Effective Nullstellensatz for strictly regular sequences

by

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IMUJ PREPRINT **1999/21** KRAKÓW

Summary. Sharp versions of the Nullsellensatz for a class of polynomial sequences are given.

1. Main results. Let $F_1, \dots, F_k : \mathbb{C}^n \to \mathbb{C}$, $1 \leqslant k \leqslant n$ be a sequence of nonconstant polynomials. Let $d_i = \deg F_i$ for $i = 1, \dots, k$.

Definition 1.1. *The sequence $F_1, \dots, F_k$ is strictly regular (s.r.) if there are linear forms $L_1, \dots, L_{n-k} : \mathbb{C}^n \to \mathbb{C}$ such that the mapping*

$$(F_1, \dots, F_k, L_1, \dots, L_{n-k}) : \mathbb{C}^n \to \mathbb{C}^n$$

is proper.

Remarks 1.2.

(1) If $k = 1$ then nonconstant F_1 is a s.r. sequence.

(2) If $k = n$ then $F_1, \dots, F_n$ is a s.r. sequence if and only if $(F_1, \dots, F_n) : \mathbb{C}^n \to \mathbb{C}^n$ is a proper mapping.

(3) Let $F_i^* : \mathbb{C}^n \to \mathbb{C}$ be a homogeneous form of degree d_i such that we have $F_i = F_i^* + (\text{terms of degree} < d_i)$ for $i = 1, \dots, k$. Suppose that the algebraic set $(F_1^*, \dots, F_k^*)^{-1}(0)$ is of dimension $n-k$, then $F_1, \dots, F_k$ is a s.r. sequence.

(4) If $F_1, \dots, F_k$ is a s.r. sequence then $(F_1, \dots, F_k)^{-1}(0)$ is of pure dimension $n-k$. In particular the algebraic cycle of zeroes $Z_{(F_1, \dots, F_k)}$ is defined (cf.[7]).

AMS Classification: 14E99, 32B99

Key words : proper polynomial mapping, geometric degree.

Work partially supported by KBN Grant 2 P03A 001 15.

Theorem 1.3. *Let $F_1, \ldots, F_k : \mathbb{C}^n \to \mathbb{C}$ be a s.r. sequence. If $G : \mathbb{C}^n \to \mathbb{C}$ is a nonconstant polynomial mapping such that $G|(F_1, \ldots, F_k)^{-1}(0) = 0$, then there is a $\mu > 0$ such that*

$$G^\mu = A_1F_1 + \ldots + A_kF_k,$$

with polynomials $A_1, \ldots, A_k : \mathbb{C}^n \to \mathbb{C}$ such that $\deg(A_iF_i) \leqslant (\prod_{i=1}^{k} d_i)(\deg G)$ for $i = 1, \ldots, k$

Remarks 1.4.

(1) Theorem 1.3 is a generalization to the case of s.r. sequence of the main result of [3].

(2) From the given formulation of the theorem it follows immmediately that $\mu \leqslant \prod_{i=1}^{k} d_i$.

Theorem 1.5. *Let $F_1, \ldots, F_k : \mathbb{C}^n \to \mathbb{C}$ be a s.r. sequence. If $F_{k+1} : \mathbb{C}^n \to \mathbb{C}$ is a polynomial of degree $d_{k+1} > 0$ such that $(F_1, \ldots, F_{k+1})^{-1}(0) = \emptyset$, then*

$$A_1F_1 + \ldots + A_{k+1}F_{k+1} = 1,$$

with polynomials $A_1, \ldots, A_{k+1} : \mathbb{C}^n \to \mathbb{C}$ such that $\deg(A_iF_i) \leqslant d_1 \cdot \ldots \cdot d_{k+1}$ for $i = 1, \ldots, k+1$

Remarks 1.6.

(1) If degrees of polynomials in Theorem 1.5 are different than 2, then theorem follows from Kollar's theorem (see [5]).

(2) If $k = n$ then [3] gives a better result.

2. Proofs. The proofs of Theorem 1.3 and Theorem 1.5 are based on the following proposition.

Proposition 2.1 (cf. [3], [6]). *Suppose that $H = (H_1, \ldots, H_n) : \mathbb{C}^n \to \mathbb{C}^n$ is a proper polynomial mapping of geometric degree μ. Then for every $G : \mathbb{C}^n \to \mathbb{C}$ of positive degree there exists a polynomial*

$$P_G(W, T) = T^\mu + P_G^{(1)}(W)T^{\mu-1} + \ldots + P_G^{(\mu)}(W),$$

in $n+1$ variables $(W, T) = (W_1, \ldots, W_n, T)$ such that:

(1) $P_G(H_1, \ldots, H_n, G) = 0$

(2) *For every $w \in \mathbb{C}^n$ we denote by $\zeta^{(1)}(w), \ldots, \zeta^{(\mu)}(w)$ the sequence of all points of $F^{-1}(0)$ counted with multiplicities. Then*

$$P_G(w, T) = \prod_{j=1}^{\mu}(T - G(\zeta^{(j)}(w))).$$

(3) $\operatorname{weight}(P_G) \leqslant (\prod_{i=1}^{n} d_i)(\deg G)$, *if* $\operatorname{weight}(W_i) = \deg H_i = d_i$ *for* $i = 1, \ldots, n$ *and* $\operatorname{weight}(T) = \deg G$.

Now let $F_1, \ldots, F_k : \mathbb{C}^n \to \mathbb{C}$ be a s.r. sequence of polynomials. Let us fix linear forms $L_1, \ldots, L_{n-k} : \mathbb{C}^n \to \mathbb{C}$ such that the mapping

$$H = (F_1, \ldots, F_k, L_1, \ldots, L_{n-k}) : \mathbb{C}^n \to \mathbb{C}^n .$$

is proper.

Proof of Theorem 1.3 Let $P(W,T) = P_G(W,T)$. From Prop. 2.1 (2) and from the assumption $G|(F_1, \ldots, F_k)^{-1}(0) = 0$ it follows that $P(0, w_{k+1}, \ldots, w_n, T) = T^\mu$ for every $(w_{k+1}, \ldots, w_n) \in \mathbb{C}^{n-k}$. This means that $P(0, W_{k+1}, \ldots, W_n, T) = T^\mu$, and consequently $P(W,T) = T^\mu + W_1 P_1(W,T) + \ldots + W_k P_k(W,T)$ with

$$\text{weight}\,(W_j P_j(W,T)) \leqslant \text{weight}\ P(W,T) \leqslant (\prod_{i=1}^{n} d_i)(\deg G) \text{ for } j = 1, \ldots, k\ .$$

Substitution $F_1, \ldots, F_k, G$ for $W_1, \ldots, W_k, T$ gives

$$G^\mu = (-P_1(F,L,G))F_1 + \ldots + (-P_k(F,L,G))F_k\,,$$

with

$$\deg\,(-P_i(F,L,G)F_i) \leqslant \text{weight}\,(-P_i(W,T)W_i) \leqslant (\prod_{i=1}^{k} d_i)(\deg G),$$

and the proof is complete.

Proof of Theorem 1.5 Let $P(W,T) = P_G(W,T)$ and $G = F_{k+1}$. Since $P(H,G) = 0$ then we get

$$-(G^\mu + P^{(1)}(F)G^{\mu-1} + \cdots + P^{(\mu-1)}(F)G) = P^{(\mu)}(H),$$

This gives

$$(*) \qquad\qquad -F_{k+1}A_{k+1} = P^{(\mu)}(H)$$

We have $P^{(\mu)}(w) = \pm \prod_{j=1}^{\mu} F_{k+1}(\zeta^{(j)}(w))$, consequently $P^{(\mu)}(0, w_1, \ldots, w_k) \neq 0$ in $\mathbb{C}^{n-k}$ and we may assume

$$P^{(\mu)}(0, w_{k+1}, \ldots, w_n) = -1\ .$$

Then $P^{(\mu)}(W) = W_1 A_1(W) + \ldots + W_k A_k(W) - 1$ with

$$\text{weight}\,(W_k A_k) \leqslant \text{weight}\,(P^{(\mu)}) \leqslant d_1 \cdot \ldots \cdot d_k\ .$$

Substitution gives $P^{(\mu)}(H) = F_1 A_1(F) + \ldots + F_k A_k(F) - 1$ with $\deg(F_i A_i(F)) \leqslant d_1 \cdot \ldots \cdot d_{k+1}$, for $i = 1, \ldots, k$. By $(*)$ we get $A_1 F_1 + \ldots + A_{k+1} F_{k+1} = 1$ with $\deg(F_i A_i(F)) \leqslant d_1 \cdot \ldots \cdot d_{k+1}$, for $i = 1, \ldots, k+1$. This ends the proof.

3. Examples. Let $F = (F_1, \dots, F_k) : \mathbb{C}^n \to \mathbb{C}^k$ be polynomial mapping. One can compare properties of the mapping F with the condition that $F_1, \dots, F_k$ is a s.r. sequence.

It is easy to see that if $F_1, \dots, F_k$ is a s.r. sequence then the mapping F is surjective and has all fibres of pure dimension $n-k$. In the next example we observe that this assumption with additional complete intersection of components of F is not enough for $F_1, \dots, F_k$ to be a s.r. sequence.

Example 3.1. Consider polynomials:

$F_1 : \mathbb{C}^3 \ni (x, y, z) \to x \in \mathbb{C}$,
$F_2 : \mathbb{C}^3 \ni (x, y, z) \to xy^2 + y \in \mathbb{C}$.

It is easy to see that F_1, F_2 is a complete intersection and all fibres of F has dimension 1. We observe that F_1, F_2 is not a s.r. sequence. Let us take a linear form $L(x, y, z) = ax + by + cz$ and suppose that the mapping $H = (F_1, F_2, L)$ is proper. Then obviously $c \neq 0$ and we can assume $c = 1$. Taking the sequence $h_\nu = (\nu^{-1}, -\nu, b\nu - a\nu^{-1})$, we have $|h_\nu| \to +\infty$ $(\nu \to +\infty)$. Observe that we obtain $H(h_\nu) = (\nu^{-1}, 0, 0) \to 0$ $(\nu \to +\infty)$, and so (F_1, F_2) is not a s.r. sequence.

Our next example says that an additional condition that all the fibres have the same degree (with multiplicity) is too strong in this situation.

Example 3.2. Consider polynomials:

$F_1 : \mathbb{C}^3 \ni (x, y, z) \to y \in \mathbb{C}$,
$F_2 : \mathbb{C}^3 \ni (x, y, z) \to z + x^2 y \in \mathbb{C}$.

It is easy see that F_1, F_2 is a s.r. sequence since the mapping (F_1, F_2, L) is a polynomial automorphism for $L(x, y, z) = x$. But $\deg F^{-1}((0, 0)) = 1$ and $\deg F^{-1}((1, 0)) = 2$.

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