POLAR INVARIANTS OF PLANE CURVES AND THE NEWTON POLYGON

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Abstract

We prove a factorization theorem for the polars of plane singularities with respect to the Newton diagram and calculate the polar quotients of nondegenerated singularities.

Introduction

Let $f = f(X, Y) \in \mathbf{C}\{X, Y\}$ be a convergent power series in two variables X, Y. In all this note we assume that f has an isolated singularity at $(0, 0) \in \mathbf{C}^2$. The polar of f with respect to the direction $(a : b) \in \mathbf{P}^1(\mathbf{C})$ of the line $t = bX - aY$ is the power series $\partial f = \frac{\partial(t,f)}{\partial(X,Y)} = a\frac{\partial f}{\partial X} + b\frac{\partial f}{\partial Y}$. If the line $bX - aY = 0$ is not tangent to the curve $f = 0$ then the polar quotients

$$\frac{(f, h)_0}{\operatorname{ord} h}, \quad h \text{ runs over irreducible divisors of } \partial f$$

(where $(f, h)_0$ is $\mathbf{C}$ codimension of the ideal generated by f, h in $\mathbf{C}\{X, Y\}$ and $\operatorname{ord} h$ is the order of h) are topological invariants of the singularity $f = 0$ (see [LMW2]).

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According to Teissier [Te1] the Łojasiewicz exponent $\mathcal{L}_0(f)$ defined to be the smallest $\theta > 0$ such that

$$|\mathrm{grad}\, f(x,y)| \geq C \max(|x|,|y|)^{\theta} \text{ near } (0,0) \in \mathbf{C}^2 ,$$

is given by the formula:

$$\mathcal{L}_0(f) = (\text{the greatest polar quotient of } f=0) - 1 \; .$$

Merle in [M] proved a theorem on a factorization of the generic polar of an algebroid irreducible curve and calculated its polar quotients in terms of the characteristic. His result was generalized by different authors (see [Ab.A], [C], [D], [G]). The aim of this note is to calculate the polar quotients of a curve $f=0$ nondegenerated in the sense of Kouchnirenko [Kou] in terms of the Newton polygon of f. Our main result gives a factorization of the generic polar curve and is parallel to Merle's theorem (see [M], Théorème 3.1). As an application we give a proof of the formula for the Łojasiewicz exponent obtained recently by the first author [Len]. The formula is an improvement of an earlier result due to Lichtin [Li].

1 Main result

Let $f \in \mathbf{C}\{X,Y\}$ be a nonzero power series without constant term. Write $f = \sum c_{\alpha\beta} X^{\alpha} Y^{\beta}$ and $\mathrm{supp}\, f = \{(\alpha,\beta) \in \mathbf{N}^2 : \; c_{\alpha\beta} \neq 0\}$. The Newton polygon $\mathcal{N}(f) = \mathcal{N}_f$ is the set of the compact faces of the boundary of the convex hull $\Delta(f)$ of $\mathrm{supp}\, f + \mathbf{N}^2$. For any segment $S \in \mathcal{N}_f$ we let $\mathrm{in}(f,S) =$ the sum of all monomials $c_{\alpha\beta} X^{\alpha} Y^{\beta}$ such that $(\alpha,\beta) \in S$. According to Kouchnirenko [Kou], the series f is nondegenerated on $S \in \mathcal{N}_f$ if the polynomial $\mathrm{in}(f,S)$ has no critical points in the set $\mathbf{C}^* \times \mathbf{C}^*$, where $\mathbf{C}^* = \mathbf{C} \setminus \{0\}$. The series in nondegenerated if it is nondegenerated on every segment of its Newton polygon. For every segment S of the Newton polygon $\mathcal{N}_f$ we denote by $|S|_1$ and $|S|_2$ the lengths of the projections of S on the horizontal and vertical axes. We call $|S|_1/|S|_2$ the inclination of S.

A segment $S \in \mathcal{N}_f$ is exceptional if $1 = |S|_1 < |S|_2$ (resp. $1 = |S|_2 < |S|_1$) and S has a vertex on the vertical (resp. horizontal) axis. Let $\mathcal{N}_f^*$ be the set of segments of $\mathcal{N}_f$ which are not exceptional. For every $S \in \mathcal{N}_f^*$ we consider the number $\varepsilon(S) \in \{-1,0,1\}$ defined as follows: $\varepsilon(S) = -1$ if $1 < |S|_1 < |S|_2$ (resp. $1 < |S|_2 < |S|_1$) and S has a vertex on the vertical (resp. horizontal) axis, $\varepsilon(S) = 1 -$ (the number of vertices of S lying on the axes) if $|S|_1 = |S|_2$; $\varepsilon(S) = 0$ for all remaining cases. Let $\frac{\alpha}{\alpha(S)} + \frac{\beta}{\beta(S)} = 1$ be the equation of the line containing S. Obviously $\alpha(S), \beta(S) > 0$ are rational numbers and $\alpha(S)/\beta(S) = |S|_1/|S|_2$.

Now we can state our main result

Theorem 1.1 *Let $f = f(X, Y) \in \mathbf{C}\{X, Y\}$ be a power series with an isolated singularity at $(0,0) \in \mathbf{C}^2$. Then for every line $bX - aY = 0$ not tangent to the curve $f = 0$ there is a factorization of the polar $\partial f = a\frac{\partial f}{\partial X} + b\frac{\partial f}{\partial Y}$:*

$$\partial f = v \prod_{S \in \mathcal{N}_f^*} g_S \quad \textit{in } \mathbf{C}\{X, Y\}$$

such that

(i) $\operatorname{ord} g_S = \min(|S|_1, |S|_2) + \varepsilon(S)$ *for $S \in \mathcal{N}_f^*$,*

(ii) if h is an irreducible factor of g_S, then

$$\frac{(f,h)_0}{\operatorname{ord} h} \geq \max(\alpha(S), \beta(S)) ,$$

the equality holding if f is nondegenerated on S.

(iii) if $\operatorname{ord} v > 0$ *then* $\operatorname{ord} v = 1$ *and* $(f, v)_0 = \operatorname{ord} f$.

The proof of the above theorem we give in Section 3 of this paper. Now, we state corollaries.

Corollary 1.2 *Let $f = f(X, Y) \in \mathbf{C}\{X, Y\}$ be a power series with an isolated singularity at $(0,0) \in \mathbf{C}^2$ nondegenerated in Kouchnirenko's sense. Then*

(i) the numbers $\max(\alpha(S), \beta(S))$ *where $S \in \mathcal{N}_f^*$ are polar quotients of $f = 0$,*

(ii) if $q \neq$ $\operatorname{ord} f$ *is a polar quotient of f then $q =$* $\max(\alpha(S), \beta(S))$ *for a segment $S \in \mathcal{N}_f^*$.*

The above corollary follows immediately from Theorem 1.1. One can complete (1.2) by noting that with the assumptions of (1.2) the number $\operatorname{ord} f$ is a polar quotient of $f = 0$ if and only if the line $\alpha + \beta = \operatorname{ord} f$ intersects $\operatorname{supp} f$ at a point not lying on the axes.

Using (1.2) and Teissier's formula for the Łojasiewicz exponent mentioned above we get the following corollary (see [Li], [F], [Len]):

Corollary 1.3 *If $f \in \mathbf{C}\{X, Y\}$ has an isolated singularity and is nondegenerated in Kouchnirenko's sense, then*

$$\mathcal{L}_0(f) = \max\{\max(\alpha(S), \beta(S)) : \ S \in \mathcal{N}_f^*\} - 1$$

if $\mathcal{N}_f^* \neq \emptyset$.

Note that $\mathcal{L}_0(f) = \operatorname{ord} f = 2$ if $\mathcal{N}_f^* = \emptyset$.

Corollary 1.4 *Let $f = \sum_{\alpha w_1 + \beta w_2 \geq w} c_{\alpha\beta} X^\alpha Y^\beta$ where $f_0 = \sum_{\alpha w_1 + \beta w_2 = w} c_{\alpha\beta} X^\alpha Y^\beta$ has an isolated singularity of order > 2. Then the number $\max(w/w_1, w/w_2)$ is a polar quotient of $f = 0$ and there are no polar quotients of $f = 0$ different from $\operatorname{ord} f$ and $\max(w/w_1, w/w_2)$.*

Proof. We have $\mathcal{N}_f = \{S\}$, where S is a segment lying on the line $\alpha w_1 + \beta w_2 = w$. It is easy to check that S is not exceptional. Hence $\partial f = v g_S$ and $\max(\alpha(S), \beta(S)) = max(w/w_1, w/w_2)$. Then we use (1.2).

Examples
1. If $f = X^a + Y^b$ where $a, b > 1$ are integers, then there is the unique polar quotient and it is equal to $\max(a, b)$.
2. If $f = X^a Y + XY^b$ where $a, b > 1$, $a \neq b$ are integers, then there are two polar quotients: $\min(a, b) + 1$ and $(ab - 1)/(\min(a, b) - 1)$.
3. If $f = Y^8 + XY^5 + X^3Y^2 + X^6$ then $\mathcal{N}_f = \{A, B, C\}$ where A is exceptional, B is of the first kind and C is of the second kind. Here $\varepsilon(B) = 0$, $\varepsilon(C) = -1$ and $\operatorname{ord} g_B = 2$, $\operatorname{ord} g_C = 1$. From $\partial f = v g_B g_C$ we get $\operatorname{ord} v = 1$. Hence there are three polar quotients: $\operatorname{ord} f = 5$, $\max(\alpha(B), \beta(B)) = \beta(B) = 6\frac{1}{2}$, $\max(\alpha(C), \beta(C)) = \alpha(C) = 6$. Moreover $\mathcal{L}_0(f) = 6\frac{1}{2} - 1 = 5\frac{1}{2}$.

2 Newton polygon of the polar curves

Let $f \in \mathbf{C}\{X, Y\}$ be a nonzero power series such that $\operatorname{ord} f > 1$. We assume that the distances d_1, d_2 from $\operatorname{supp} f$ to the axes are less or equal to 1. Every series with an isolated singularity satisfies the above conditions.

Let $\partial = a\frac{\partial}{\partial X} + b\frac{\partial}{\partial Y}$ be such that $ab \neq 0$ and $\partial(\mathrm{in} f) \neq 0$ (note that $\partial(\mathrm{in} f) = 0$ if and only if $\mathrm{in} f = const.(bX - aY)^{\operatorname{ord} f}$). To avoid a trivial case we assume that f is not generic with respect to X, Y that is $\operatorname{ord} f(X, 0) \neq \operatorname{ord} f$ or $\operatorname{ord} f(0, Y) \neq \operatorname{ord} f$.

Our purpose is to compare the Newton polygons of the series f and ∂f. We divide the nonexceptional segments of $\mathcal{N}_f$ into two classes. A segment $S \in \mathcal{N}_f$ is *of the first kind* if 1) $|S|_1 < |S|_2$ and S has no vertex on the vertical axis or 2) $|S|_2 < |S|_1$ and S has no vertex on the horizontal axis or

3) $|S|_1 = |S|_2$ and S has no vertices on the axes. Clearly, every segment of the first kind is not exceptional. A segment $S \in \mathcal{N}_f$ is *of the second kind* if it is neither of the first kind nor exceptional.

Let S be a segment of the first kind with vertices (α_1, β_1) and (α_2, β_2) where $\alpha_1 < \alpha_2$ (then $\beta_1 > \beta_2$). We denote by ∂S the segment with vertices $(\alpha_1 - 1, \beta_1)$ and $(\alpha_2 - 1, \beta_2)$ if $|S|_1 < |S|_2$, the segment with vertices $(\alpha_1, \beta_1 - 1)$ and $(\alpha_2, \beta_2 - 1)$ if $|S|_2 < |S|_1$ and the segment with vertices $(\alpha_1 - 1, \beta_1)$ and $(\alpha_2, \beta_2 - 1)$ if $|S|_1 = |S|_2$. Clearly S and ∂S are parallel.

A vertex of the Newton diagram $\Delta(f)$ is principal if it is lying on the line $\alpha + \beta = \operatorname{ord} f$. If there is exactly one principal vertex (α_0, β_0) and it is not lying on the axes (that is $\alpha_0 > 0$ and $\beta_0 > 0$) then we denote by V the segment with vertices $(\alpha_0 - 1, \beta_0)$ and $(\alpha_0, \beta_0 - 1)$. Clearly $|V|_1 = |V|_2 = 1$. We set $\mathcal{V} = \{V\}$ if V exists and $\mathcal{V} = \emptyset$, otherwise.

The *first* (*last*) segment of $\mathcal{N}_f$ is the segment nearest to the vertical (horizontal) axis. Every segment of the second kind is the first or is the last.

Let $\delta_1, \delta_2 \geq 0$ be the distances from $\operatorname{supp}(\partial f)$ to the axes. We can state now the main result of this section.

Theorem 2.1 *With the assumptions and definitions introduced above, we have*

(1) If $S \in \mathcal{N}_f$ is of the first kind then $\partial S \in \mathcal{N}(\partial f)$. If $\Delta(f)$ has exactly one principal vertex and it is not lying on the axes then $V \in \mathcal{N}(\partial f)$. The segments ∂S ($S \in \mathcal{N}_f$ of the first kind) and the segment V form the sequence of succesive segments when ordered in such a way that the sequence of their inclinations is increased. If the first segment (the last) is of the first kind or it is exceptional, then $\delta_1 = 0$ ($\delta_2 = 0$).

(2) Suppose that the first segment $F \in \mathcal{N}_f$ is of inclination ≤ 1 and F has a vertex on the vertical axis. There is a set $\mathcal{T}_1$ of segments of $\mathcal{N}(\partial f)$ such that

(i) if $T \in \mathcal{T}_1$ then $|T|_1/|T|_2 \leq |F|_1/|F|_2$,

(ii) $\delta_1 + \sum_{T \in \mathcal{T}_1} |T|_1 = |F|_1 + \varepsilon(F)$,

(iii) a line supporting $\Delta(f)$ parallel to a segment $T \in \mathcal{T}_1$ passes through $(0, \beta(F))$.

If the first segment is of inclination > 1 or has no vertex on the vertical axis we put $\mathcal{T}_1 = \emptyset$.

(3) Suppose that the last segment $L \in \mathcal{N}_f$ is of inclination≥ 1 and L has a vertex on the horizontal axis. There is a set $\mathcal{T}_2$ of segments of $\mathcal{N}(\partial f)$ such that

(i') if $T \in \mathcal{T}_2$ then $|L|_1/|L|_2 \le |T|_1/|T|_2$,

(ii') $\delta_2 + \sum_{T \in \mathcal{T}_2} |T|_2 = |L|_2 + \varepsilon(L)$,

(iii') a line supporting $\Delta(f)$ parallel to a segment $T \in \mathcal{T}_2$ passes through $(\alpha(L), 0)$.

If the last segment is of inclination < 1 or has no vertex on the horizontal axis we put $\mathcal{T}_2 = \emptyset$.

(4)

$$\mathcal{N}(\partial f) = \{\partial S : S \in \mathcal{N}_f \textit{is of the first kind}\} \cup \mathcal{V} \cup \mathcal{T}_1 \cup \mathcal{T}_2 \ .$$

(5) Let $S \in \mathcal{N}_f$ and $T \in \mathcal{N}(\partial f)$ be parallel. Then

$$\mathrm{in}(\partial f, T) = \begin{cases} a\frac{\partial}{\partial X}\mathrm{in}(f, S) \ \textit{if}\ |S|_1 < |S|_2 \ , \\ b\frac{\partial}{\partial Y}\mathrm{in}(f, S) \ \textit{if}\ |S|_2 < |S|_1 \ , \\ \partial \mathrm{in}(f, S) \ \textit{if}\ |S|_1 = |S|_1 \ . \end{cases}$$

In order to prove theorem 2.1 we need some preliminary notions. An application $v : \mathbf{C}\{X, Y\} \setminus \{0\} \to \mathbf{N}$ is a *weight* of $\mathbf{C}\{X, Y\}$ if for every nonzero $f = \sum c_{\alpha\beta} X^\alpha Y^\beta \in \mathbf{C}\{X, Y\}$ we have $v(f) = \inf\{\alpha v(X) + \beta v(Y) : (\alpha, \beta) \in \mathrm{supp}\, f\}$, where $v(X) > 0, v(Y) > 0$ are integers. We put $v(0) = +\infty$ where the symbol $+\infty$ has usual properties. Moreover we put

$$\mathrm{in}_v f = \sum_{\alpha v(X) + \beta v(Y) = v(f)} c_{\alpha\beta} X^\alpha Y^\beta \ .$$

For every $S \in \mathcal{N}_f$ we consider the weight v_S associated to S by putting $v_S(X) = |S|_2$ and $v_S(Y) = |S_1|$. Clearly $\alpha|S|_2 + \beta|S|_1 = v_S(f)$ is the equation of the line l_S containing the segment S, therefore $v_S(f) = |S|_2\alpha(S) = |S|_1\beta(S)$. A line $l \subset \mathbf{R}^2$ is a *barrier* of $\Delta(f)$ if it has equation $v_1\alpha + v_2\beta = w$ where $v_1, v_2, w > 0$ are integers and satisfies two conditions: 1. if $(\alpha, \beta) \in \Delta(f)$, then $v_1\alpha + v_2\beta \ge w$, 2. $\Delta(f) \cap l \ne \emptyset$. We call v_2/v_1 the inclination of l. Clearly, the inclination of l_S is equal to the inclination of S.

In the sequel we use the following simple facts:
1) $p \in \mathrm{supp}\, f$ is a vertex of $\Delta(f)$ if and only if there is a barrier l of $\Delta(f)$ such that $\Delta(f) \cap l = \{p\}$.
2) vertices p, p' of $\Delta(f)$ are succesive if and only if the line passing through p and p' is a barrier of $\Delta(f)$.

If v is a weight of $\mathbf{C}\{X, Y\}$, then the line $l_v :\ v(X)\alpha + v(Y)\beta = v(f)$ is a barrier of $\Delta(f)$ and $\Delta(f) \cap l_v = \mathrm{conv}\,(\mathrm{supp}\, \mathrm{in}_v f)$. If l is a barrier of $\Delta(f)$, then $l = l_v$ for a weight v. We omit the easy proof of the following lemma.

Lemma 2.2 *Let* $\partial = a\frac{\partial}{\partial X} + b\frac{\partial}{\partial Y}$ *with* $(a,b) \neq (0,0)$.

(i) If $a \neq 0$ *and* v *is a weight such that* $v(Y) < v(X)$ *and* $\mathrm{in}_v f \neq const. Y^q$ *then* $v(\partial f) = v(f) - v(X)$ *and* $\mathrm{in}_v(\partial f) = a\frac{\partial}{\partial X}\mathrm{in}_v f$,

(ii) If $b \neq 0$ *and* v *is a weight such that* $v(X) < v(Y)$ *and* $\mathrm{in}_v f \neq const. X^p$ *then* $v(\partial f) = v(f) - v(Y)$ *and* $\mathrm{in}_v(\partial f) = b\frac{\partial}{\partial Y}\mathrm{in}_v f$,

(iii) If $\mathrm{in} f \neq const.(bY - aX)^m$ *and* $ab \neq 0$ *then* $\mathrm{ord}\,(\partial f) = \mathrm{ord}\, f - 1$ *and* $\mathrm{in}(\partial f) = \partial(\mathrm{in} f)$.

For every $Z \subset \mathbf{R}^2$ we denote by $Z - (1,0)$ resp. $Z - (0,1)$ the image of Z by the translation $(\alpha,\beta) \mapsto (\alpha - 1, \beta)$ resp. $(\alpha,\beta) \mapsto (\alpha, \beta - 1)$. Using the notion of barrier we can rewrite lemma 2.2 as follows.

Lemma 2.3

(i) Let $\partial = a\frac{\partial}{\partial X} + b\frac{\partial}{\partial Y}$ *with* $a \neq 0$ *and let* l *be a barrier of* $\Delta(f)$ *of inclination* < 1 *such that* $l \cap \mathrm{supp}\, f$ *contains a point not lying on the vertical axis. Then* $l - (1,0)$ *is a barrier of* $\Delta(\partial f)$ *and*

$$(l-(1,0)) \cap \mathrm{supp}\,(\partial f) = \{(\alpha-1,\beta) \in \mathbf{N}^2 : \alpha > 0 \textit{ and } (\alpha,\beta) \in l \cap \mathrm{supp}\, f\}\ .$$

(ii) Let $\partial = a\frac{\partial}{\partial X} + b\frac{\partial}{\partial Y}$ *with* $b \neq 0$ *and let* l *be a barrier of* $\Delta(f)$ *of inclination* > 1 *such that* $l \cap \mathrm{supp}\, f$ *contains a point not lying on the horizontal axis. Then* $l - (0,1)$ *is a barrier of* $\Delta(\partial f)$ *and*

$$(l-(0,1)) \cap \mathrm{supp}\,(\partial f) = \{(\alpha,\beta-1) \in \mathbf{N}^2 : \beta > 0 \textit{ and } (\alpha,\beta) \in l \cap \mathrm{supp}\, f\}\ .$$

(iii) Let $\partial = a\frac{\partial}{\partial X} + b\frac{\partial}{\partial Y}$ *be such that* $\mathrm{in} f \neq const.(bX - aY)^m$. *Then the line* $k : \alpha + \beta = \mathrm{ord}\, f - 1$ *is a barrier of* $\Delta(\partial f)$ *and*

$$\begin{aligned} &k \cap \mathrm{supp}\,(\partial f) \\ &\quad \subset \{(\alpha - 1, \beta) \in \mathbf{N}^2 : \alpha > 0,\ (\alpha,\beta) \in \mathrm{supp}\, f \textit{ and } \alpha + \beta = \mathrm{ord}\, f\} \\ &\quad \cup \{(\alpha, \beta - 1) \in \mathbf{N}^2 : \beta > 0,\ (\alpha,\beta) \in \mathrm{supp}\, f \textit{ and } \alpha + \beta = \mathrm{ord}\, f\} \end{aligned}$$

Now we can prove Theorem 2.1.

Proof of (1). Let $S \in \mathcal{N}_f$ be a segment of the first kind. Assume that $|S|_1 < |S|_2$ (the cases $|S|_2 < |S|_1$ and $|S|_1 = |S|_2$ are treated similarly). Then $\partial S = S - (1,0)$. By Lemma 2.3 (i) the line $l_S - (1,0)$ is a barrier of $\mathrm{supp}\,(\partial f)$ and $l_S - (1,0) = l_{\partial S}$. Therefore ∂S is a segment of $\mathcal{N}(\partial f)$.

To check the second part of (1) consider two succesive segments S, S' of $\mathcal{N}_f$. It suffices to observe that
1. If S, S' are both of inclination ≤ 1 or ≥ 1, then $\partial S, \partial S'$ are succesive segments of $\mathcal{N}(\partial f)$,
2. If $|S|_1/|S|_2 < 1 < |S'|_1/|S'|_2$, then the common vertex of S and S' is principal and ∂S, V, $\partial S'$ are succesive segments of $\mathcal{N}(\partial f)$.
3. If $|S|_1 < |S|_2$, the lower vertex of S does not lie on the horizontal axis and is principal (resp. $|S|_2 < |S|_1$ and the upper vertex of S does not lie on the vertical axis), then $\partial S, V$ (resp. $V, \partial S$) are succesive segments of $\mathcal{N}(\partial f)$.
To end the proof of (1) let us observe that if F is of the first kind then the segment ∂F has a vertex on the vertical axis. If F is exceptional with the lower vertex $(1,\beta)$, then the point $(0,\beta)$ is a vertex of $\mathcal{N}(\partial f)$. Therefore $\delta_1 = 0$. Analogously we treat the case of the last segment.

Proof of (2). Let F be the first segment of $\mathcal{N}_f$. Suppose that F is not exceptional, $|F|_1 \leq |F|_2$ and F has a vertex on the vertical axis. Let $(\alpha_0, \beta_0) = (0, \beta_0) = (0, \beta(F))$ and $(\alpha_1, \beta_1) = (|F|_1, \beta_1)$ be vertices of F. Let $(\alpha_1', \beta_1') = (\alpha_1 - 1, \beta_1)$ if $|F|_1 < |F|_2$ and $(\alpha_1', \beta_1') = (\alpha_1, \beta_1 - 1)$ if $|F|_1 = |F|_2$. We claim that (α_1', β_1') is a vertex od $\Delta(\partial f)$.

Indeed, there is a barrier l of $\Delta(f)$ such that $l \cap \operatorname{supp} f = \{(\alpha_1, \beta_1)\}$ and the inclination of l is < 1 if $|F|_1 < |F|_2$ and > 1 if $|F|_1 = |F|_2$. Therefore $(l - (1,0)) \cap \operatorname{supp}(\partial f) = \{(\alpha_1', \beta_1')\}$ if $|F|_1 < |F|_2$, $(l - (0,1)) \cap \operatorname{supp}(\partial f) = \{(\alpha_1', \beta_1')\}$ if $|F|_1 = |F|_2$ and (α_1', β_1') is a vertex of $\Delta(\partial f)$ by Lemma 2.3. Let $\mathcal{T}_1$ be the set of all segments $T \in \mathcal{N}(\partial f)$ lying above the vertex (α_1', β_1'). We will check properties (i), (ii), (iii) when $|F|_1 < |F|_2$ (the proofs when $|F|_1 = |F|_2$ are similar). By Lemma 2.3 the line $l_F - (1,0)$ is a barrier of $\Delta(\partial f)$ passing through the vertex $(\alpha_1 - 1, \beta_1)$. Let $T \in \mathcal{T}_1$ and consider the barrier l_F of $\Delta(\partial f)$. Clearly $|T|_1/|T|_2 =$ inclination of $l_T \leq$ inclination of $l_F - (1,0) = |F|_1/|F|_2$, that is (i) holds.

Property (ii) follows from the calculation:

$$\delta_1 + \sum_{T \in \mathcal{T}_1} |T|_1 = \alpha_1 - 1 = |F|_1 - 1 = |F|_1 + \varepsilon(F) \, .$$

To check (iii) take a barrier of $\Delta(f)$ parallel to a segment of $T \in \mathcal{T}_1$. It is of inclination $|T|_1/|T|_2 \leq |F|_1/|F|_2$. Therefore it passes through $(0, \beta_0) = (0, \beta(F))$.

Proof of (3) is analogous to the proof of (2).

Proof of (4) follows easily from (1), (2) and (3).

Proof of (5). Consider two parallel segments $S \in \mathcal{N}(f)$ and $T \in \mathcal{N}(\partial f)$. Let v_S be the weight defined by putting $v_S(X) = |S|_2$ and $v_S(Y) = |S|_1$. Assume $|S|_1 < |S|_2$. Clearly $\operatorname{in}_{v_S} f = \operatorname{in}(f, S) \neq$ a monomial and by Lemma 2.2

(i): $\mathrm{in}_{v_S}(\partial f) = a\frac{\partial}{\partial X}\mathrm{in}(f,S)$. On the other hand $\mathrm{in}_{v_S}(\partial f) = \mathrm{in}_{v_T}(\partial f)$ for S and T are parallel and we get

$$\mathrm{in}(\partial f, T) = \mathrm{in}_{v_T}(\partial f) = \mathrm{in}_{v_S}(\partial f) = a\frac{\partial}{\partial X}\mathrm{in}(f,S)\ .$$

If $|S|_2 < |S|_1$ or $|S|_2 = |S|_1$, then the proof is similar.

3 Proof of the main result

We need some preliminary facts. The following is well-known (see [Te2]).

Lemma 3.1 *Let $f = f(X,Y)$ be a nonzero power series without constant term. Then there is a factorization*

$$f = uX^{d_1}Y^{d_2}\prod_{S\in\mathcal{N}_f} f_S$$

in $\mathbf{C}\{X,Y\}$ where u is a unit such that

(i) $\mathcal{N}(f_S) = \{S'\}$ *where S' is the segment with vertices $(|S|_1, 0)$ and $(0, |S|_2)$,*

(ii) $\mathrm{in}(f,S) = (a\ monomial)\mathrm{in}(f_S, S')$ *in $\mathbf{C}\{X,Y\}$.*

Recall that a pair of power series f, g without constant term is nondegenerated if for every pair of parallel segments $S \in \mathcal{N}_f$ and $T \in \mathcal{N}_g$ the system of equations $\mathrm{in}(f,S) = \mathrm{in}(g,T) = 0$ has no solution in $\mathbf{C}^* \times \mathbf{C}^*$. Let

$$\frac{\alpha}{\alpha(T,\mathcal{N}_f)} + \frac{\beta}{\beta(T,\mathcal{N}_f)} = 1$$

be the equation of the barrier of $\Delta(f)$ parallel to the segment T of (possibly) another Newton polygon. Clearly, if $S \in \mathcal{N}_f$, then $\alpha(S,\mathcal{N}_f) = \alpha(S)$, $\beta(S,\mathcal{N}_f) = \beta(S)$.

Lemma 3.2 *Let $f \in \mathbf{C}\{X,Y\}$ be a nonzero power series without constant term and let $h \in \mathbf{C}\{X,Y\}$ be a power series such that $\mathcal{N}_h = \{T\}$ where T is a segment with vertices on the axes. Then*

$$\frac{(f,h)_0}{\mathrm{ord}\, h} \geq \max\left(\alpha(T,\mathcal{N}_f), \beta(T,\mathcal{N}_f)\right)\ ,$$

the equality holding if the pair f, h is nondegenerated.

Recall the following (see, for example [Pł2]):

Property. *Let f, g be nonzero power series without constant term. Then for every weight v:*

$$(f,g)_0 \geq \frac{v(f)v(g)}{v(X)v(Y)} \ ,$$

the equality holding if the system of equations $\mathrm{in}_v f = \mathrm{in}_v g = 0$ *has no solution* $\neq (0,0)$.

To check lemma 3.2 we apply the above property to f, h and to the weight v_T given by $v_T(X) = |T|_2$ and $v_T(Y) = |T|_1$. It is easy to see that $v_T(h) = |T|_1|T|_2$, $v_T(f) = |T|_2\alpha(T,\mathcal{N}_f) = |T|_1\beta(T,\mathcal{N}_f)$ and $\mathrm{ord}\, h = \min(|T|_1, |T|_2)$. Therefore

$$\begin{aligned}\frac{(f,h)_0}{\mathrm{ord}\, h} &\geq \frac{v_T(f)}{\min(|T|_1,|T|_2)} = \max\left(\frac{v_T(f)}{|T|_1}, \frac{v_T(f)}{|T|_2}\right) \\ &= \max(\alpha(T,\mathcal{N}_f), \beta(T,\mathcal{N}_f)) \ ,\end{aligned}$$

the equality holding if the pair f, h is nondegenerated.

Lemma 3.3 *Every polar quotient of the curve $f = 0$ is greater than or equal to* $\mathrm{ord}\, f$. *If the curve $f = 0$ has* $\mathrm{ord}\, f$ *tangents paiwise different, then there is only one polar quotient of $f = 0$ and it is equal to* $\mathrm{ord}\, f$.

Proof. Suppose that the line $bX - aY = 0$ is not tangent to $f = 0$. The first part follows from the inequality $(f,h)_0 \geq (\mathrm{ord}\, f)(\mathrm{ord}\, h)$. It is easy to check that if f has $\mathrm{ord}\, f$ tangents, then the curves $f = 0$ and $\partial f = a\partial f/\partial X + b\partial f/\partial Y = 0$ have no common tangent. Therefore for every irreducible factor h of ∂f we get

$$\frac{(f,h)_0}{\mathrm{ord}\, h} = \mathrm{ord}\, f \ .$$

Proof of Theorem 1.1. Let $f = f(X,Y) \in \mathbf{C}\{X,Y\}$ be a power series with an isolated singularity at $(0,0) \in \mathbf{C}^2$. If $\mathcal{N}_f = \{S\}$, $|S|_1 = |S|_2$ or $\mathcal{N}_f^* = \emptyset$ then the theorem follows easily from Lemma 3.3. Therefore we assume in the sequel that $\mathcal{N}_f$ contains a segment of inclination $\neq 1$.

Let $\partial = a\partial/\partial X + b\partial/\partial Y$ with $a\, b\, \mathrm{in} f(a,b) \neq 0$. Then $\partial(\mathrm{in} f) \neq 0$. By Lemma 3.1 we may write

$$\partial f = uX^{\delta_1}Y^{\delta_2} \prod_{T\in\mathcal{N}(\partial f)} (\partial f)_T \ . \tag{1}$$

For every nonexceptional segment $S \in \mathcal{N}_f$ we define the series g_S as follows. If S is a segment of the first kind, then $g_S = (\partial f)_{\partial S}$. If F is the first segment of inclination ≤ 1 and with a vertex on the horizontal axis, then we put

$$g_F = X^{\delta_1} \prod_{T \in \mathcal{T}_1} (\partial f)_T \ .$$

Similarly, if L is the last segment of inclination ≥ 1 and it is of the second kind, then

$$g_L = Y^{\delta_2} \prod_{T \in \mathcal{T}_2} (\partial f)_T \ .$$

Moreover, we put $v = u(\partial f)_V$ if $\mathcal{V} = \{V\}$ and $v = u$ if $\mathcal{V} = \emptyset$. By Theorem 2.1 and (1) we get

$$\partial f = v \prod_{S \in \mathcal{N}_f^*} g_S \ . \tag{2}$$

To check Theorem 1.1 we have to verify three cases.

Case 1. S is a segment of the first kind. Recall that for every segment of the first kind, $\varepsilon(S) = 0$ if $|S|_1 \neq |S|_2$ and $\varepsilon(S) = 1$ if $|S|_1 = |S|_2$. Therefore

$$\operatorname{ord} g_S = \operatorname{ord} (\partial f)_{\partial S} = \min (|\partial S|_1, |\partial S|_2) = \min (|S|_1, |S|_2) + \varepsilon(S) \ .$$

This proves part (i) of the theorem.

Let h be an irreducible factor of $g_S = (\partial f)_{\partial S}$. Then $\mathcal{N}_h = \{T\}$, T is parallel to ∂S and consequently to S. By Lemma 3.2 we get

$$\begin{aligned} \frac{(f,h)_0}{\operatorname{ord} h} &\geq \max (\alpha(T, \mathcal{N}_f), \beta(T, \mathcal{N}_f)) = \max (\alpha(S, \mathcal{N}_f), \beta(S, \mathcal{N}_f)) \\ &= \max (\alpha(S), \beta(S)) \ . \end{aligned}$$

If f is nondegenerated on S, then by Part (5) of Theorem 2.1 the pair f, h is nondegenerated and we get

$$\frac{(f,h)_0}{\operatorname{ord} h} = \max (\alpha(S), \beta(S)) \ .$$

Case 2. The first segment F, $|F|_1 \leq |F|_2$ of $\mathcal{N}_f$ is of the second kind. We get by the second part of Theorem 2.1:

$$\begin{aligned} \operatorname{ord} g_F &= \operatorname{ord} \left(X_1^{\delta} \prod_{T \in \mathcal{T}_1} (\partial f)_T \right) = \delta_1 + \sum_{T \in \mathcal{T}_1} \operatorname{ord} (\partial f)_T \\ &= \delta_1 + \sum_{T \in \mathcal{T}_1} |T|_1 = |F|_1 + \varepsilon(F) = \min (|F|_1, |F|_2) + \varepsilon(F) \ . \end{aligned}$$

Let h be an irreducible factor of g_F. If h divides $(\partial f)_T$, then $\mathcal{N}_h = \{T'\}$ where T' is parallel to T. Thus $\alpha(T', \mathcal{N}_f) = \alpha(T, \mathcal{N}_f)$, $\beta(T', \mathcal{N}_f) = \beta(T, \mathcal{N}_f)$ and we get

$$\frac{(f,h)_0}{\operatorname{ord} h} \geq \max (\alpha(T', \mathcal{N}_f), \beta(T', \mathcal{N}_f)) = \beta(T, \mathcal{N}_f) = \beta(F)$$

by the second part of Theorem 2.1. The equality for f nondegenerated on S we check like in Case 1. If $h = X$, then

$$\frac{(f,h)_0}{\operatorname{ord} h} = \operatorname{ord} f(0,Y) = \beta(F)$$

since the first segment F has a vertex on the vertical axis.

Case 3. The last segment L of $\mathcal{N}_f$, $|L|_2 \leq |L|_1$ is of the second kind. The proof is analogous to the proof above.

To finish the proof it suffices to remark that if $\operatorname{ord} v > 0$, then $\operatorname{ord} v = \max(|V|_1, |V|_2) = 1$ and

$$(f,v)_0 = \max(\alpha(V,\mathcal{N}_f), \beta(V,\mathcal{N}_f)) = \operatorname{ord} f$$

since $|V|_1 = |V|_2$ and the pair f, v is nondegenerated.

We have proved our theorem for polars $\partial f = a\partial/\partial X + b\partial/\partial Y$ in the direction $(a:b)$ such that $ab\operatorname{in} f(a,b) \neq 0$. If $a = 0$ or $b = 0$ and $\operatorname{in} f(a,b) \neq 0$ then the theorem is still true but the proof needs some modification: instead of using Theorem 2.1 we use the description of the polygons $\mathcal{N}(\partial f/\partial X)$, $\mathcal{N}(\partial f/\partial Y)$ given in [Len] (Theorems 5.3 and 5.5).

Remark

One can also prove factorization theorems similar to our main result for special polars $\partial f/\partial X$ and $\partial f/\partial Y$.

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