FLATNESS AND FIBRED POWERS OVER SMOOTH VARIETIES

ANDRÉ GALLIGO AND MICHAŁ KWIECIŃSKI

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ABSTRACT. We prove the equivalence between flatness of a complex algebraic map to a smooth variety of dimension n and torsion freeness of its n-th fibred power, under the assumption that the source space is of pure dimension. This generalizes the corresponding result for finite maps due to Auslander and proves a conjecture of Vasconcelos for a large class of objects. We develop several constructions in local analytic geometry and commutative algebra useful for our proof.

1. Introduction and Main Result

The simplest examples of non-flat complex algebraic maps $f : X \to Y$ are those which have **vertical components**, i.e. isolated or embedded components of X which are mapped to a set with empty interior in Y (Y being reduced). Most of the time non flatness occurs because of more subtle reasons, but in this paper we show quite precisely how non-flatness phenomena can be amplified using fibred powers to eventually produce vertical components. By a complex algebraic map, we mean a morphism which is locally of finite type, between schemes which are locally of finite type over $\mathbb{C}$. As special cases this includes maps in the affine or quasiprojective categories, allowing for non-reduced objects as fibred products. Our main result is :

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Theorem 1.1. *Let $f : X \to Y$ be a complex algebraic map. Suppose that X and Y are reduced, that X is of pure dimension and that Y is smooth. Let $n = \dim Y$. Then the following conditions are equivalent :*

- *f is flat,*
- *the canonical map $\underbrace{X \underset{Y}{\times} \cdots \underset{Y}{\times} X}_{n \text{ times}} \to Y$, has no (isolated or embedded) vertical components.*

Translated into commutative algebra, our theorem says that if we are given a finitely generated regular $\mathbb{C}$-algebra R of dimension n and a finitely generated R-algebra A, equidimensional over $\mathbb{C}$, then A is R-flat iff the n-th tensor power $A \underset{R}{\otimes} \cdots \underset{R}{\otimes} A$ is R-torsion free.

It is in a similar algebraic setup that Vasconcelos [15], [16] conjectured the above statement (without the equidimensionality assumption) and proved it in the case $\dim R = 2$ ($\dim Y = 2$). Almost 40 years ago, M.Auslander ([1], Theorem 3.2) established the corresponding result for finite maps (i.e. A is a finitely generated R-module). A less powerful result appeared in [11]: flatness is equivalent to the torsion freeness of *all* fibred powers. A somewhat parallel result can be found in [12]: the openness of f is equivalent to the absence of isolated (but perhaps not embedded) vertical components in the n-th fibred power.

The purpose of proving Theorem 1.1 is twofold. First of all, we wish to contribute to a better geometric understanding of flatness, together with classical and recent papers on the matter (see e.g. [3], [4], [7], [9], [10]). Secondly, at least for affine algebraic maps, our result is interesting from an algorithmic point of view (see e.g. [5] and [3]) as it establishes a link between flatness and primary decomposition via the simple algorithmic construction of the n-th tensor power.

Auslander's proof [1] relies on properties, specific to finitely generated modules, which are no longer true in our setup. Nevertheless, we will follow parts of the general pattern of that proof.

Our theorem remains valid in the local analytic category (in a suitable wording) and we will state and prove it that way (Theorem 6.1). This allows us to develop algebraic and transcendental-geometric tools. For induction purposes, we will deal not only with rings (local rings of analytic spaces) but also with a type of modules that we call "almost finitely generated". We show that the "flat dimension" of these modules behaves much like the projective dimension of finitely generated modules and extend some of Auslander's results on vanishing of Tor functors. In the geometric part, for an element of an almost finitely generated module, we construct its "support" which is a "hypergerm", i.e. a family of germs of sets, directed by the "$\supset$" relation. Such a support, whose construction is transcendental, enjoys many invariance properties and provides a finer geometric substitute to torsion. In particular, it allows us to overcome the fact that for non finitely generated modules M and N, $\mathrm{Tor}_i^R(M,N)$ is not necessarily a torsion R-module. Armed with these tools we proceed with our proof.

Throughout the paper, $\widehat{\otimes}$ will denote analytic tensor products and $\widehat{\mathrm{Tor}}$ the corresponding derived functors. We will use germs of analytic spaces, analytic sets and analytic maps but also germs of arbitrary subsets of analytic spaces (see e.g. [14]). This makes sense, even taking germs at points not belonging to those subsets.

2. Almost finitely generated modules and flat dimension

Let $t = (t_1, \ldots, t_n)$ be variables and let $R = \mathbb{C}\{t\}$ denote the ring of convergent power series over $\mathbb{C}$ in the variables t. The results of this section and the following one remain valid if we replace the ring of convergent power series by the ring of formal power series. In what follows, $\mathbf{m}$ will always denote the maximal ideal of R. By a **local analytic R-algebra**, we mean a ring A of the type $A = \mathbb{C}\{t,x\}/I$, where $x = (x_1, \ldots, x_s)$ are new variables and I is an ideal in $\mathbb{C}\{t,x\}$. We will be interested in R-modules M which are also finitely generated A modules, for some local analytic R-algebra A. By lack of a classical name, we call such modules **almost finitely generated** R-modules.

Proposition 2.1. *Let M be an almost finitely generated R-module. Then the following properties hold.*

1. *If $M \subset \mathbf{m}M$, then $M = 0$.*
2. *The set of zero divisors of M in R is a finite union of prime ideals.*
3. *If every element of $\mathbf{m}$ is a zero divisor of M, then there exists an element $m \in M$ such that $\mathbf{m} = \mathrm{Ann}_R(m)$.*

Sketch of proof of Proposition 2.1. Let A be a local analytic R-algebra, such that M is a finitely generated A-module. Assertion n^o 1 follows from Nakayama's lemma applied to M over A. To establish assertion n^o 2, notice that the set of zero divisors of M in R is the sum of the contractions to R of the associated primes of M in A. Finally, assertion n^o 3 follows from assertion n^o 2, the prime avoidance theorem and the above remark on associated primes. □

Proposition 2.2. *Let M and N be almost finitely generated R-modules. Then the following properties hold.*

1. *There exists a local analytic R-algebra A, which is a regular local ring, which is R-flat and such that M and N are finitely generated A-modules.*
2. *$M \widehat{\otimes}_R N$ is almost finitely generated.*
3. *The modules $\widehat{\mathrm{Tor}}_i^R(M,N)$ are almost finitely generated. More precisely, for A as in assertion n^o 1,*

$$\widehat{\mathrm{Tor}}_i^R(M,N) = \mathrm{Tor}_i^{A \widehat{\otimes}_R A}(M \widehat{\otimes}_R A, A \widehat{\otimes}_R N) .$$

4. $\widehat{\mathrm{Tor}}$ *is rigid, i.e. if* $\widehat{\mathrm{Tor}}^R_{i_0}(M,N) = 0$, *then* $\widehat{\mathrm{Tor}}^R_i(M,N) = 0$ *for all* $i \geq i_0$.

Sketch of proof of Proposition 2.2. Let A and B be local analytic R-algebras, such that M (resp. N) is a finitely generated A-module (resp. B-module). If $A = \mathbb{C}\{t,x\}/I$ and $B = \mathbb{C}\{t,y\}/J$, then one can replace both A and B by $\mathbb{C}\{t,x,y\}$. This shows assertion n^o 1. The second assertion follows from the formula

$$M \underset{R}{\widehat{\otimes}} N \cong \left(M \underset{R}{\widehat{\otimes}} B\right) \underset{A \underset{R}{\widehat{\otimes}} B}{\otimes} \left(A \underset{R}{\widehat{\otimes}} N\right) .$$

To show the third assertion, choose A as in 1. Let F_* be an A-free resolution of M. Since A is R-flat, F_* is also an R-flat resolution of M, and hence $\widehat{\mathrm{Tor}}^R_i(M,N)$ can be calculated as the cohomology of the complex $F_* \underset{R}{\widehat{\otimes}} N$. According to the proof of assertion n^o 2, this is the same as the cohomology of the complex

$$\left(F_* \underset{R}{\widehat{\otimes}} A\right) \underset{A \underset{R}{\widehat{\otimes}} A}{\otimes} \left(A \underset{R}{\widehat{\otimes}} N\right) .$$

Now, since A is R-flat, $F_* \underset{R}{\widehat{\otimes}} A$ is an $A \underset{R}{\widehat{\otimes}} A$-flat resolution of $M \underset{R}{\widehat{\otimes}} A$. Piecing this together yields the result.

To prove rigidity, notice that in assertion n^o 3 we may choose A to be $\mathbb{C}\{t,x\}$ and then $A \underset{R}{\widehat{\otimes}} A \cong \mathbb{C}\{t,x,x'\}$, where x' is another set of n variables. In particular $A \underset{R}{\widehat{\otimes}} A$ is a regular local ring. The assertion then follows from the formula in assertion n^o 3, together with Auslander's rigidity theorem for finitely generated modules over a regular local ring ([1], Theorem 2.1). $\square$

Lemma 2.3. *Let A be a local analytic R-algebra. Suppose, that we are given a short exact sequence of finitely generated A-modules (i.e. with A-linear homomorphisms):*

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

and an almost finitely generated R-module N. Then, there exists a local analytic R-algebra B, with the following properties.

1. *All modules* $\widehat{\mathrm{Tor}}^R_i(M_j,N)$ *are finitely generated B-modules,*
2. *The R-linear homomorphisms in the long exact sequence of* $\widehat{\mathrm{Tor}}^R_i(M_j,N)$ *are B-linear.*

The proof is similar to that of Proposition 2.2. $\square$

As usual, an M **- sequence** means a sequence $r_1,\ldots,r_p \in \mathbf{m}$, such that r_i is not a zero divisor in $M/(r_1,\ldots,r_{i-1})M$ for all i.

Lemma 2.4. *If M is an almost finitely generated R module, then all maximal M-sequences are of the same length. We call that length the* **depth** *of M and denote it* $\operatorname{depth}(M)$.

The classical proof of Northcott - Rees for finitely generated modules (see e.g. [13], VI, Proposition 3.1) carries over to this case, as the only properties of finitely generated modules used in that proof are among those established in Proposition 2.1 for almost finitely generated modules.

Definition 2.5. *The* **flat dimension** *of an R-module M is defined as the minimal length of a flat resolution of M (i.e. a resolution by R-flat modules). It is denoted by* $\operatorname{fd}(M)$.

Lemma 2.6. *If M is an almost finitely generated R-module, then*

1. $\operatorname{fd}(M)$ *is equal to the biggest index i such that $\widehat{\operatorname{Tor}}_i^R(M,N)$ is nonzero for some N,*
2. *in definition 2.5 it is enough to take into account resolutions by flat almost finitely generated modules.*

Proof. We will prove the two assertions simultaneously. Let A be as in Proposition 2.2, i.e. such that M is a finitely generated A-module and A is R-flat. Let F_* be an A-free (and hence R-flat) resolution of M. Now, the above lemma is an easy consequence of the following well known characterization of flatness which applies in the case of almost finitely generated modules: M is R-flat iff $\widehat{\operatorname{Tor}}_1^R(M, R/\mathbf{m}) = 0$ (see e.g. [9], 6, Proposition 2). Indeed, the lemma follows from the above characterization applied to the kernels of maps appearing in the resolution F_*. □

The flat dimension verifies the following Auslander - Buchsbaum type formula. Recall, that we have assumed once and for all that $R = \mathbb{C}\{t\}$ with $t = (t_1, \ldots, t_n)$.

Theorem 2.7. *If M is an almost finitely generated R-module, then*

$$\operatorname{fd}(M) + \operatorname{depth}(M) = n .$$

Note about the proof of Theorem 2.7. The proof can be adapted from the proof of the usual Auslander - Buchsbaum equality for finitely generated modules; for example from the one presented in [13] (Chapter VII, Proposition 1.12). This is left to the reader, with the following two comments.

Firstly, the proof in [13] relies on a lemma ([13], VII, Lemma 1.13) which states that if $r \in R$ is not a zero divisor of a finitely generated R-module M, then M is R-free if and only if M/rM is $R/(r)$-free. If we replace "free" by "flat", then this lemma remains true if M is an almost finitely generated R-module (see e.g. [9], 6, Proposition 7.)

Secondly, the following lemma allows us to perform induction on the dimension of R without leaving the rings of power series over $\mathbb{C}$ and in particular making the statement for $n = 0$ trivial.

Lemma 2.8. *Let M be an almost finitely generated R module. Suppose, that there exists an element of* $\mathbf{m}$ *which is not a zero divisor of M. Then, there exists a $\mathbb{C}$-linear combination of the variables $t_1, \ldots, t_n$ which is not a zero divisor of M.*

Proof of lemma 2.8. Suppose that the required linear combination does not exist. This means that the $\mathbb{C}$-linear space $L = \mathbb{C}t_1 + \cdots + \mathbb{C}t_n$ consists entirely of zero divisors of M. Thus, by n^o 2 of our Proposition 2.1, it is contained in the set-theoretic sum of primes giving the set of zero divisors. As the primes are also $\mathbb{C}$-vector spaces, L must be contained in one such prime P. Therefore $t_1, \ldots, t_n \in P$ and hence $\mathbf{m} \subset P$. This means in particular that $\mathbf{m}$ consists only of zero divisors of M, contrary to our assumption. $\square$

Corollary 2.9. *If M is a torsion free, almost finitely generated R-module then* $\mathrm{fd}\,(M) \leq n-1$.

Indeed, a torsion free module has depth at least one. $\square$

Armed with Propositions 2.1 and 2.2, Lemma 2.3, the notion of flat dimension and Theorem 2.7, we can rewrite section 1 of Auslander's paper [1], replacing "finitely generated" by "almost finitely generated", "free" by "flat", "homological dimension" by "flat dimension" and restricting ourselves to rings of power series over $\mathbb{C}$ (notice also that "depth" is called "codimension" in [1]). Indeed, we have established all the properties needed and the rewriting is purely formal.

In particular we obtain the following generalization of Corollary 1.3 of [1].

Proposition 2.10. *Suppose M and N are almost finitely generated R-modules. If $\widehat{\mathrm{Tor}}_i^R(M,N) = 0$ for all $i \geq 1$, then* $\mathrm{fd}\,(M) + \mathrm{fd}\,(N) = \mathrm{fd}\left(M \widehat{\otimes}_R N\right)$.

The subsequent sections of Auslander's paper [1] cannot however be automatically rewritten, since they make use of additional properties of finitely generated modules. These include the possibility to embed a torsion free finitely generated module in a flat finitely generated module and the fact that the Tor modules are torsion modules. In the next two sections, we will introduce some constructions to overcome these difficulties.

3. Hypergerms and supports

As in the previous section, let $t = (t_1, \ldots, t_n)$ be variables, let $R = \mathbb{C}\{t\}$ and let A be a local analytic R-algebra. By $\mathrm{Specan}\, A$, we will denote the unique germ of an analytic space X_ξ, with $A = \mathcal{O}_{X_\xi}$. If we denote $Y_\eta = \mathrm{Specan}\, R$, then the R-algebra structure of A corresponds to a germ of an analytic map $f_\xi : X_\xi \to Y_\eta$.

Further, let M be a finitely generated A-module and let $m \in M$. The purpose of this section is to define the support of m in Y_η and establish its properties which will be used in the proof of theorem 6.1. The property of having support with empty interior mimics the notion of being a torsion element.

In general, the image of a germ of an analytic space by a germ of an analytic map is not a well defined germ (think of a germ of a blowup map). So, we first introduce the notion of a hypergerm in a germ X_ξ of an analytic space X at a point ξ.

Definition 3.1. *Let X_ξ be a germ of an analytic space. A* **pre-hypergerm in** X_ξ *is a family $\{Z_\xi^\alpha\}_\alpha$ of (not necessarily locally analytic) germs of sets $Z_\xi^\alpha \subset X_\xi$, on which the inverse inclusion relation "$\supset$" satisfies the Moore-Smith property: for every two germs $Z_\xi^{\alpha_1}, Z_\xi^{\alpha_2}$, there exists a third germ $Z_\xi^{\alpha_3}$, such that $Z_\xi^{\alpha_1} \supset Z_\xi^{\alpha_3}$ and $Z_\xi^{\alpha_2} \supset Z_\xi^{\alpha_3}$.*

Two pre-hypergerms $\{Z_\xi^\alpha\}_\alpha$ and $\{W_\xi^\beta\}_\beta$ are said to be equivalent iff for every Z_ξ^α there exists W_ξ^β, such that $Z_\xi^\alpha \supset W_\xi^\beta$ and symmetrically: for every W_ξ^β there exists Z_ξ^α, such that $W_\xi^\beta \supset Z_\xi^\alpha$.

Definition 3.2. *A* **hypergerm** Γ_ξ *in X_ξ is the equivalence class of a pre-hypergerm in X_ξ in the above equivalence relation.*

Any germ is a hypergerm in an evident way (as the equivalence class of a one-element family). The advantage of hypergerms over germs is that one can define the image of a hypergerm as a hypergerm. Indeed, let $f_\xi : X_\xi \to Y_\eta$ be a germ of an analytic map. Let Γ_ξ be a hypergerm in X_ξ. To define a hypergerm $f_\xi(\Gamma_\xi)$ in Y_η we need to consider representatives X, Y and $f : X \to Y$ of X_ξ, Y_η and f_ξ; a fundamental system of neighbourhoods $\{U_\alpha\}_\alpha$ of ξ in X and a pre-hypergerm $\{Z_\xi^\beta\}_\beta$ - representative of Γ_ξ. Furthermore, for each β, let $\tilde{Z}^\beta$ be a representative of Z_ξ^β.

Definition 3.3. *The image $f_\xi(\Gamma_\xi)$ is defined as the hypergerm represented by the pre-hypergerm $\left\{\left(f(U_\alpha \cap \tilde{Z}^\beta)\right)_\eta\right\}_{(\alpha,\beta)}$.*

It is a simple exercise to prove that the above definition is independent of all the choices made. The same goes for the definition below.

Definition 3.4. *Let Γ_ξ and Ξ_ξ be hypergerms and $\{Z_\xi^\alpha\}_\alpha$ and $\{W_\xi^\beta\}_\beta$ their respective representatives. We will write*

$$\Gamma_\xi \supset \Xi_\xi$$

iff every Z_ξ^α contains some W_ξ^β.

Definition 3.5. *With the notations stated at the beginning of this section, we define* $\operatorname{supp}(m)$ *as a hypergerm in Y_η as follows. Let $Z_\xi \subset X_\xi$ be the zero-set germ of the annihilator ideal* $\operatorname{Ann}_A(m)$. *Let* $\operatorname{supp}(m)$ *be the image hypergerm $f_\xi(Z_\xi) \subset Y_\eta$.*

We remark that $\operatorname{supp}(m)$ is contained in (but generally not equal to) the zero set germ of $\operatorname{Ann}_R(m)$. Below, we show that $\operatorname{supp}(m)$ depends only on the R-module structure of M. This will be used tacitly in further reasoning.

Proposition 3.6. *The hypergerm* $\operatorname{supp}(m)$ *is independent of the choice of* A.

For the purpose of the proof, we shall denote $\operatorname{supp}_A(m)$ the support defined using A. As a first step towards the proof of Proposition 3.6, we prove the following lemma.

Lemma 3.7. *If* M *is a finitely generated* A*-module and also a finitely generated* R*-module and* $m \in M$*, then* $\operatorname{supp}_A(m) = \operatorname{supp}_R(m)$*. In particular, in this case* $\operatorname{supp}_A(m)$ *is equal to the zero-set germ of* $\operatorname{Ann}_R(m)$.

Proof of Lemma 3.7. Let $I = \operatorname{Ann}_A(m)$ and let Z_ξ be the zero set germ of I. The natural monomorphism (multiplication by m) $A/I \to M$ makes A/I into an A-submodule of M and hence into an R-submodule of M. Thus, A/I is a finitely generated R-module so the germ of the restriction $f_\xi|_{Z_\xi} : Z_\xi \to Y_\eta$ is a finite map. Thus, by the Remmert Rank Theorem (see e.g. [14]) its image hypergerm is actually a germ of a locally analytic set and is given as the zero-set germ of the inverse image of I by the map $R \to A$. As this last ideal is simply $\operatorname{Ann}_R(m)$ the lemma is proved. □

Proof of Proposition 3.6. We have to show that if we are given two locally analytic R-algebras A and B and if our R-module M is at the same time a finitely generated A-module and a finitely generated B-module, then $\operatorname{supp}_A(m) = \operatorname{supp}_B(m)$.

But M is a finitely generated $A\widehat{\underset{R}{\otimes}}B$-module and so the proof will be finished if we show that $\operatorname{supp}_A(m) = \operatorname{supp}_{A\widehat{\otimes}_R B}(m)$ (and symmetrically for $\operatorname{supp}_B(m)$). This equality of supports follows from lemma 3.7 (with R and A replaced by A and $A\widehat{\underset{R}{\otimes}}B$). □

Remark 3.8. *Let* A *be a locally analytic* R*-algebra and* $\lambda : M \to N$ *a monomorphism of finitely generated* A*-modules. Then* $\operatorname{supp}(m) = \operatorname{supp}(\lambda(m))$.

The remark follows directly from the fact that $\operatorname{Ann}_A(m) = \operatorname{Ann}_A(\lambda(m))$. □

Lemma 3.9. *Let* $\mu : A \to B$ *be a monomorphism of local analytic* R*-algebras and* $m \in A$*. Then* $\operatorname{supp}(m) \supset \operatorname{supp}(\mu(m))$.

Proof. The map $\operatorname{Specan} B \to \operatorname{Specan} R$ factors through $\operatorname{Specan} A$. The image of the zero-set germ of $\operatorname{Ann}_B(\mu(m))$ in $\operatorname{Specan} A$ is contained in the zero-set germ of $\mu^{-1}(\operatorname{Ann}_B(\mu(m))) = \operatorname{Ann}_A(m)$. This gives the result. □

4. Vertical components

To treat vertical components of germs of analytic maps we need the following notion of empty interior for hypergerms.

Definition 4.1. *Let* Γ_η *be a hypergerm in* Y_η*. We say that* Γ_η *has* **empty interior** *iff at least one of the germs in a representative pre-hypergerm (and hence in all representative pre-hypergerms) of* Γ_η *has empty interior in* Y_η.

Definition 4.2. *Let $f_\xi : X_\xi \to Y_\eta$ be a germ of an analytic mapping. A component (isolated or embedded) Z_ξ of X_ξ is called a* **vertical component** *of f_ξ iff the hypergerm $f(Z_\xi)$ has empty interior in Y_η.*

Elementary considerations on primary decompositions lead to the following lemma.

Lemma 4.3. *The following conditions are equivalent.*

- *f_ξ has no vertical components*
- *for every nonzero element $m \in \mathcal{O}_{X_\xi}$, the hypergerm* $\operatorname{supp}(m)$ *has nonempty interior in Y_η.*

The above lemma is a local analytic analogue of the equivalence between torsion - freeness and "without vertical components" in the algebraic case. Let us remark, that Osgood's map ([8]) has (only) vertical components but is torsion-free.

We use lemma 4.3 to extend the property of the absence of vertical components to almost finitely generated modules.

Definition 4.4. *Let M be an almost finitely generated R-module. We say that M has no vertical components, iff for any $m \in M$, with $m \neq 0$,* $\operatorname{supp}(m)$ *has nonempty interior in* Specan R.

This can of course be rephrased in terms of A-associated primes, if we want to mention A, but we shall not need it.

Proposition 4.5. *If M and N are almost finitely generated R-modules, $i > 0$ and $m \in \widehat{\operatorname{Tor}}_i^R(M,N)$, then the hypergerm* $\operatorname{supp}(m)$ *has empty interior in* Specan R.

Proof of Proposition 4.5. By the proof of Proposition 2.2 we may suppose that M and N are finitely generated A-modules where $A = \mathbb{C}\{t,x\}$. Taking representatives, we can now extend the germs $X_\xi = \operatorname{Specan} A$ and $Y_\eta = \operatorname{Specan} R$ to analytic spaces X and Y, extending also the map between the germs to a map $f : X \to Y$ and extending the modules M and N to coherent sheaves $\mathcal{M}$ and $\mathcal{N}$ on X. Consider $X \underset{Y}{\times} X$ with projections π_1 and π_2 on X. By n^o 3 of Proposition 2.2, for any $\xi' \in X$,

$$\widehat{\operatorname{Tor}}_i^{\mathcal{O}_{Y_{f(\xi')}}}(\mathcal{M}_{\xi'},\mathcal{N}_{\xi'}) \;=\; \operatorname{Tor}_i^{\mathcal{O}_{(X \underset{Y}{\times} X)(\xi',\xi')}}\left((\pi_1^*\mathcal{M})_{(\xi',\xi')},(\pi_2^*\mathcal{N})_{(\xi',\xi')}\right) .$$

We can now also extend the element $m \in \widehat{\operatorname{Tor}}_i^A(M,N)$, to a section $\tilde{m}$ of the coherent sheaf $\operatorname{Tor}_i^{\mathcal{O}_{X \underset{Y}{\times} X}}(\pi_1^*\mathcal{M},\pi_2^*\mathcal{N})$. Then $\operatorname{supp}(m)$ is just the image hypergerm of the germ of the (usual) support of the section $\tilde{m}$. By a theorem of Frisch (see [6] or [2] V. th. 4.5) there exists a set $\Sigma \subset Y$, with empty interior in Y, such that for any point $\eta' \notin \Sigma$ and for any $\xi' \in f^{-1}(\eta')$ the stalk $\mathcal{M}_{\xi'}$ is $\mathcal{O}_{Y,\eta'}$-flat. Hence, the support of the section $\tilde{m}$, must be contained in $f^{-1}(\Sigma)$ and thus $\operatorname{supp}(m) \subset \Sigma_\eta$. In particular $\operatorname{supp}(m)$ has empty interior in Y_η. $\square$

Lemma 4.6. *If Y_η is reduced, then any flat map $f_\xi : X_\xi \to Y_\eta$ has no vertical component.*

Proof of Lemma 4.6. Let $A = \mathcal{O}_{X_\xi}$ and take $m \in A$, $m \neq 0$. We have to show that $\operatorname{supp}(m)$ has nonempty interior in Y_η. Extend X_η and Y_η to analytic spaces X and Y, extending the map f_ξ to a flat analytic map $f : X \to Y$ (this is possible by the openness of the set of points of flatness [2] V, th.4.2). Shrinking X and Y if necessary, extend m to $\tilde{m} \in \mathcal{O}_X(X)$. Let S be the (usual) support of $\tilde{m}$ in X. Now S_ξ is the zero set-germ of the ideal $\operatorname{Ann}_A(m)$ and hence $\operatorname{supp}(m) = f_\xi(S_\xi)$. Now, in an any neighbourhood of ξ, one can find a point ξ', such that all nonempty fibres of the restriction $f|_S$ are of the same dimension in a neighbourhood of ξ'. Indeed, to find the set of such points ξ' it is enough to take the smooth part of the set S and throw away the subset where the rank of $f|_S$ drops. Now by the Remmert rank theorem (see e.g. [14]), the image $f_{\xi'}(S_{\xi'})$ is a germ of an analytic subset of Y. Hence, it must be equal to $Y_{f(\xi')}$, otherwise $\tilde{m}_{\xi'}$ would be a torsion element of $\mathcal{O}_{X_{\xi'}}$ over $\mathcal{O}_{Y_{f(\xi')}}$, which is impossible by the flatness of f. In summary, we have shown that in an aribitrary neighbourhood of ξ in X, we can find a point $\xi' \in S$, such that $f_{\xi'}(S_{\xi'})$ has nonempty interior in $Y_{f(\xi')}$. It is easy to deduce that $f_\xi(S_\xi)$ must have nonempty interior in $Y_{f(\xi)}$, which is what we had to show. □

5. Equidimensional maps - structure and homological properties.

Proposition 5.1. *Let X_ξ, Y_η be germs of reduced analytic spaces. Assume that X_ξ is of pure dimension and that Y_η is smooth. Let $f_\xi : X_\xi \to Y_\eta$ be a germ of an equidimensional analytic map (i.e. such that all nonempty fibers are of dimension $\dim X_\xi - \dim Y_\eta$). Then f_ξ factors as*

$$X_\xi \xrightarrow{\phi_\xi} Z_\zeta \xrightarrow{\pi_\zeta} Y_\eta \ ,$$

where

- *Z_ζ is a germ of a smooth analytic space,*
- *π_ζ is flat,*
- *ϕ_ξ is finite and torsion-free.*

Sketch of proof of Proposition 5.1. Take an installation of f_ξ, i.e. a factorization:

$$f_\xi = pr_1 \circ \iota$$

where ι is a a germ of a locally closed embedding $X_\xi \to Y_\eta \times \mathbb{C}^s_0$ with $\xi = (\eta, 0)$, and pr_1 is the natural projection $Y_\eta \times \mathbb{C}^s_0 \to Y_\eta$. Such a factorization is easily obtained for example by embedding first X_ξ in an $\mathbb{C}^s_0$ and then the graph of f_ξ in $Y_\eta \times \mathbb{C}^s_0$.

Now, let $p : \mathbb{C}^s \to \mathbb{C}^d$ be a generic linear projection and consider the composition

$$X_\xi \xrightarrow{\iota} Y_\eta \times \mathbb{C}^s_0 \xrightarrow{\mathrm{id}_{Y_\eta} \times p} Y_\eta \times \mathbb{C}^d_0 \xrightarrow{\mathrm{pr}_1} Y_\eta .$$

Taking $Z_\zeta = Y_\eta \times \mathbb{C}_0^d$, $\pi_\zeta = \mathrm{pr}_1$, and $\phi_\xi = (\mathrm{id}_{Y_\eta} \times p) \circ \iota$ one obtains a factorization with the desired properties. In particular, by the equidimensionality assumption and since ϕ_ξ is finite, the image by ϕ_ξ of any isolated component of X_ξ contains a germ of a neighbourhood of $(\eta, 0)$ in $Y_\eta \times \mathbb{C}_0^d$. Since X_ξ is reduced, this implies that ϕ_ξ has no vertical components and hence that ϕ_ξ is torsion free. $\square$

Lemma 5.2. *Under the assumptions of Proposition 5.1, consider the factorization from that Proposition. Then, there exists a finitely generated, free $\mathcal{O}_{Z_\zeta}$-module F and an $\mathcal{O}_{Z_\zeta}$-linear monomorphism $\mathcal{O}_{X_\xi} \to F$.*

Proof of lemma 5.2. The proof follows directly from the following well known lemma.

Lemma 5.3. *Let A be a noetherian integral domain and M - a torsion free, finitely generated A-module. Then there exists a finitely generated, free A-module F, and an A-linear monomorphism $M \to F$.*

Such a monomorphism is easily constructed by composing the natural map $M \to (M^*)^*$, which is injective in this case, with the dual of a presentation of M^*. $\square$

Proposition 5.4. *Let X_ξ and Y_η be germs of reduced complex analytic spaces. Suppose that X_ξ is of pure dimension and that Y_η is smooth. Let $f_\xi : X_\xi \to Y_\eta$ be a germ of an equidimensional analytic map. Denote $R = \mathcal{O}_{Y_\eta}$ and $M = \mathcal{O}_{X_\xi}$. Let N be an almost finitely generated R-module.*

If $M \widehat{\otimes}_R N$ has no vertical components (as understood in definition 4.4) then:

1. $\widehat{\mathrm{Tor}}_i^R(M, N) = 0$ *for all* $i > 0$,
2. $\mathrm{fd}\,(M) + \mathrm{fd}\,(N) = \mathrm{fd}\left(M \widehat{\otimes}_R N\right)$.

Proof of Proposition 5.4. Apply Proposition 5.1 and let $A = \mathcal{O}_{Z_\zeta}$. By lemma 5.2, we have an exact sequence of finitely generated A-modules, where F is A-free:

$$0 \to M \to F \to F/M \to 0\ .$$

In particular F is R-flat. Tensoring this sequence with N, we get an exact sequence:

$$0 \to \widehat{\mathrm{Tor}}_1^R(F/M, N) \to M \widehat{\otimes}_R N \to F \widehat{\otimes}_R N \to (F/A) \widehat{\otimes}_R N \to 0\ ,$$

and isomorphisms

$$\widehat{\mathrm{Tor}}_{i+1}^R(F/M, N) \cong \widehat{\mathrm{Tor}}_i^R(M, N)$$

for all $i \geq 1$. We now show that $\widehat{\mathrm{Tor}}_1^R(F/M, N) = 0$. Let $m \in \widehat{\mathrm{Tor}}_1^R(F/M, N)$, by Proposition 4.5, $\mathrm{supp}(m)$ has empty interior in Y_η. Furthermore, by lemma 2.3 the monomorphism $\lambda : \widehat{\mathrm{Tor}}_1^R(F/M, N) \to M \widehat{\otimes}_R N$ is a monomorphism of finitely generated B-modules for some local analytic R-algebra B. By remark 3.8, $\mathrm{supp}(\lambda(m))$ is the same as $\mathrm{supp}(m)$ hence has empty interior. By the assumption on $M \widehat{\otimes}_R N$, this implies that $\lambda(m) = 0$ and hence $m = 0$. We have therefore proved that $\widehat{\mathrm{Tor}}_1^R(F/M, N) = 0$. By rigidity ($n^o$ 4 of Proposition 2.2), we get $\widehat{\mathrm{Tor}}_i^R(F/M, N) = 0$ for all $i \geq 1$, hence $\widehat{\mathrm{Tor}}_i^R(M, N) = 0$ for all $i \geq 1$. The second assertion now follows from Proposition 2.10. □

6. Flatness in analytic maps.

Below we state and prove the main theorem in the local analytic category. The algebraic case (theorem 1.1) follows easily by standard faithful flatness arguments (as the one detailed in [11]). A completely analogous global analytic statement is also true. We do not state it here, referring to [11] for a formal definition of global components (especially embedded components) for analytic spaces. We remark that a map $f : X \to Y$ has no vertical components if no germ $f_\xi : X_\xi \to Y_{f(\xi)}$ has vertical components.

We will use the following notation. The point $\underbrace{(\xi, \dots, \xi)}_{i \text{ times}}$ will be denoted by $\xi^{\{i\}}$. The fibred power germ $\underbrace{X_\xi \times_{Y_\eta} \cdots \times_{Y_\eta} X_\xi}_{i \text{ times}}$ will be denoted by $X_{\xi^{\{i\}}}^{\{i\}}$ and the canonical map germ $X_{\xi^{\{i\}}}^{\{i\}} \to Y_\eta$ will be denoted by $f_{\xi^{\{i\}}}^{\{i\}}$. The meaning of vertical components for a germ of an analytic map is given in definition 4.2.

Theorem 6.1. *Let $f : X_\xi \to Y_\eta$ be a germ of a complex analytic map of germs of complex analytic spaces. Suppose that X_ξ and Y_η are reduced, that X_ξ is of pure dimension and that Y_η is smooth. Let $n = \dim Y_\eta$. Then the following conditions are equivalent :*
- *f_ξ is flat,*
- *the canonical map $f_{\xi^{\{n\}}}^{\{n\}} : X_{\xi^{\{n\}}}^{\{n\}} \to Y_\eta$ has no (isolated or embedded) vertical components.*

We begin the proof with the following lemma.

Lemma 6.2. *If $f_{\xi^{\{n\}}}^{\{n\}} : X_{\xi^{\{n\}}}^{\{n\}} \to Y_\eta$ has no vertical components, then for any $i \leq n$, the map $f_{\xi^{\{i\}}}^{\{i\}} : X_{\xi^{\{i\}}}^{\{i\}} \to Y_\eta$ has no vertical components.*

Proof of lemma 6.2. Let $R = \mathcal{O}_{Y,\eta}$, $M = \mathcal{O}_{X,\xi}$. By lemma 4.3, $M^{\widehat{\otimes}_R n}$ has no vertical components. Fix $i \leq n$ and consider the natural homomorphism of R-algebras

$$M^{\widehat{\otimes}_R i} \to M^{\widehat{\otimes}_R n}$$

$$m_1 \otimes \cdots \otimes m_i \mapsto m_1 \otimes \cdots \otimes m_i \otimes 1 \otimes \cdots \otimes 1 \ .$$

It is injective, since it has a left inverse:

$$m_1 \otimes \cdots \otimes m_{i-1} \otimes (m_i m_{i+1} \cdots m_n) \leftarrow m_1 \otimes \cdots \otimes m_n \ .$$

Therefore, by Lemma 3.9 and Lemma 4.3 $f_{\xi\{i\}}^{\{i\}}$ has no vertical components. □

Proof of Theorem 6.1.

If f_ξ is flat, each of its fibred powers is also flat hence, by lemma 4.6, cannot have vertical components.

Suppose therefore that $f_{\xi\{n\}}^{\{n\}}$ has no vertical components. We may also assume that $\dim Y_\eta > 0$.

First, we prove that f_ξ is equidimensional recalling an argument of Kwieciński and Tworzewski ([12], Theorem 3.2). Since, by lemma 6.2, f_ξ itself has no vertical components, by the previously cited theorem of Frisch there is a dense open subgerm $V_\eta \subset Y_\eta$, such that for $U_\xi = f_\xi^{-1}(V_\eta)$, the restriction $f_\xi|_{U_\eta}$ is a germ of an equidimensional map. Since $f_{\xi\{n\}}^{\{n\}}$ has no vertical isolated components, $U_{\xi\{n\}}^{\{n\}}$ is dense in $X_{\xi\{n\}}^{\{n\}}$ hence

$$\dim X_{\xi\{n\}}^{\{n\}} = \dim U_{\xi\{n\}}^{\{n\}} = \dim Y_\eta + n(\dim X_\xi - \dim Y_\eta) = n(\dim X_\xi - n + 1) \ .$$

But, also because of the lack of vertical isolated components in $f_{\xi\{n\}}^{\{n\}}$, we have:

$$\dim X_{\xi\{n\}}^{\{n\}} \ > \ \dim\left(\left(f_{\xi\{n\}}^{\{n\}}\right)^{-1}(\eta)\right) \ = \ \dim\left((f_\xi^{-1}(\eta))^n\right) \ = \ n(\dim f_\xi^{-1}(\eta)) \ .$$

Thus $\dim f_\xi^{-1}(\eta) < \dim X_\xi - n + 1$ and hence $\dim f_\xi^{-1}(\eta) = \dim X_\xi - \dim Y_\eta$ which means precisely that f_ξ is equidimensional.

We now prove that f_ξ is flat. Let $R = \mathcal{O}_{Y,\eta}$, $M = \mathcal{O}_{X,\xi}$. By lemma 4.3, $M^{\widehat{\otimes}_R n}$ has no vertical components. For $1 \leq i \leq n$, by lemma 6.2, $M^{\widehat{\otimes}_R i}$ has no vertical components. Thus we can apply lemma 5.4 inductively to all $N = M^{\widehat{\otimes}_R i}$, to get

$$\mathrm{fd}\left(M^{\widehat{\otimes}_R n}\right) \ = \ n\mathrm{fd}\,(M) \ .$$

On the other hand, by Corollary 2.9 we have

$$\mathrm{fd}\left(M^{\widehat{\otimes}_R n}\right) < \dim R = n \ .$$

Therefore $n\mathrm{fd}\,(M) < n$ and hence $\mathrm{fd}\,(M) = 0$. In other words, M is R flat which means that f_ξ is flat. This ends the proof of Theorem 6.1. □

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References

[1] M. Auslander, *Modules over unramified regular local rings*, Illinois J. Math. **5** (1961), 631–647.

[2] C. Bănică, O. Stănăşilă, "Algebraic methods in the global theory of complex spaces", Editura Academiei Bucharest; John Wiley and Sons, London - New York - Sydney, 1976.

[3] D. Bayer, A. Galligo, M. Stillman, *Groebner bases and extension of scalars*, in "Computational algebraic geometry and commutative algebra", Proc. Cortona 1991, D.Eisenbud et al. (ed.), Cambridge University Press Symp. Math. 34 (1993), pp. 198-215.

[4] E. Bierstone, P.D. Milman, "The local geometry of analytic mappings", Dottorato di Ricerca in Matematica, ETS Editrice, Pisa, 1988.

[5] D.Eisenbud, C.Huneke, W.V.Vasconcelos, *Direct methods for primary decomposition*, Invent. Math. **110** (1992), pp. 207-235.

[6] J. Frisch, *Points de platitude d'un morphisme d'espaces analytiques complexes*, Invent. Math. **4** (1967), pp. 118–138.

[7] A. Galligo, *Théorème de division et stabilité en géométrie analytique locale*, Annales Inst. Fourier **29** (1979), pp. 107-184.

[8] H.Grauert, R.Remmert, "Analytische Stellenalgebren", Springer Verlag, 1971.

[9] H. Hironaka, *Stratifications and Flatness*, in "Real and Complex Singularities", Proc. Oslo 1976, ed. Per Holm, Stijthof and Noordhof 1977, 199–265.

[10] J. Kollár, *Flatness criteria*, J. of Algebra, **175** (1995), pp. 715-727.

[11] M. Kwieciński, *Flatness and fibred powers*, Manuscripta Mathematica **97**,n^o 2 (1998), pp. 163-173.

[12] M. Kwieciński, P. Tworzewski, *Fibres of analytic maps* , Bull. Polish Acad. Sci. Math. **47** (1999)n^o 3, to appear (earlier e-print version: alg-geom/9606008).

[13] E. Kunz, "Introduction to Commutative Algebra and Algebraic Geometry", Birkhäuser, 1985.

[14] S. Łojasiewicz, "Introduction to Complex Analytic Geometry", Birkhäuser, Basel, 1991.

[15] W.V. Vasconcelos, *Flatness testing and torsionfree morphisms*, Journal of Pure and Applied Algebra **122**, 313-321 (1997).

[16] W.V. Vasconcelos, "Computational methods in commutative algebra and algebraic geometry", Algorithms and Computation in Mathematics 2, Springer Verlag, 1998.

Université de Nice - Sophia-Antipolis, Laboratoire Jean-Alexandre Dieudonné, Parc Valrose, 06108 Nice, France.

E-mail address: galligo@math.unice.fr

Uniwersytet Jagielloński, Instytut Matematyki, ul. Reymonta 4, 30-059 Kraków, Poland.

E-mail address: kwiecins@im.uj.edu.pl