}^m$ is called polynomial if $f = (f_1, ..., f_m)$ where mappings f_i are polynomial mappings. More generally, if $X \subset \mathbb{P}^n(\mathbb{R})$, $Y \subset \mathbb{P}^m(\mathbb{R})$ are semialgebraic sets and $f : X \to Y$ is a regular mapping then

f is called polynomial if it has an extension to a regular mapping $\mathbb{C}f : \mathbb{C}X \to \mathbb{C}Y$.

For an algebraic set $X \subset \mathbb{R}^n$ we consider the coordinate ring of X: $\mathbb{R}[X] := \mathbb{R}/I(X)$, where $I(X) = \{f \in \mathbb{R}[x_1, ..., x_n] : f(X) = \{0\}\}$.

A polynomial mapping $f : X \to \mathbb{R}^m$ is called generically finite if for generic $x \in X$ the fiber $f^{-1}(f(x))$ is finite. It is easy to see that f is generically-finite if and only if the mapping $f : \mathbb{C}X \to \mathbb{C}^m$ is generically-finite.

If $g = \sum_{|\alpha|\le n} a_\alpha x^\alpha \in \mathbb{C}[x_1, ..., x_n]$, then $\overline{g} := \sum_{|\alpha|\le n} \overline{a_\alpha} x^\alpha \in \mathbb{C}[x_1, ..., x_n]$, where the overband denotes the complex conjugate.

If $f = (f_1, ..., f_m) : \mathbb{R}^n \to \mathbb{R}^m$ is a polynomial mapping, then by deg f we mean the number $max_{j=1}^m$ deg f_j.

3 Preliminaries.

In the beginning we recall some basic facts about polynomial mappings of $\mathbb{C}^2$. We start with the theorem about surfaces which contain $\mathbb{C}^2$ as an open subset ([6]):

Theorem 3.1 *Let X be a complete normal surface. Assume, that X contains a plane $\mathbb{C}^2$ as an open, dense subset. Let $W_1, W_2 \subset X \setminus \mathbb{C}^2$ be two connected, complete curves without common components. Then the intersection $W_1 \cap W_2$ is either the empty set or it is a point.*

Definition 3.2 Let Γ be an affine curve such that there is a surjective polynomial mapping $\phi : \mathbb{C} \to \Gamma$. Then Γ is called to be *an affine parametric line* and the mapping ϕ is called to be *a parametrization of* Γ.

In the sequel we need also the following (see [6]):

Proposition 3.3 *Let X be an irreducible affine variety of dimension ≥ 1. The following conditions are equivalent:*

1) for every point $x \in X$ there is a parametric affine line in X going through x;

2) there exists a Zariski-open, non-empty subset U of X, such that for every point $x \in U$ there is a parametric affine line in X going through x;

3) there exists a subset U of X of the second Baire's category, such that for every point $x \in U$ there is a parametric affine line in X going through x;

4) there exists an affine variety W with $\dim W = \dim X - 1$ *and a dominant polynomial mapping $\phi : W \times \mathbb{C} \to X$.*

Definition 3.4 An affine irreducible variety X is called $\mathbb{C}$-*uniruled* if it is of dimension ≥ 1, and satisfies one of equivalent conditions $1) - 4)$ listed in Proposition 3.7. More generally, an affine variety X is called $\mathbb{C}$-*uniruled if all its irreducible components are* $\mathbb{C}$*-uniruled.*

The next result was proved in our papers [5], [6]:

Theorem 3.5 *Let $f = (f_1, ..., f_n) : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ be a dominant polynomial map. Then the set S_f of points at which f is either empty, or it is a $\mathbb{C}$-uniruled hypersurface. Moreover, its degree is not greater than*

$$\frac{\prod_{i=1}^{n} \deg f_i - \mu(f)}{\min_{i=1,\dots,n} \deg\ f_i},$$

where $\mu(f)$ denotes the maximal number of points in (finite) fibers of f.

4 Mappings of the real plane.

In this section we study generically-finite polynomial mappings $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^m$. We start with the following:

Definition 4.1 Let $\Gamma \subset \mathbb{R}^n$ be a semi-algebraic closed curve, such that there exists a polynomial surjective mapping $\phi : \mathbb{R} \to \Gamma$. Then Γ will be called *a parametric semi-line* and the mapping ϕ will be called *a parametrization of* Γ. If a curve Γ is algebraic it will be called *a parametric line.*

We come to our first result:

Theorem 4.2 *Let $f = (f_1, ..., f_m) : \mathbb{R}^2 \longrightarrow \mathbb{R}^m$ be a generically finite polynomial mapping. Then the set S_f of points at which f is not proper is a union of finite number (possibly 0) of parametric semi-lines.*

Proof. We can treat the mapping f as a rational mapping $f : \mathbb{P}^2(\mathbb{R}) \to \mathbb{P}^n(\mathbb{R})$. Now resolve points of indeterminacy of f by sequence of blowing up's. In this way we obtain a smooth surface Z and a regular mapping $F : Z \to \mathbb{P}^n(\mathbb{R})$, such that $res_{\mathbb{R}^2} F = f$.

We can also consider the surface $\mathbb{C}Z$ and the mapping $\mathbb{C}F$, which are canonical complexifications of Z and F (i.e. to obtain Z, F we consider complex extension of real blow up's). Hence the mapping $\mathbb{C}F$ is regular in a neighborhood of Z. Without loss of generality we can assume that the mapping $\mathbb{C}F$ is regular on the whole of $\mathbb{C}Z$ (later we resolve only complex points of indeterminacy).

The set $R := Z \setminus \mathbb{R}^2$ is connected and it is a union $\bigcup R_i$ of circles $R_i \cong \mathbb{P}^1(\mathbb{R})$. Moreover, the complexification $\mathbb{C}R_i := S_i$ is isomorphic to $\mathbb{P}^1(\mathbb{C})$ and it contains R_i as $\mathbb{P}^1(\mathbb{R})$, i.e., there is a (complex) biregular mapping $\Psi_i : \mathbb{P}^1(\mathbb{C}) \to S_i$ such that it induces a (real) biregular mapping $\psi_i : \mathbb{P}^1(\mathbb{R}) \to R_i$. It follows from elementary properties of blowing up.

Let us denote by L_∞ the hyperplane at infinity in $\mathbb{R}^m$ and take $Q := F^{-1}(L_\infty)$. Let R_j be a component of R which is not in Q. We show that R_j has at most one common point with Q. Indeed, the curve $Q' := \mathbb{C}F^{-1}(\mathbb{C}L_\infty)$ is connected as the complement of semi-affine surface $\mathbb{C}F^{-1}(\mathbb{C}^m)$ (see [6]). Hence our assertion follows from Theorem 3.1 (applied to the surface $\mathbb{C}Z$).

Now let us note that the set S_f is exactly the set $F(R \setminus Q)$. Let H be a connected (non-empty) component of the set S_f. Hence there are indices $j_1, ..., j_k$ such that $H = \bigcup_{i=1}^k F(R_{j_i} \setminus Q)$. On the other hand we know that for every j the curve $R_j \cong \mathbb{P}^1(\mathbb{R})$ has at most one common point with Q. Consequently, the restriction of the mapping F to $R_j \setminus Q$ gives a regular mapping $\phi_j : \mathbb{R} \to H \subset \mathbb{R}^m$. Here we identify $\mathbb{R}$ with $R_j \setminus Q$ or (if $R_j \cap Q = \emptyset$) with R_j minus one point. Without restriction of generality we can assume that all mappings ϕ_j are non-constant. Indeed, since the set R is connected, it means that the set $F(R) \subset S_f \cup L_\infty$ is connected, too, hence H cannot be an isolated point. Consequently, we can excluded all constant mappings ϕ_j. Now, we show that the mapping ϕ_j must be polynomial.

Since the mapping ϕ_j is regular, we have $\phi_j = (P_1/Q_1, ..., P_m/Q_m)$, where P_i, Q_i are relatively prime real polynomials from $\mathbb{R}[t]$ with $Q_i \neq 0$ everywhere. (see [1], 3.1.9).

Moreover, the mapping ϕ_j extends uniquely to the mapping $\Phi_j : S_j \cong \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^m(\mathbb{C})$. If $\prod_{i=1}^m Q_i \neq const$, then S_j would have two or more (conjugate) complex common points with $Q' = \mathbb{C}F^{-1}(\mathbb{C}L_\infty)$ (namely zeros of some of Q_i). By Theorem 3.1 it is a contradiction.

Hence, the mapping ϕ_j is indeed a polynomial mapping. This also implies that $R_j \cap Q \neq \emptyset$ (i.e. $R_j \setminus Q \cong \mathbb{R}$), and consequently $H = \bigcup_{i=1}^k \phi_{j_i}(\mathbb{R})$.$\blacksquare$

Remark 4.3 Let $S \subset \mathbb{R}^m$, where $m \geq 2$, be a parametric semi-line. We prove in the section 7 that there exists a polynomial mapping $f : \mathbb{R}^2 \to \mathbb{R}^m$ with finite fibers for which S is exactly the set of points at which f is not proper (i.e., $S = S_f$).

5 Auxiliary results.

For a sequel we need some auxiliary results.

Theorem 5.1 *Let $X \subset \mathbb{R}^m$ be a real affine variety and $Y \subset X$ be a closed subvariety. Let $f : Y \to \mathbb{R}^n$ be a polynomial mapping. Assume that* dim $X \leq n$. *Then there exists a polynomial mapping $F : X \to \mathbb{R}^n$ such that*

1) $res_Y F = f$

2) the mapping $res_{\mathbb{C}X \setminus \mathbb{C}Y} \mathbb{C}F : \mathbb{C}X \setminus \mathbb{C}Y \to \mathbb{C}^n$ is quasi-finite.

In particular if the mapping f is quasi-finite, then the mapping F is quasi-finite, too.

Proof. Let $I(Y) = (h_1, ..., h_r)$ denote the ideal of Y in $\mathbb{R}[X]$. Then polynomials $(h_1, ..., h_r)$ are also generators of the ideal of $\mathbb{C}Y$ in $\mathbb{C}[\mathbb{C}X]$.

Let $g = (g_1, \ldots, g_n)$ be a polynomial extension of f to X. Let $x_1, \ldots, x_m$ be a coordinate polynomials in $\mathbb{R}^m$.

Take $H_{ij} := h_i \cdot x_j;\ i = 1, ..., r,\ j = 1, ..., m$; and consider the map $H = (g_1, \ldots, g_n, h_1, ..., h_r, H_{11}, ..., H_{1m}, ..., H_{r1}, \ldots, H_{rm}) : X \to \mathbb{R}^{n+r+mr}$. By construction $\mathbb{C}H$ is injective outside $\mathbb{C}Y$, $H(\mathbb{C}Y) \subset \mathbb{C}^n \times \{0\}$ and dim $H(\mathbb{C}X) =$ dim $\mathbb{C}X \leq n$. Take $\Gamma := cl(H(\mathbb{C}X))$, it is a closed subset of $\mathbb{C}^{n+r+mr}$. It is easy to see that there exists a proper linear projection $\pi : \Gamma \to \mathbb{C}^n \times \{0\}$, such that $\pi(H(X)) \subset \mathbb{R}^n \times \{0\}$.

Indeed, we have the canonical inclusion $\mathbb{C}^{n+r+mr} \subset \mathbb{P}^{n+r+mr}(\mathbb{C})$ and the set W of points at infinity of $\mathbb{C}Y$ has dimension $\leq n-1$. Hence we can find a real linear subspace L of the hyperplane at infinity of dimension $r+mr-1$ which is disjoint from W and from $cl(\mathbb{C}^n \times \{0\})$. Let $\pi_{\mathbb{C}L} : \mathbb{P}^{n+r+mr}(\mathbb{C}) \to cl(\mathbb{C}^n \times \{0\})$ be the projection determined by $\mathbb{C}L$. Since L is a real subspace, we have

$\pi_{\mathbb{C}L} : \mathbb{P}^{n+r+mr}(\mathbb{R}) \to cl(\mathbb{R}^n \times \{0\})$ The restriction of $\pi_{\mathbb{C}L}$ to Γ is a finite map of Γ into $\mathbb{C}^n \times \{0\}$ and it is the projection π we were looking for.

Now it is enough to take $F = \pi \circ H$.■

Also the following lemma will be useful:

Lemma 5.2 *Let $\{a_1, ..., a_n\} \subset \mathbb{C}^2$. Then there exists a polynomial automorphism $Q = (Q_1, Q_2) : \mathbb{C}^2 \to \mathbb{C}^2$ such that*

1) $Q_1, Q_2 \in \mathbb{R}[x, y]$

2) $Q(a_i) \in \{(x, y) \in \mathbb{C}^2 : x = 0\}$.

Proof. Let $a_i = (x_i, y_i)$. There is a real linear transformation T of $\mathbb{C}^2$ such that that $T(a_i) = (x_i', y_i')$ and $x_i' \neq x_j'$ for $i \neq j$ and $x_i' \neq \overline{x_j'}$ for $a_i \neq \overline{a_j}$. Indeed, let M be a non-zero real number such that $M \neq x_i - x_j/(y_i - y_j)$ for every $y_i \neq y_j$ and $M \neq x_i - \overline{x_j}/(y_i - \overline{y_j})$ for every $y_i \neq \overline{y_j}$. It is easy to see that the linear transformation $T(x, y) = (x + My, y)$ has desired properties.

Hence, we can assume that $x_i \neq x_j$ for every $i \neq j$ and if $x_i = \overline{x_j}$ then $y_i = \overline{y_j}$. Now let ϕ be a polynomial of a minimal degree such that $\phi(x_i) = y_i$ and $\phi(\overline{x_i}) = \overline{y_i}$. By our assumption this polynomial exists and it has real coefficients (the latter we get computing these coefficients by Cramer's rules). Now it is enough to take $Q(x, y) = (x, y - \phi(x))$.■

Lemma 5.3 *Let $X \subset \mathbb{R}^n$ be a real algebraic set. Let $F : \mathbb{C}X \to \mathbb{C}^m$ be a polynomial mapping. If $F(X) \subset \mathbb{R}^m$ then the mapping $f : X \ni x \to F(x) \in \mathbb{R}^m$ is a real polynomial mapping. In particular F has a real extension to $\mathbb{C}^n$.*

Proof. Let $F = (F_1, ..., F_m) : \mathbb{C}^n \to \mathbb{C}^m$ and let $\overline{F} = (\overline{F_1}, ..., \overline{F_m}) : \mathbb{C}^n \to \mathbb{C}^m$. By our assumption mappings F and $\overline{F}$ coincide on X. If we take $G = (F + \overline{F})/2$ then $G : \mathbb{R}^n \to \mathbb{R}^m$ is a real polynomial mapping and $res_X G = f$. Since X is Zariski dense in $\mathbb{C}X$ we have that $F = res_{\mathbb{C}X} G$, i.e., F has an extension to a real polynomial mapping G.■

Proposition 5.4 *Let $\Lambda \subset \mathbb{R}^m$ be a real algebraic curve. Then there exists a real algebraic curve $X \subset \mathbb{R}^2$ and a real polynomial mappings $f : X \to \Lambda$ such that the mapping $\mathbb{C}f : \mathbb{C}X \to \mathbb{C}\Lambda$ is an embedding and the set $\Lambda \setminus f(X)$ is finite.*

Proof. We can assume that $\mathbb{C}\Lambda$ is smooth and affine. It is a standard result that $\mathbb{C}\Lambda$ can be embedded in $\mathbb{C}^2$ in a birational way. More precisely there exists a proper regular and birational map $\Phi : \mathbb{C}\Lambda \to \mathbb{C}^2$, such that $\Lambda' := \Phi(\mathbb{C}\Lambda)$ has only finite number of singular points. Moreover, we can assume that the mapping Φ is a composition of (real) projections, hence it is described by real polynomials.

By Lemma 5.2 we can assume that all singular points of Λ' lie on the line $\{x = 0\}$. Moreover, we can assume that $(0, 0) \notin R := \{x = 0\} \cap \Lambda'$.

Let us consider the map $q : \mathbb{C}^2 \ni (x, y) \to (x, xy) \in \mathbb{C}^2$. Take $X' = q^{-1}(\Lambda')$ and $X = X' \cap \mathbb{R}^2$. Now $\Psi = q \circ (\Phi)^{-1} : X' \to \mathbb{C}^n$ is a closed embedding of X' in $\mathbb{C}\Lambda$. Moreover, we have $\Psi(X) \subset \mathbb{R}^n$. Indeed, assume that $x \in X$ and $y = \Psi(x) \notin \mathbb{R}^n$. We have $\Phi(y) = \Phi(\overline{y}) = q(x)$ and $y \neq \overline{y}$, which contradicts the fact that Φ induces an injection $\Phi^{-1}(X') \to X'$. Finally, we can take $f = res_X \Psi : X \to \Lambda$.■

Corollary 5.5 *Let $\Lambda \subset \mathbb{R}^m$ be a real algebraic curve. Then there exist a finite subset $S \subset \Lambda$ and a real algebraic curve $X \subset \mathbb{R}^2$, such that the curve X is biregulary equivalent with the curve $\Lambda \setminus S$.*

Definition 5.6 Let $S \subset \mathbb{P}^n(\mathbb{R})$ be a closed semi-algebraic curve. The curve S is called $\mathbb{R}$-rational if there is a regular surjective mapping $f : \mathbb{P}^1(\mathbb{R}) \to S$, which can be extended to a regular mapping $\mathbb{C}f : \mathbb{P}^1(\mathbb{C}) \to \mathbb{C}S$.

Theorem 5.7 *Let X be obtained from $\mathbb{P}^2(\mathbb{R})$ by a sequence of blowing up's with centers at infinity. Then for every $x \in X$ and for every non-empty Zariski open subset $U \subset \mathbb{R}^2$ there is a $\mathbb{R}$-rational curve Γ such that*

1) $x \in \Gamma$

2) Γ contains a point from U.

Proof. We use induction on the number p of elementary blowing up's. For $X_0 = \mathbb{P}^2(\mathbb{R})$ our theorem is obvious, hence for $p = 0$ theorem is true. Moreover every point $x \in X_0$ has the following important property:

$*$) there is an affine neighborhood U_x of the point x, which is polynomially isomorphic to $\mathbb{R}^2$, i.e., there is an isomorphism $\phi_x : \mathbb{R}^2 \to U_x$ which extends to an isomorphism $\mathbb{C}\phi_x : \mathbb{C}^2 \to \mathbb{C}U_x \subset \mathbb{C}X_0$.

Let us note that if the surface $X_p \subset \mathbb{P}^N(\mathbb{R})$ has the property $*$) then also the blown surface $X_{p+1} \subset \mathbb{P}^M(\mathbb{R})$ has the same property. It follows from the well-known properties of σ process. Now our theorem is obvious.■

Corollary 5.8 *Let $X \subset \mathbb{R}^n$ be an algebraic set, which is polynomially isomorphic with $\mathbb{R}^k$, $k \geq 1$. Let $a \in cl(X) \subset \mathbb{P}^n(\mathbb{R})$, (where $cl(X)$ denotes the closure in the strong topology) and $U \subset X$ be a non-empty Zariski open subset. Then there is a $\mathbb{R}$-rational curve Γ such that*

1) $a \in \Gamma \subset cl(X)$

2) Γ contains a point from U.

Proof. We can assume that $k > 1$. Let $a \in cl(X) \subset \mathbb{P}^n(\mathbb{R})$. By a curve selecting lemma there is an algebraic curve $G \subset X$ such that $a \in cl(G) \subset \mathbb{P}^n(\mathbb{R})$. Now by Proposition 5.4 there is a real algebraic curve $X \subset \mathbb{R}^2$ and a real polynomial mappings $f : X \to \Lambda$ such that the mapping $\mathbb{C}f : \mathbb{C}X \to \mathbb{C}\Lambda$ is an embedding and the set $\Lambda \setminus f(X)$ is finite. By Theorem 5.1 we can extend the mapping f to a polynomial quasi-finite mapping $F : \mathbb{R}^2 \to X \subset \mathbb{P}^n(\mathbb{R})$. Moreover, we can assume that $F(\mathbb{R}^2) \cap U \neq \emptyset$.

By a construction we have $a \in cl(F(X))$. We can resolve indeterminacy of the mapping F by a sequence of blowing up's. Consequently we obtain a blown projective plane X_p and a polynomial mapping $F' : X_p \to cl(X)$ such that $a \in F'(X_p)$. Now the statement follows from Theorem 5.7.$\blacksquare$

To prove the next result we need also some facts about extensions of finite mappings.

Definition 5.9 *Let $X \subset \mathbb{R}^n$, $Y \subset \mathbb{R}^m$ be irreducible algebraic sets. Let $f = (f_1, ..., f_m) : X \to Y$ be a polynomial mapping and let $f^* : \mathbb{R}[Y] \ni h \to h \circ f \in \mathbb{R}[X]$ be an induced mapping.*

The mapping f is called finite if the ring $\mathbb{R}[X]$ is a finite $f^*(\mathbb{R}[Y])$ algebra (or equivalently the ring $\mathbb{R}[X]$ is a finite $\mathbb{R}[f_1, ..., f_m]$ algebra).

Remark 5.10 a) A finite mapping is proper and it has finite fibers.

b) If a mapping f is proper then the mapping $\mathbb{C}f : \mathbb{C}X \to \mathbb{C}Y$ is also finite.

c) A composition of finite mappings is a finite mapping.

d) A non-constant polynomial mapping $\phi : \mathbb{R} \to \mathbb{R}^m$ is finite.

Theorem 5.11 *Let $X \subset \mathbb{R}^m$ be a real affine algebraic variety and let Γ be a closed algebraic subvariety of X. Let $f : \Gamma \to \mathbb{R}^n$ be a finite polynomial map and assume $n \geq \dim X$. Then there exists a finite polynomial map $F : X \to \mathbb{R}^n$, such that $\mathrm{res}_\Gamma F = f$.*

Proof. We follow closely our method from [5]. Let $I(\Gamma)$ denote the ideal of Γ in $\mathbb{R}[X]$. Let $g = (g_1, \ldots, g_n)$ be a polynomial extension of f to X. Let $x_1, \ldots, x_m$ be coordinate polynomial functions in $\mathbb{R}^m$.

It follows from our assumptions that there exist polynomial functions $a^i_j \in \mathbb{R}[X_1, \ldots, X_n]$ such that $H_i = \sum_{j=0}^{n_i} a^i_j(g)x_i^{n_i-j} = 0 \mod I(\Gamma)$, and $a^i_0(g) = \text{const} \neq 0$ for $i = 1, \ldots, m$.

Consider the map $H = (g_1, \ldots, g_n, H_1, \ldots, H_m) : X \to \mathbb{R}^{n+m}$. By construction H is a finite polynomial mapping and $H(\Gamma) \subset \mathbb{R}^n \times \{0\}$ and dim $H(X)$ =dim $X \leq n$. Of course the set $H(X)$ is a closed (in Euclidian topology) subset of $\mathbb{R}^{n+m}$. Let $Y := cl(H(X))$ be a Zariski closure of $H(X)$ in $\mathbb{R}^{n+m}$. Then dim Y = dim X. It is easy to see that there exists a finite linear projection $\pi : Y \to \mathbb{R}^n \times \{0\}$.

Indeed, we have the canonical inclusion $\mathbb{R}^{n+m} \subset \mathbb{P}^{n+m}(\mathbb{R})$ and the set W of points at infinity of Y has dimension $\leq n-1$. Hence we can find a linear subspace L of the hyperplane at infinity of dimension $m-1$ which is disjoint from $W \cup cl(\mathbb{R}^n \times \{0\})$. Let $\pi_L : \mathbb{P}^{n+m}(\mathbb{R}) \to cl(\mathbb{R}^n \times \{0\})$ be the projection determined by L. The restriction of π_L to Y is a finite mapping of Y into $\mathbb{R}^n \times \{0\}$ and it is the projection π we were looking for.

Now it is enough to take $F = p \circ H$ to obtain a finite extension of f to the whole of X.■

6 General case.

In the beginning we compare the real set S_f with the complex one. We have the following simple but basic fact:

Proposition 6.1 *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a dominant polynomial map and let $\mathbb{C}f : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ be its complexification. Then the set S_f of points at which f is not proper is contained in the set $S_{\mathbb{C}f}$ of points at which the mapping $\mathbb{C}f$ is not proper.*

Proof. The (easy) purely topological proof is left to the reader.■

In the sequel the following definition will be useful:

Definition 6.2 Let $S \subset \mathbb{R}^n$ be a semialgebraic set. The set S is called $\mathbb{R}$-uniruled if for every point $a \in S$ there is a parametric semi-line in S through the point a. Moreover, the set S is called generically $\mathbb{R}$-uniruled if there is an open and dense subset $U \subset S$, such that for every point $a \in U$ there is a parametric semi-line in S through this point.

Remark 6.3 Of course, every $\mathbb{R}$-uniruled set is generically $\mathbb{R}$-uniruled. Moreover, if a set S is generically $\mathbb{R}$-uniruled, then the set $\mathbb{C}S$ is $\mathbb{C}$-uniruled. It follows immediately from Proposition 3.3.

Lemma 6.4 *Let $S \subset \mathbb{P}^n(\mathbb{R})$ be a $\mathbb{R}$-rational curve, such that $S' := S \cap \mathbb{R}^n \neq \emptyset$ and S is not completely contained in $\mathbb{R}^n$. Then there is an algebraic curve $X \subset \mathbb{R}^2$ and a finite and surjective polynomial mapping $G : X \to S'$.*

Proof. Let $f : \mathbb{P}^1(\mathbb{R}) \to S$, be a real surjective regular mapping, which can be extended to a regular mapping $\mathbb{C}f : \mathbb{P}^1(\mathbb{C}) \to \mathbb{C}S$. Let $b \in \mathbb{P}^1(\mathbb{R})$ be a point such that $f(b) \notin \mathbb{R}^n$. Hence in fact we have a rational surjective mapping $f : \mathbb{R} \to S'$, which can be extended to a rational mapping $F := \mathbb{C}f : \mathbb{C} \to \mathbb{C}S'$.

Let $P := \{a_1, ..., a_n\}$ be a set of poles of the mapping $\mathbb{C}f : \mathbb{C} \to \mathbb{C}S'$. Let us note that if $a_i \in P$ then also $\overline{a_i} \in P$. Consequently the rational mapping $\phi : \mathbb{C} \ni t \to (t, 1/\prod_{j=1}^n (t - a_j) \in \mathbb{C}^2$ has real coefficients. Let $X' = \phi(\mathbb{C} \setminus P)$. Then X' is a closed curve in $\mathbb{C}^2$ and we have an isomorphism $\psi : X' \to \mathbb{C} \setminus P$. If we take $G := F \circ \psi$ then G has real coefficients and it gives a finite mapping $G : X' \to \mathbb{C}S'$. Now we can pass to the real part of X' and lemma follows.$\blacksquare$

Now we are ready to prove our main theorem:

Theorem 6.5 *Let $f = (f_1, ..., f_m) : \mathbb{R}^n \to \mathbb{R}^m$ be a generically finite polynomial mapping. Then the set S_f is closed, semi-algebraic and for every non-empty connected component $S \subset S_f$ we have $1 \leq \dim\, S \leq n - 1$.*

Moreover, the set S_f is $\mathbb{R}$-uniruled. Further, there is a real polynomial $P \in \mathbb{R}[x_1, ..., x_m]$ of degree at most $(\deg f)^{n-1} - 1$, such that $S_f \subset P^{-1}(0)$.

Proof. Let $\overline{X}$ be a closure of the set $X := graph(f) \subset \mathbb{R}^n \times \mathbb{R}^m$ in $\mathbb{P}^n(\mathbb{R}) \times \mathbb{R}^m$. Let us note that X is polynomially isomorphic to $\mathbb{R}^n$. The set $\overline{X}$ is semi-algebraic, hence the set $R := \overline{X} \setminus X$ is semialgebraic, too. Moreover, if $\pi : \mathbb{P}^n(\mathbb{R}) \times \mathbb{R}^m \to \mathbb{R}^m$ is the canonical projection we have $S_f = \pi(R)$, consequently the set S_f is closed and semi-algebraic.

Let $b \in S_f$. Then there is a point $a \in R$ such that $b = \pi(a)$. By Corollary 5.8 there is a $\mathbb{R}$-rational curve $\Lambda \subset cl(X) \subset \mathbb{P}^n(\mathbb{R}) \times \mathbb{P}^m(\mathbb{R})$, which contains the point a. We can assume that Λ contains a point $c \in X$, which is smooth with respect to f. Let $\Lambda' := \Lambda \cap X$. By Lemma 6.4 there exist an algebraic curve $\Gamma \subset \mathbb{R}^2$ and a finite and surjective polynomial mapping $q : \Gamma \to \Lambda'$. By Theorem 5.11 we can extend q to a finite map $Q : \mathbb{R}^2 \to \mathbb{R}^n \cong X$. Let us take $F = \phi \circ Q$, where $\phi : X \to \mathbb{R}^m$ is a canonical projection. It is easy to see that $S_F \subset S_f$ and $b \in S_F$. Now by Theorem 4.2 we get that there is a parametric semi-line in S_f through b.

Let $\pi : \mathbb{R}^m \to \{0\} \times \mathbb{R}^n$ be a projection which is finite on $cl_Z(f(X))$ and take $h = \pi \circ f$. Then $h : \mathbb{R}^n \to \mathbb{R}^n$ is a polynomial mapping and $S_h = \pi(S_f)$. Let us consider the mapping $\mathbb{C}h : \mathbb{C}^n \to \mathbb{C}^n$. We know by Theorem 6.1 that $S_h \subset S_{\mathbb{C}h}$. By Theorem 3.5 the set $S_{\mathbb{C}h}$ is described by a polynomial Q of degree less than $(\deg f)^{n-1}$. Moreover, since the mapping h is real, we have that the polynomial Q is also real (it can be deduced from [4]). Now, the set S_f is contained in the zero set of polynomial $P := Q \circ \pi$ and theorem follows.$\blacksquare$

Corollary 6.6 *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a generically finite polynomial mapping. Then every non-empty connected component of S_f is $\mathbb{R}$-uniruled, in particular unbounded.*

Corollary 6.7 *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a generically finite polynomial mapping. Let Q be a non-empty connected component of $\mathbb{R}^n \setminus S_f$. Then $H_{n-1}(Q, \mathbb{Z}) = 0$. In particular for $n = 2$ all these components are simply connected.*

Corollary 6.8 *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a generically finite polynomial mapping. Assume that S is an irreducible algebraic component of a (real) Zariski closure of S_f. Then the algebraic variety $\mathbb{C}S$ is $\mathbb{C}$-uniruled. Further , if $S \subset S_f$ and either S is smooth and connected or S is smooth and has pure dimension $n-1$, then S is generically $\mathbb{R}$-uniruled.*

Remark 6.9 If $f : \mathbb{R}^n \to \mathbb{R}^m$ is a generically finite polynomial mapping, then the real algebraic closure set S_f can be not generically $\mathbb{R}$-uniruled. In fact Gwoździewicz in [4] show that if f is the Pinchuk mapping, then the algebraic closure of the set S_f has a point as an isolated component.

7 Examples.

In this section we give some examples of the set S_f, as well as some applications of our results.

Theorem 7.1 *Let $S \subset \mathbb{R}^m$ be a semi-algebraic set and assume that there is a finite and surjective polynomial mapping $\phi : \mathbb{R}^{n-1} \to S$. Assume that $m \geq n$. Then there exists a polynomial mapping $f : \mathbb{R}^n \to \mathbb{R}^m$ with finite fibers for which S is exactly the set of points at which f is not proper (i.e., $S = S_f$).*

Proof. Indeed, let $V = \{(x_1, x_2, x_3, ..., x_{n+1}) \in \mathbb{R}^{n+1} : x_3^2 - x_1x_2 - 1 = 0\}$. Then V is an affine variety and there exists a (non-proper) embedding $i : \mathbb{R}^n \ni (u_1, v_2) \to (u_1(2 + u_1u_2), u_2, 1 + u_1u_2, u_3, ..., u_n) \in V$. Note that $V \setminus i(\mathbb{R}^n)$ is the linear subspace $l = \{x_3 = -1, x_2 = 0\} \cong \mathbb{R}^{n-1}$. If $\phi : l \ni (x_1, 0, -1, x_4, ..., x_{n+1}) \to \phi(x) \in S$ is a finite polynomial parametrization of S, then by Theorem 5.11, the mapping ϕ can be extended to a finite polynomial mapping $\Phi : V \to \mathbb{R}^m$ and it is enough to take $f = \Phi \circ i$.$\blacksquare$

Example 7.2 a) There is no polynomial mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ such that $f(\mathbb{R}^n) = \{(x_1, ..., x_n) \in \mathbb{R}^n : x_2 > 0, x_3 > 0, ..., x_n > 0,\ x_1 \cdot ... \cdot x_n > 1\}$. Indeed, let $U = f(\mathbb{R}^n)$ and $S = bd(U)$ (a border of U). Then $\mathbb{C}S$ has dimension $n - 1$ and consequently it must be a component of the set $S_{\mathbb{C}f}$. In particular it must be $\mathbb{C}$-uniruled. It means that the hypersurface $\{(x_1, ..., x_n) \in \mathbb{C}^n : \ x_1 \cdot ... \cdot x_n = 1\}$ is $\mathbb{C}$-uniruled, and we get a contradiction.

b) There is no polynomial mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ such that $f(\mathbb{R}^n) = \{(x_1, ..., x_n) \in \mathbb{R}^n : \ \sum_{i=1}^n x_i^2 > 1\}$. Indeed, let $U = f(\mathbb{R}^n)$ and $S = bd(U) = \{(x_1, ..., x_n) \in \mathbb{R}^n : \ \sum_{i=1}^n x_i^2 = 1\}$. Then S is a real algebraic set of dimension $n - 1$ and consequently it must be an irreducible component of the set $cl_Z(S_f)$. In particular it must be generically $\mathbb{R}$-uniruled, hence unbounded, a contradiction.

c) More generally, let $U \subset \mathbb{R}^n$ be an open domain. Assume that $bd(U)$ contains an algebraic set Γ of dimension $n-1$, such that either Γ is bounded, or Γ is smooth, connected and not generically $\mathbb{R}$-uniruled. Then there is no polynomial mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ such that $f(\mathbb{R}^n) = U$.

Example 7.3 Let us consider a polynomial mapping $f : \mathbb{R}^n \to \mathbb{R}^n$. If $n = 2$, then the set S_f (if non-empty) has codimension 1. But for $n > 2$ it is not longer true. Indeed, let

$$f_k : \mathbb{R}^n \ni (x_1, ..., x_n) \to (x_1, ..., x_{n-1}, (x_1^2 + ... + x_k^2)x_n),$$

where $k = 1, ..., n - 1$. Then $S_{f_k} = \{x \in \mathbb{R}^n : x_1 = x_2 = ... = x_k = 0\}$. In particular codim $S_{f_k} = k$, where $k = 1, ..., n - 1$.

8 Real Jacobian Conjecture

At the end of this paper is to worth to consider the case of a real polynomial mapping with the Jacobian which nowhere vanishes. We have the following:

Theorem 8.1 *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a real polynomial mapping with the Jacobian which nowhere vanishes. If* codim $S_f \geq 3$ *then f is a bijection (and consequently $S_f = \emptyset$).*

Proof. Let us consider sets $X := \mathbb{R}^n \setminus S_f$ and $Y := \mathbb{R}^n \setminus f^{-1}(S_f)$. Since codim $S_f \geq 3$ we have that the set X is simply connected. Moreover, the mapping $f : Y \to X$ is proper and unramified, hence it is a topological covering. In particular it is a homeomorphism. Consequently, the general fiber of the mapping f is a one point. Since the mapping f is unramified, this implies that f is an injection. Now our statements follows by theorem of Białynicki-Rosenlicht - see [2].■

On the other hand the example of Pinchuk shows that there are real polynomial mapping with the Jacobian which nowhere vanishes with codim $S_f = 1$. Hence the only interesting case is the case of codim $S_f > 1$ and we can state:

Real Jacobian Conjecture. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a real polynomial mapping with the Jacobian which nowhere vanishes. If* codim $S_f > 1$ *then f is a bijection (and consequently $S_f = \emptyset$).*

By Theorem 4.2 this conjecture is true in dimension two. The Real Jacobian Conjecture is closely connected with the following famous Jacobian Conjecture:

Jacobian Conjecture. *Let $f : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial mapping with the Jacobian which nowhere vanishes. Then f is an isomorphism.*

In fact we have:

Proposition 8.2 *The Real Jacobian Conjecture in dimension $2n$ implies the Jacobian Conjecture in (complex) dimension n.*

Proof. Indeed, let $f : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial mapping with a non-zero Jacobian. We can treat the mapping f as a real polynomial mapping $f : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$. Assume that f is not an isomorphism. In particular the mapping f can not be finite. Thus by [5], [6] we get that the set S_f has a complex codimension 1, hence it has a real codimension 2. Consequently, if the Real Jacobian Conjecture holds in dimension $2n$, this gives a contradiction.■

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