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On a certain Banach space related to the Schauder system.

Anna Maria Pelczar

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Summary: This paper deals with a certain Banach space derived by taking all continuous functions on the unit interval represented with respect to the Schauder system by an unconditionally convergent series. Another proof is provided for the fact that this space contains Hölder functions of degree $\alpha \leq 1$.

1 Preliminary remarks.

Let X be a Banach space and let $\{x_n\}$ be a Schauder basis in X. In this paper we will deal with unconditional Schauder bases which can be defined in many ways - for example - as follows:

Definition 1.1 *A Schauder basis $\{x_n\}$ is called unconditional when for each sequence of scalars $\{a_n\}$ the series $\sum_1^\infty a_n x_n$ is convergent if and only if the series $\sum_1^\infty |a_n| x_n$ is convergent.*

There is a general procedure of constructing the unconditional bases starting from the arbitrary ones ([3], [4]). Suppose that $\{x_n\}$ is an arbitrary Schauder basis in a Banach space X. Let

$$X_o = \{x : \sum_1^\infty a_n x_n\}$$

where $\{a_n\}$ is a sequence of reals, which is eventually 0.

In the space X_o we define a new norm $\|\cdot\|^\bullet$ by the formula

$$\|\sum_{1}^{\infty} a_n x_n\|^{\bullet} = \sup\{\|\sum_{1}^{\infty} \varepsilon_n a_n x_n\| : \varepsilon_n \in \{+1,-1\}\}$$

We denote by $X^{\bullet}$ the completion of X_o with respect to the norm $\|\cdot\|^{\bullet}$. It is not hard to see that the sequence $\{x_n\}$ is an unconditional basis in $X^{\bullet}$, and we denote this new basic sequence by $\{x_n\}^{\bullet}$.

In the case when $\{x_n\}$ is unconditional the spaces X and $X^{\bullet}$ coincide and the sequences $\{x_n\}$ and $\{x_n\}^{\bullet}$ are equivalent. In general the space $X^{\bullet}$ may be very "far" from X and then practically useless to study X itself.

The situation becomes a little more interesting, when we start from a basis which is of the so-called type UL*. This type of basic sequences was considered in [3] and [4] and we recall

Definition 1.2 *A basic sequence $\{x_n\}$ is of the type UL* if for each sequence of scalars $\{a_n\}$ the convergence of the series $\sum_1^{\infty} |a_n| x_n$ implies the convergence of the series $\sum_1^{\infty} a_n x_n$.*

Clearly, there are basic sequences which are not UL* (for example the canonical basis in the James space J), and there are basic sequences which are UL* without being unconditional (for example the canonical basis in J*). The sequences of the type UL* are more interesting with respect to the operation $\bullet$ because in this situation the space $X^{\bullet}$ is "closer" to the space X than in general.

Let us denote by X^{+} the cone of those elements from X, which have non-negative coordinates with respect to the basis $\{x_n\}$ i.e.

$$X^{+} = \{x \in X : x_n^*(x) \geq 0\}, n \in \mathbf{N}\}.$$

If $\{x_n\}$ is of the type UL* one may easily verify that

$$X^{\bullet} = X^{+} - X^{+}$$

and the formula

$$\|x\|^{\diamond} = \inf\{\|u\| + \|v\| : x = u - v, u \geq 0, v \geq 0\}$$

defines in $X^{\bullet}$ a norm equivalent to the norm $\|\cdot\|^{\bullet}$. In other words the spaces X and $X^{\bullet}$ have the same cone of positive vectors with respect to the basis $\{x_n\}$ (namely X^{+}).

This means that we may look at X^{+} with its original topology as a subset of a "smaller" space, namely $X^{\bullet}$.

2 Some properties of the space $C([0,1])^\bullet$.

In the present section we examine the space $X^\bullet$ when $X = C([0,1])$, the space of continuous functions on the unit interval considered with the Schauder system.

Let us recall the definition of the Schauder system of "hat" functions; we start with some notation.

Let $\{a_{n,k} = \frac{k}{2^n} : n \in \mathbf{N}, k \in \{0,1,...,2^n\}\}$ be the set of dyadic points of the interval [0,1]. Then following equations hold: $a_{n,k} = a_{n-1,\frac{k}{2}}$ for $k \in 2\mathbf{N}$ and $a_{n,k} = \frac{1}{2}(a_{n-1,\frac{k-1}{2}} + a_{n-1,\frac{k+1}{2}})$ for any $k \in 2\mathbf{N}+1$. In order to avoid repetition of points we will consider the set $\{a_{n,k} : n \in \mathbf{N}, k \in L_n\}$, where $L_0 = \{0,1\}$ and $L_n = \{1,...,2^n\} \cap 2\mathbf{N}+1$ for $n \geq 1$. The set $\{a_{n,k}\}_{k\in L_n}$ is called the n-th net.

It is well known that the Schauder system of "hat" functions $\{F_{n,k} : n \in \mathbf{N}, k \in L_n\}$ given by formulas: $F_{0,0}(x) = 1$, $F_{0,1}(x) = x$,

$$F_{n,k}(x) = \begin{cases} \frac{x-a_{n,k-1}}{a_{n,k}-a_{n,k-1}} & \text{for } x \in [a_{n,k-1}, a_{n,k}], \\ \frac{a_{n,k+1}-x}{a_{n,k+1}-a_{n,k}} & \text{for } x \in [a_{n,k}, a_{n,k+1}], \\ 0 & \text{otherwise} \end{cases} \quad \text{for } n \geq 1, k \in L_n$$

forms a Schauder basis in the space of continuous functions on the unit interval $C([0,1])$ considered with the supremum norm $\|\cdot\|_\infty$. Moreover, this basis is of the type UL* (Corollary 3.2, [4]).

By the general construction described in the previous section, the Schauder system forms unconditional basis in the space $C([0,1])^\bullet$. Obviously the space $C([0,1])^\bullet$ is essentially contained in the space $C([0,1])$, since there is no unconditional basis in the space $C([0,1])$ (see [2]).

The space $C([0,1])^\bullet$ contains a large class of continuous functions; for instance the Weierstrass function given by the formula $f = \sum_{n=1}^{\infty} \frac{1}{2^n} \sum_{k\in L_n} F_{n,k}$, as a uniformly convergent series of "hat" functions with positive coeficients, belongs to $C([0,1])^\bullet$. From the well known theorem characterizing Hölder functions in terms of their coefficients in the Schauder system ([1]) follows

Theorem 2.1 *Every Hölder function of degree $\alpha \leq 1$ (in particular any C^1 function) belongs to the space $C([0,1])^\bullet$.*

Below is presented another, geometric proof of this fact. First let us introduce some notation.

For a non-negative continuous function f and for any $m \geq 0$ by ω_m denote the modulus of continuity $\omega_m = \max\{|f(x) - f(y)|, |x-y| \leq \frac{1}{2^m}\}$ with regard to $\frac{1}{2^m}$. Then the following holds true

Lemma 2.1 *If there exists an increasing sequence of natural numbers* $\{s_n\}_{n=1}^{\infty}$ *such that the series*

$$\sum_{n=1}^{\infty} 2^{s_n - n}\omega_{s_n}, \quad \sum_{n=1}^{\infty} 2^{n-s_n} s_{n+1}$$

are convergent, then the function f *belongs to the space* $C([0,1])^{\bullet}$

Notice first that the lemma implies the theorem. Indeed, if f satisfies Hölder condition: $|f(x) - f(y)| \le C|x-y|^{\alpha}$ for some $C > 0$ and $0 < \alpha \le 1$, then $\omega_m = \frac{C}{2^{\alpha n}}$. It is easy to check that the sequence $s_n = n + [\sqrt{n}]$ satisfies assumptions of the lemma, and this ends the proof of the theorem.

The proof of the lemma is divided into three steps.

Step 1.

We will show that a non-negative function f such that supp $f = [a_{n,k}, a_{n,k+1}]$ for some $n, k \in \mathbf{N}$ and $f|_{\text{supp } f}$ is a concave function, belongs to $C([0,1])^{+}$.

Proof of step 1. Notice first that arbitrary non-negative concave function f on the interval $[0,1]$ belongs to $C([0,1])^{+}$. Indeed, put $p_{0,0} = f(0) \ge 0$, $p_{0,1} = f(1) \ge 0$, $p_{1,1} = (f - p_{0,0}F_{0,0} - p_{0,1}F_{0,1})(\frac{1}{2})$, notice that $p_{1,1} \ge 0$ by concavity of f and proceed by induction. The series $\sum_{n,k=0}^{\infty} p_{n,k}F_{n,k}$ converges to f pointwise (not only in dyadic points) since its partial sums are concave. Moreover, this series converges increasingly to f, hence by Dini's theorem uniformly to f. Thus f is a uniformly convergent series of "hat" functions with positive coeficients.

Now take a non-negative function whose restriction to its support of the form $[a_{n,k}, a_{n,k+1}]$ is concave. Analogous argument shows that f belongs to the cone $C([0,1])^{+}$ (notice that the support of any "hat" function is either contained in the interval $[a_{n,k}, a_{n,k+1}]$ or disjoint with the open interval $(a_{n,k}, a_{n,k+1})$).

Step 2.

Let L be the set of piecewise affine functions with nodes at dyadic points;

$$L = \{f \in C([0,1]); |; \exists n \in \mathbf{N}: \ \forall k \in \mathbf{N}, 1 \le k \le 2^n: \ \ f|_{[a_{n,k-1}, a_{n,k}]} \text{ affine}\}$$

We will show that the space $C([0,1])^{\bullet}$ contains the space L. Moreover, a non-negative function $f \in L$ with nodes on the set $\{a_{n,k}\}_{k=1}^{2^n}$ has a decomposition $f = f^{+} - f^{-}$, where $f^{+}, f^{-} \in C([0,1])^{+}$, such that $\|f^{-}\|_{\infty} \le (n+2)\|f\|_{\infty}$.

Proof of step 2. Take a function $f \in L$ as above. For any natural number $0 \le k \le 2^n$ put $b_{n,k} = a_{n+1,2k-1}$, $c_{n,k} = a_{n+1,2k+1}$ (dyadic points at the $(n+1)$-th level, which are closest to $a_{n,k}$, but different from $a_{n,k}$) and:

$$f_0(x) = \begin{cases} f(0)\frac{c_{n,1}-x}{c_{n,1}} & 0 \le x \le c_{n,1} \\ 0 & \text{otherwise} \end{cases}$$

$$f_{2^n}(x) = \begin{cases} f(1)\frac{x-b_{n,2^n}}{1-b_{n,2^n}} & b_{n,2^n} \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_k(x) = \begin{cases} f(a_{n,k})\frac{x-b_{n,k}}{a_{n,k}-b_{n,k}} & \text{for } b_{n,k} \le x \le a_{n,k} \\ f(a_{n,k})\frac{c_{n,k}-x}{c_{n,k}-a_{n,k}} & \text{for } a_{n,k} \le x \le c_{n,k} \\ 0 & \text{otherwise} \end{cases} \quad \text{for } 0 < k < 2^n$$

Then $f_k \le f$. Notice that the function $h = f - \sum_{k=0}^{2^n} f_k$ is a finite sum of functions described in the step 1., hence it belongs to the cone $C([0,1])^+$.

Every function f_k belongs to the space $C([0,1])^\bullet$, more precisely f_k has a decomposition $f_k = h_k - g_k$, where the positive part h_k is the "hat" function with peak at $a_{n,k}$ multiplied by $f(a_{n,k})$, and the negative part g_k is a sum of two functions g_k^1 and g_k^2 such that supp $g_k^1 = [a_{n,k-1}, a_{n,k}]$, supp $g_k^2 = [a_{n,k}, a_{n,k+1}]$ and $g_k^i|_{\text{supp } g_k^i}$ is concave for $i = 1, 2$. Hence, by the step 1., the function g_k belongs to $C([0,1])^+$. Thus the function f has a decomposition $f = f^+ - f^-$, where $f^+ = h + \sum_{k=1}^{2^n} h_k \in C([0,1])^+$ and $f^- = \sum_{k=1}^{2^n} g_k \in C([0,1])^+$.

Now we will estimate $\|f^-\|_\infty$. Functions h_k are multiplied by scalars "hat" functions with peaks only at points in first $(n+1)$ nets. Hence any point of the interval $[0,1]$ belongs to at most $(n+2)$ supports of functions h_k, and thus (since supp $g_k \subset$ supp h_k) to at most $(n+2)$ supports of functions g_k. Since $\|g_k\|_\infty \le \|f_k\|_\infty \le \|f\|_\infty$, it follows that $\|f^-\|_\infty = \|\sum_{k=1}^{2^n} g_k\|_\infty \le (n+2)\|f\|_\infty$, which completes the proof of step 2.

Step 3.

Proof of the Lemma.

Fix $s \in \mathbf{N}$ and $k \in L_s$. Let $M = \max\{f(x) : a_{s,k-1} \le x \le a_{s,k}\}$. Then by the definition of the continuity modulus for any $x \in [a_{s,k-1}, a_{s,k}]$ we have $M - \omega_s \le f(x) \le M$. Hence there is a covering of the graph of f consisting of rectangles $\{P_k\}_{k=1}^{2^s} = [a_{s,k-1}, a_{s,k}] \times [p_k, q_k]$, where $q_k - p_k \le \omega_s$.

Let h be a function such that for any natural number $1 \le k \le 2^s$ we have $h(a_{s,k}) = \max\{\min\{p_k, p_{k+1}\}, 0\}$ and $h|_{[a_{s,k-1}, a_{s,k}]}$ is an affine function. By the choice of p_k and q_k it follows that $\|h - f\|_\infty \le 2\omega_s$ (the double height of rectangles). Moreover, the Lipschitz constant C_h of function h satisfies $C_h \le 2^s\omega_s$. By step 2, function h belongs to the space $C([0,1])^\bullet$ and has a decomposition $h = h^+ - h^-$, where $h^+, h^- \in C([0,1])^+$, such that $\|h^-\|_\infty \le (s+2)\|h\|_\infty \le (s+2)\|f\|_\infty$

Notice that the function $g = f - h$ is non-negative and satisfies $\|g\|_\infty \le 2\omega_s$, with continuity modulus $\omega_m^g \le \omega_m + \frac{1}{2^m}C_h \le \omega_m + \frac{2^s\omega_s}{2^m}$.

Let $\{s_n\}_{n=1}^\infty$ be the sequence satisfying the assumptions of the Lemma. Applying the construction described above define by induction sequences of functions $\{h_n\}_{n=1}^\infty$, $\{f_n\}_{n=0}^\infty$ satisfying the following conditions:

1. $f_{n-1} = f_n + h_n$, $f_n \geq 0$, $h_n \geq 0$ for $n \geq 1$,
2. $||f_n||_\infty \leq 2\omega_{s_n}^{n-1}$ for $n \geq 1$,
3. $\omega_m^n = \omega_m^{f_n} \leq \omega_m + \frac{1}{2^m}\sum_{i=1}^n 2^{n-i+s_i}\omega_{s_i}$ for $n \geq 1$, $m \geq 0$,
4. $h_n \in L$ and $||h_n^-||_\infty \leq (s_n+2)||f_{n-1}||_\infty$ for $n \geq 1$.

The first step of the induction procedure: apply the previous construction to the function f and scalar s_1 and put $f_0 = f$, $h_1 = h$, $f_1 = g$, $\omega_m^0 = \omega_m$, $\omega_m^1 = \omega_m^{f_1}$ for $m \geq 0$.

The n-th step: apply the previous construction to the function f_{n-1} and scalar s_n and put $h_n = h$, $f_n = g$. By the step 2. function h_n belongs to the space $C([0,1])^\bullet$ and

$$||h_n^-||_\infty \leq (s_n+2)||h_n||_\infty \leq (s_n+2)||f_{n-1}||_\infty \quad \text{(condition 4.)}$$

Moreover, by the condition 2. for $(n-1)$-th step and construction, Lipschitz constant C_{h_n} of the function h_n satisfies

$$C_{h_n} \leq 2^{s_n}\omega_{s_n}^{n-1} \leq 2^{s_n}\omega_{s_n} + \sum_{i=1}^{n-1} 2^{n-1-i+s_i}\omega_{s_i}.$$

By the previous argument we have that:

$$||f_n||_\infty \leq 2\omega_{s_n}^{n-1} \quad \text{(condition 2.)}$$

and:

$$\omega_m^n \leq \omega_m^{n-1} + \frac{1}{2^m}C_{h_n} \leq \omega_m + \frac{1}{2^m}\left(\sum_{i=1}^{n-1} 2^{n-1-i+s_i}\omega_{s_i} + 2^{s_n}\omega_{s_n} + \sum_{i=1}^{n-1} 2^{n-1-i+s_i}\omega_{s_i}\right) =$$

$$= \omega_m + \frac{1}{2^m}\sum_{i=1}^{n} 2^{n-i+s_i}\omega_{s_i} \quad \text{(condition 3.)}$$

hence functions f_n and h_n satisfy conditions 1., 2., 3. and 4., which completes the induction procedure.

For such sequences we have

$$(\star) \quad f - \sum_{k=1}^{n} h_k = f_n$$

$$(\star\star) \quad \sum_{n=1}^{\infty} ||h_n^-||_\infty \leq \sum_{n=1}^{\infty}(s_n+2)||f_{n-1}||_\infty \leq (s_1+2)||f||_\infty + 2\sum_{n=2}^{\infty}(s_n+2)\omega_{s_{n-1}}^{n-2}$$

Now it is enough to prove that the series $\sum_{n=2}^{\infty} s_n \omega_{s_{n-1}}^{n-2}$ is convergent. Then in particular $||f_n||_\infty \to 0$, therefore by $(\star)$ the function series $\sum_{n=1}^{\infty} h_n$ converges uniformly to f. Moreover by $(\star\star)$ the function series $\sum_{n=1}^{\infty} h_n^-$ converges uniformly to some function in the cone $C([0,1])^+$, hence

$$f = \sum_{n=1}^{\infty} h_n = \sum_{n=1}^{\infty} (h_n^+ - h_n^-) = \sum_{n=1}^{\infty} h_n^+ - \sum_{n=1}^{\infty} h_n^-,$$

where all function series converge uniformly. Therefore the function f is represented as a difference of two functions from the cone $C([0,1])^+$, which ends the proof of the Lemma.

Consider now the series $\sum_{n=0}^{\infty} s_{n+2} \omega_{s_{n+1}}^n$. By the condition 3., monotonicity of the sequence $\{s_n\}$ we have:

$$\sum_{n=0}^{\infty} s_{n+2} \omega_{s_{n+1}}^n \leq \sum_{n=0}^{\infty} s_{n+2} (\omega_{s_{n+1}} + \sum_{i=1}^{n} 2^{n-i+s_i-s_{n+1}} \omega_{s_i}) \leq$$

$$\leq \sum_{n=0}^{\infty} s_{n+2} \omega_{s_{n+1}} + \sum_{n=0}^{\infty} \sum_{i=1}^{n} s_{n+2} 2^{n-i+s_i-s_{n+1}} \omega_{s_i} \leq \sum_{n=1}^{\infty} s_{n+1} \omega_{s_n} +$$

$$+ \sum_{i=1}^{\infty} \omega_{s_i} 2^{s_i - i} \sum_{n=0}^{\infty} s_{n+2} 2^{n-s_{n+1}} = \sum_{n=1}^{\infty} s_{n+1} \omega_{s_n} + \frac{1}{2} \sum_{n=1}^{\infty} 2^{s_n - n} \omega_{s_n} \sum_{n=1}^{\infty} 2^{n-s_n} s_{n+1},$$

The series $\sum 2^{s_n - n} \omega_{s_n}$ and $\sum 2^{n-s_n} s_{n+1}$ are convergent by the choice of $\{s_n\}$, hence the series $\sum s_{n+1} \omega_{s_n}$ also and thus the proof is completed.

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Institute of Mathematics
Jagiellonian University
Reymonta 4, 30-059 Krakøw
e-mail: apelczar@im.uj.edu.pl