Stabilization of evolution equations by noise with application to partial differential equations of parabolic type.[1]

ANNA A. KWIECIŃSKA

IM UJ PREPRINT 1999/22

Abstract

We consider a deterministic equation of evolution $X'(t) = AX(t)dt$, in a separable Hilbert space. We prove that if A generates a C_0-semigroup, then this equation can be stabilized, in terms of Lyapunov exponents, by noise. Then we apply this abstract result to partial differential equations of parabolic type. We also compute the Lyapunov exponents of these PDEs, both deterministic and stochastic, as functions of the eigenvalues of the operator A.

1. Introduction.

The present paper is a development of [10], where we have constructed an example of a class of partial differential equations being stabilized (in

[1] Research partially supported by KBN grant 2 P03A 016 16

terms of Lyapunov exponents) by noise. In this paper we provide sufficient conditions for exponential stabilization of abstract evolution equations. We apply these results to parabolic partial differential equations extending thus the example from [10].

First, in a separable Hilbert space H, we study an equation of evolution of the form

$$X'(t) = AX(t),$$

with the initial condition f (see Section 2). If $f \in \mathcal{D}(A)$ and A generates a C_0-semigroup, the general theory ensures the existence of a unique solution to the above problem, which we denote by $x^{\mathrm{det}}(f)$. For $f \in \mathcal{D}(A)$, $f \neq 0$, the Lyapunov exponent of the deterministic equation is defined as:

$$\lambda^{\mathrm{det}}(f) = \limsup_{t\to\infty} \frac{1}{t} \log \|x^{\mathrm{det}}(t)\|.$$

We prove that $\lambda^{\mathrm{det}}(f) < \infty$ (see Lemma 1). Next, we study the Dirichlet problem for a stochastic perturbation of the above equation, namely

$$dX = AX\,dt + \sigma \sum_{k=1}^{N} B_k X\, d\beta_k,$$

with the initial condition f. If $f \in \mathcal{D}(A)$, A generates a C_0-semigroup and the operators B_k satisfy the assumptions (A1) (A3) and (A4), there exists a unique strong solution to the above stochastic problem, which we denote by $x^{\mathrm{st}}(f)$. For $f \in \mathcal{D}(A), f \neq 0$, the Lyapunov exponents of the stochastic equation are defined as

$$\lambda^{\mathrm{st}}(f,\omega) = \limsup_{t\to\infty} \frac{1}{t} \log \|x^{\mathrm{st}}(t,\omega)\|.$$

In Theorem 1 we prove an inequality relating the deterministic and stochastic Lyapunov exponents for a fixed initial condition f and almost all ω. Under our assumptions the stochastic Lyapunov exponents turn out to be smaller, almost surely, than the corresponding deterministic one. It means that the deterministic system is made more stable by adding a term with white noise. Moreover, there exists σ_0 such that, for $\sigma \geq \sigma_0$ the stochastic Lyapunov exponent is strictly smaller than zero with probability one. This is especially interesting, because it means that for σ big enough the stochastic system is exponentially stable, although the deterministic one may be even exponentially unstable. Then taking $B_k = b_k\mathrm{Id}$, $k = 1, \dots, N$, we prove the main result of our paper, that is Theorem 2. This theorem provides a sufficient condition for stabilization of deterministic evolution equations by noise (see Corollary 1), which is both fairly general and natural in the semigroup approach. We prove that, if the operator A generates a C_0-semigroup, then the equation of evolution can be made stable by noise, i.e. all the Lyapunov exponents become smaller than zero with probability one.

In Section 3, these abstract results are applied to partial differential equations of parabolic type. We study the Dirichlet problem for the equation

$$\frac{\partial u}{\partial t} = Au,$$

with the initial condition f, where A is a second order, symmetric differential operator satisfying suitable assumptions (for a formal setting see Section 3). We prove that the operator A generates a C_0-semigroup in the space $L^2(\mathcal{O})$. Then we consider the Dirichlet problem for the following stochastic perturbation of the above equation

$$dv(t) = Av(t)dt + \sigma \sum_{k=1}^{N} b_k v(t) d\beta_k(t),$$

with the initial condition f. It follows from the abstract theory that for σ big enough the stochastic system is exponentially stable. In addition, we obtain bounds on the Lyapunov exponents, both deterministic and stochastic, which turn out to be sharp (see Section 4).

In Section 4 we obtain stronger results concerning the Lyapunov exponents of the parabolic equations studied in Section 3 than those following from the abstract theory. We prove that these Lyapunov exponents exist as limits and in the stochastic case they are also non-random. What is more, in both cases they can be computed as functions of the eigenvalues of the operator A, see Theorem 4 and Theorem 5. Finally, we give an example, where the Lyapunov exponents can be computed explicitely.

The problem of stabilization of ordinary differential equations by noise, in terms of almost sure Lyapunov exponents, has been thoroughly studied, see for instance [2], [3], [4], [14] and [15]. However, to our knowledge, there are very few results concerning stochastic stabilization in the infinite-dimensional case. Apart from our previous paper, almost sure exponential stabilization has been studied very recently by Caraballo, Liu and Mao, see [5] and other types of stabilization have been studied by Leha, Maslowki and Ritter, see [12]. For a survey on stochastic stability see [13].

2. The abstract problem.

Let H be a separable Hilbert space and let A be a linear operator from $\mathcal{D}(A) \subset H$ into H. We consider the following deterministic equation of evolution

$$X'(t) = AX(t) \tag{1}$$

with the initial condition

$$X(0) = f \in \mathcal{D}(A). \tag{2}$$

Following [16] by a solution to the above Cauchy problem we mean an H valued function $x(t)$ such that $x(t)$ is continuous for $t \geq 0$, continuously differentiable, $u(t) \in \mathcal{D}(A)$ for $t > 0$ and (1)-(2) is satisfied. If $A : \mathcal{D}(A) \subset H \to H$ generates a C_0-semigroup $S(t)$ and if $f \in \mathcal{D}(A)$ the above Cauchy problem has a unique solution $x(t) = S(t)f$. The solution of the above problem will be denoted by $x^{\det}(t)$. For $f \neq 0$, $f \in \mathcal{D}(A)$ the Lyapunov exponent of the problem (1)-(2) is defined as

$$\lambda^{\det}(f) = \limsup_{t\to\infty} \frac{1}{t} \log \|x^{\det}(t)\|.$$

Lemma 1 *If the operator A generates a C_0-semigroup $S(t)$, then for any initial value f*

$$\lambda^{\det}(f) \leq \lambda,$$

where $\lambda = const > 0$ and does not depend on f.

Proof. It follows from the fact that since $S(t)$ is a C_0-semigroup, there exist constants $\lambda > 0$ and and $M \geq 1$ such that

$$\|S(t)\| \leq M \exp(\lambda t),$$

see [16], Theorem 2.2, p.4. □

Let us now consider the following stochastic problem

$$dX = AXdt + \sigma \sum_{k=1}^{N} B_k X d\beta_k, \tag{3}$$

$$X(0) = f \in H, \tag{4}$$

where

$$A : \mathcal{D}(A) \subset H \to H, \quad B_k : \mathcal{D}(B_k) \subset H \to H, \quad k = 1, 2, \ldots, N$$

and $\beta_1(\cdot), \ldots, \beta_N(\cdot)$, $N < \infty$, are independent, real Wiener processes. Let us assume the following:

(1) The operator A generates a semigroup $S(t)$ and the operators $\sigma B_1, \ldots, \sigma B_N$ generate mutually commuting C_0-groups $S_k(t)$, $k = 1, \ldots, N$.

(2) For $k = 1, \ldots, N$, $\mathcal{D}(\sigma^2 B_k^2) \supset \mathcal{D}(A)$ and $\bigcap_{k=1}^{N} \mathcal{D}(((\sigma B_k)^*)^2)$ is a dense subset of H, where $(\sigma B_k)^*$ denotes the adjoint of σB_k.

(3) The operator $C = A - \frac{1}{2}\sigma^2 \sum_{k=1}^{N} B_k^2$, $\mathcal{D}(C) = \mathcal{D}(A)$ is the infinitesimal generator of a C_0-semigroup $S_0(t)$, $t \geq 0$.

(4) There exist $\lambda_0 \in \mathbf{R}$ and bounded operators $K_1, \ldots, K_N$ such that

$$(\lambda_0 - C)\sigma B_k (\lambda_0 - C)^{-1} = \sigma B_k + K_k, \quad k = 1, \ldots, N.$$

Then, if $f \in \mathcal{D}(C)$, the above problem has a unique strong solution, as defined in [6] p.152, on any interval $[0, T]$, where $T > 0$ – see [6], Proposition 6.29, p. 176 and Theorem 6.30, p. 177.

We will study next an evolution equation of the above form, and we assume that:

(**A1**) The operators B_k are diagonalizable in the same basis and are bounded from above and below, i.e there exist $\{e_j\}$ a basis of the Hilbert space H and constants d_i^k, where $k = 1, \ldots, N$ and $i = 1, 2, \ldots$, such that

$$\langle B_k e_i, e_j \rangle = d_k^i \delta_{ij}, \quad k = 1, \ldots, N; \quad i, j = 1, 2, \ldots,$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in H. The assumption that the operators B_k are bounded from above and below is then equivalent to the condition that there exist constants $m_k > 0$ and $M_k > 0$, $k = 1, \ldots, N$, such that

$$m_k \leq |d_i^k| \leq M_k, \quad k = 1, \ldots, N; \quad i = 1, 2, \ldots.$$

Obviously the operators B_k also mutually commute.
(**A2**) The operator A generates a C_0-semigroup.
(**A3**) There exist $\lambda_0 \in \mathbf{R}$ and bounded operators $K_1, \ldots, K_N$ such that

$$(\lambda_0 - C)\sigma B_k(\lambda_0 - C)^{-1} = \sigma B_k + K_k, \quad k = 1, \ldots, N,$$

where $C = A - \frac{1}{2}\sigma^2 \sum_{k=1}^{N} B_k^2$.
(**A4**) The operators A and B_k, $k = 1, \ldots, N$, commute.

Proposition 1 *The general assumptions (1)-(4) follow from (A1)-(A4).*

To prove the above proposition we will need two lemmas.

Lemma 2 *Let A and B be the infinitesimal generators of C_0-semigroups $S(t)$ and $T(t)$ respectively. If A and B commute, then $S(t)$ and $T(t)$ also commute.*

Proof. Since A and B are generators of C_0-semigroups we have:

$$S(t)x = \lim_{\lambda \to \infty} \exp(tA_\lambda)x, \quad x \in H$$

and

$$T(t)x = \lim_{\gamma \to \infty} \exp(tB_\gamma)x, \quad x \in H,$$

where A_λ and B_γ are the Yosida approximations of A and B respectively, i.e. $A_\lambda = \lambda AR(\lambda : A)$ and $B_\gamma = \gamma BR(\gamma : B)$; see [16] Theorem 5.5, p.21. There exist $\bar{\lambda} > 0$ and $\bar{\gamma} > 0$ such that for $\lambda > \bar{\lambda}$ and $\gamma > \bar{\gamma}$ the operators A_λ and B_γ are well defined and bounded. In what follows we will consider the operators A_λ and B_γ with $\lambda > \bar{\lambda}$ and $\gamma > \bar{\gamma}$. As the operators A and B commute, the operators A_λ and B_γ also commute.

Let us fix arbitrary λ and γ. We put

$$S_{A_\lambda}^N = \sum_{n=1}^{N} \frac{(tA_\lambda)^n}{n!}$$

and

$$S_{B_\gamma}^N = \sum_{n=1}^{N} \frac{(tB_\gamma)^n}{n!}$$

Since the operators A_λ and B_γ commute, we have

$$S^N_{A_\lambda} S^N_{B_\gamma} = S^N_{B_\gamma} S^N_{A_\lambda}$$

and passing to the limit with N we get

$$\exp(tA_\lambda)\exp(tB_\gamma) = \exp(tB_\gamma)\exp(tA_\lambda).$$

From the above equality we obtain for arbitrary $x \in H$:

$$\lim_{\lambda\to\infty} \exp(tA_\lambda)\exp(tB_\gamma)x = \lim_{\lambda\to\infty} \exp(tB_\gamma)\exp(tA_\lambda)x$$

and we deduce that

$$S(t)\exp(tB_\gamma)x = \exp(tB_\gamma)S(t)x.$$

Passing again to the limit with γ we deduce that

$$S(t)T(t)x = T(t)S(t)x.$$

$\square$

Lemma 3 *Let $S(t)$ and $T(t)$ be two commuting C_0-semigroups. We put $R(t) = S(t)T(t)$. Then $R(t)$ is also a C_0-semigroup.*

Proof. We have

$$R(0) = S(0)T(0) = \mathrm{Id}$$

and since the semigroups $S(t)$ and $R(t)$ commute

$$R(t+s) = S(t+s)T(t+s) = S(t)S(s)T(t)T(s) = S(t)T(t)S(s)T(s) = R(t)R(s).$$

To prove that

$$\lim_{t\downarrow 0} R(t)x = x, \quad \forall x \in H,$$

we estimate

$$|T(t)S(t)x - x| = |T(t)S(t)x - T(t)x + T(t)x - x|$$

$$\leq |T(t)S(t)x - T(t)x| + |T(t)x - x| \leq \|T(t)\||S(t)x - x| + |T(t)x - x|,$$

where by $|\cdot|$ we have denoted the norm in the Hilbert space H and by $\|\cdot\|$ the operator norm. It follows from [16], Theorem 2.2, p.4 that there exist constants $\lambda > 0$ and $M \geq 1$ such that for $t \in [0,1]$

$$\|T(t)\| \leq M \exp \lambda,$$

we can therefore conclude that

$$\lim_{t \downarrow 0} |T(t)S(t)x - x| = 0$$

□

Proof of the Proposition. (1) Follows from Lemma 2 and (A1).
(2) We have $\mathcal{D}(B_k) = \mathcal{D}(B_k^*) = H$, $k = 1, \ldots, N$, which implies (2).
(3) Obviously $\mathcal{D}(C) = \mathcal{D}(A)$, where $C = A - \frac{1}{2}\sigma^2 \sum_{k=1}^N B_k^2$. The semigroup generated by the operator C is given by the following formula

$$S_0(t) = S(t) \prod_{k=1}^N \exp(-\frac{1}{2} t \sigma^2 B_k^2).$$

We have assumed that the operator A generates a C_0-semigroup. On the other hand the operators B_k^2 are bounded and defined on the whole space H and therefore $\exp(-\frac{1}{2}t\sigma^2 B_k^2)$, $k = 1, \ldots, N$, define C_0-semigroups. Due to Lemma 2 these semigroups are mutually commuting since their generators are mutually commuting (the assumptions (A1) and (A4)), so Lemma 3 enables us to conclude that the operator C generates a C_0-semigroup.
(4) It is the assumption (A3). □

So Proposition 6.29 and Theorem 6.30 from [6] can be applied and if $f \in \mathcal{D}(C) = \mathcal{D}(A)$ there exists a unique solution to the problem (3)-(4). This solution is given by the following formula

$$x^{\mathrm{st}}(t) = S_0(t) \prod_{k=1}^N \exp[\sigma B_k \beta_k(t)] f. \tag{5}$$

For $f \neq 0$, $f \in \mathcal{D}(A)$ the Lyapunov exponent of the system (3)-(4) is defined as:

$$\lambda^{\mathrm{st}}(f, \omega) = \limsup_{t \to \infty} \frac{1}{t} \log \|x^{\mathrm{st}}(t, \omega)\|.$$

From Lemma 1 and the next theorem it will follow that $\lambda^{\mathrm{st}}(f, \omega) < \infty$ a.s.

Theorem 1 *Let $f \in \mathcal{D}(A)$, $f \neq 0$. Under the assumptions (A1)-(A4) the following inequality holds:*

$$\lambda^{\rm st}(f,\omega) \leq \lambda^{\rm det}(f) - \sigma^2\alpha \quad a.s. \tag{6}$$

where

$$\alpha = -\limsup_{t\to\infty} \frac{1}{t} \log \| \prod_{k=1}^{N} \exp[(-\frac{1}{2}B_k^2)t] \| > 0.$$

It follows that there exists σ_0 such that for $\sigma \geq \sigma_0$

$$\lambda^{\rm st}(f,\omega) < 0 \quad a.s. \tag{7}$$

and σ_0 does not depend on f.

Proof. We have

$$\lambda^{\rm st}(f,\omega) = \limsup_{t\to\infty} \frac{1}{t} \log \|S_0(t) \prod_{k=1}^{N} \exp[\sigma B_k \beta_k(t)] f\|$$

$$\leq \limsup_{t\to\infty} \frac{1}{t} \log \| \prod_{k=1}^{N} \exp[(-\frac{1}{2}\sigma^2 B_k^2)t] \prod_{k=1}^{N} \exp[\sigma B_k \beta_k(t)]\| \|S(t)f\|$$

$$\leq \limsup_{t\to\infty} \frac{1}{t} \log \| \prod_{k=1}^{N} \exp[(-\frac{1}{2}\sigma^2 B_k^2)t] \prod_{k=1}^{N} \exp[\sigma B_k \beta_k(t)]\| + \limsup_{t\to\infty} \frac{1}{t} \log \|S(t)f\|$$

$$\leq \lambda^{\rm det}(f) + \limsup_{t\to\infty} \frac{1}{t} \log \| \prod_{k=1}^{N} \exp[(-\frac{1}{2}\sigma^2 B_k^2)t]\| \| \prod_{k=1}^{N} \exp[\sigma B_k \beta_k(t)]\|$$

$$\leq \lambda^{\rm det}(f) + \limsup_{t\to\infty} \frac{1}{t} \log \| \prod_{k=1}^{N} \exp[(-\frac{1}{2}\sigma^2 B_k^2)t]\| + \limsup_{t\to\infty} \frac{1}{t} \log \| \prod_{k=1}^{N} \exp[\sigma B_k \beta_k(t)]\|.$$

Since the operators B_k, $k = 1, \ldots, N$, are bounded from above by constants $M_k > 0$, $k = 1, \ldots, N$, we have

$$\| \prod_{k=1}^{N} \exp[\sigma B_k \beta_k(t)]\| \leq \prod_{k=1}^{N} \exp[\sigma M_k |\beta_k(t)|],$$

and hence

$$\limsup_{t\to\infty} \frac{1}{t} \log \| \prod_{k=1}^{N} \exp[\sigma B_k \beta_k(t)]\| \leq 0,$$

since by the strong law of large numbers $\lim_{t\to\infty}\beta_k(t)/t = 0$ with probability 1, $k = 1, \dots, N$ (see [1], page 46).

On the other hand, since all the operators B_k are diagonal and there exist constants $m_k > 0$ and $M_k > 0$ such that

$$m_k \leq \|B_k\| \leq M_k,$$

an easy computation gives the following bounds:

$$-\frac{1}{2}\sigma^2(\sum_{k=1}^{N} M_k^2) \leq \frac{1}{t}\log \| \prod_{k=1}^{N} \exp[-\frac{1}{2}\sigma^2(B_k^2)t]\| =$$

$$= \sigma^2\frac{1}{t}\log \| \prod_{k=1}^{N} \exp[-\frac{1}{2}(B_k^2)t]\| \leq -\frac{1}{2}\sigma^2(\sum_{k=1}^{N} m_k^2)$$

and hence there exists $\alpha > 0$ such that

$$\limsup_{t\to\infty} \frac{1}{t}\log \| \exp[(-\frac{1}{2}\sigma^2 \sum_{k=1}^{N} B_k^2)t]\| = -\sigma^2\alpha.$$

Now the first part of the statement of the theorem follows.

The second part follows directly from the first part and Lemma 1. $\square$

Let us now consider the problem (3)-(4) with $B_k = b_k\mathrm{Id}$, where $b_k \neq 0$, $k = 1, \dots, N$, i.e.

$$dX(t) = AX(t)dt + \sigma \sum_{k=1}^{N} b_k X(t) d\beta_k(t) \tag{8}$$

$$X(0) = f \in \mathcal{D}(A) \tag{9}$$

We denote by $\lambda^{\mathrm{st}}(f,\omega)$ the Lyapunov exponent of the stochastic problem (8)-(9) and by $\lambda^{\mathrm{det}}(f)$ the Lyapunov exponent of the deterministic problem (1)-(2).

Theorem 2 *Let $f \neq 0$, $f \in \mathcal{D}(A)$. If the operator A generates a C_0-semigroup, solutions to (1)-(2) and (8)-(9) exist, are unique and we have the following estimate:*

$$\lambda^{\mathrm{st}}(f,\omega) \leq \lambda^{\mathrm{det}}(f) - \frac{1}{2}\sigma^2 \sum_{k=1}^{N} b_k^2 \quad a.s.$$

Proof. The existence and uniqueness of solution to the deterministic problem (1)-(2) follow from the fact that A generates a C_0-semigroup. For the stochastic problem it is enough to check that the conditions (A1)-(A4) hold.

The assumptions (A1), (A2) and (A4) are obviously satisfied. We have to verify that (A3) holds. We compute:

$$(\lambda_0 \mathrm{Id} - C) = (\lambda_0 \mathrm{Id} - (A - \frac{1}{2}\sigma^2 \sum_{k=1}^{N} b_k^2 \mathrm{Id})) = ((\lambda_0 - \frac{1}{2}\sigma^2 \sum_{k=1}^{N} b_k^2)\mathrm{Id} - A).$$

Since A generates a C_0-semigroup the operator A is closed and there exists $\delta \geq 0$ such that the resolvent set $\varrho(A)$ of A contains the ray (δ, ∞), see[16], Theorem 5.3, p.20, see also Section 3. It follows that if $\gamma > \delta$, then $R(\gamma : A)$ is well defined. Therefore, if we take λ_0 satisfying

$$\lambda_0 - \frac{1}{2}\sigma^2 \sum_{k=1}^{N} b_k^2 > \delta$$

we have that

$$(\lambda_0 \mathrm{Id} - C) b_k \mathrm{Id} (\lambda_0 \mathrm{Id} - C)^{-1} = b_k (\lambda_0 \mathrm{Id} - C)(\lambda_0 \mathrm{Id} - C)^{-1} = b_k \mathrm{Id}.$$

It means that (A3) holds with $K_k = 0$, for $k = 1, 2, \ldots, N$.

We can now apply Theorem 1. It is easy to check that $\alpha = \frac{1}{2}\sum_{k=1}^{N} b_k^2$. □

Corollary 1 *If the operator A generates a C_0-semigroup, the deterministic system (1)-(2) can be stabilized by noise.*

Proof. Let us fix $f \neq 0$, $f \in \mathcal{D}(A)$. Lemma 1 ensures the existence of a constant $\lambda > 0$ such that

$$\lambda^{\mathrm{det}}(f) \leq \lambda.$$

Let σ_0 be such that

$$\frac{1}{2}\sigma_0^2 \sum_{k=1}^{N} b_k^2 > \lambda,$$

then, for $\sigma \geq \sigma_0$ we have $\lambda^{\mathrm{st}}(f, \omega) < 0$ for almost all ω. □

3. Application to partial differential equations of parabolic type.

Let $\mathcal{O}$ be a bounded domain in $\mathbf{R}^d$: a ball or a set that can be mapped into a ball by a regular mapping of class $C^2(\bar{\mathcal{O}})$.

We will study the following parabolic equation:

$$\frac{\partial u}{\partial t} = Au, \tag{10}$$

where $u = u(t,x)$, $t \in \mathbf{R}^+$ (where $\mathbf{R}^+$ denotes the interval $[0,\infty)$), $x \in \mathcal{O}$. We set the initial condition

$$u(0,x) = f(x) \tag{11}$$

and the Dirichlet boundary condition

$$u(t,x) = 0, \quad x \in \partial\mathcal{O}. \tag{12}$$

The operator A is of the form

$$Au = \sum_{i,j} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + a(x)u,$$

where the coefficients satisfy the following assumptions:
(C1) a_{ij} are differentiable
(C2) $a_{ij} = a_{ji}$
(C3) $\mu_1\xi^2 \le \sum_{i,j} a_{ij}\xi_i\xi_j \le \mu_2\xi^2$
(C4) $\left|\frac{\partial a_{ij}}{\partial x_k}\right| \le \mu_3$
(C5) $|a| \le \mu_4$,
where μ_l are constants and $\mu_l > 0$, $l = 1,\ldots,4$.

It is easy to verify that the operator A is symmetric on the space $H_0^1(\mathcal{O})$.

The function f in (11) takes real values. In this section we consider only deterministic real-valued initial conditions $f \in H_0^1(\mathcal{O}) \cap H^2(\mathcal{O})$.

We will need the following subspace of the space $H^2(\mathcal{O})$:
$W_{2,0}^2(\mathcal{O})$ - a subspace of $H^2(\mathcal{O})$ generated by functions belonging to $C^2(\bar{\mathcal{O}})$ and vanishing on $\partial\mathcal{O}$.

Later on we will prove that in our case $W_{2,0}^2(\mathcal{O}) = H_0^1(\mathcal{O}) \cap H^2(\mathcal{O})$.

There exists an orthonormal basis of $L^2(\mathcal{O})$, $\{e_j\}$, $j = 1,2,\ldots$, consisting of eigenvectors of the operator A (see [11], p.181) such that

$$Ae_j = \lambda_j e_j, \quad \text{where} \quad \lambda_j \searrow -\infty \ (j \to \infty)$$

and

$$[e_k, e_l] = (\lambda_0 - \lambda_k)^2 \delta_k^l,$$

where $[u,u]^{\frac{1}{2}}$ is equivalent to the norm in $W^2_{2,0}(\mathcal{O})$. The constant λ_0 is chosen such that $\lambda_k < \lambda_0$, $k = 1, 2, \ldots$.

Lemma 4 *Under our assumptions we have the following characterisation of the space* $W^2_{2,0}(\mathcal{O})$

$$W^2_{2,0}(\mathcal{O}) = \{u = \sum_{j=1}^{\infty} u_j e_j(x) \in L^2(\mathcal{O}) : \sum_{j=1}^{\infty} u_j^2 (\lambda_0 - \lambda_j)^2 < \infty\}$$

Proof. Obviously $\left\{\frac{1}{\lambda_0 - \lambda_j} e_j\right\}$ is an orthonormal basis of $W^2_{2,0}(\mathcal{O})$ with the scalar product $[\cdot,\cdot]$.

If $u \in L^2(\mathcal{O})$ we can write

$$u = \sum_{j=1}^{\infty} u_j e_j = \sum_{j=1}^{\infty} (\lambda_0 - \lambda_j) u_j \left(\frac{1}{\lambda_0 - \lambda_j} e_j \right)$$

and the equality follows. □

Proposition 2 *Under our assumptions the following equality holds:*

$$W^2_{2,0}(\mathcal{O}) = H^1_0(\mathcal{O}) \cap H^2(\mathcal{O}).$$

Proof. Obviously we have

$$W^2_{2,0}(\mathcal{O}) \subset H^2(\mathcal{O}) \quad (*)$$

The inclusion

$$W^2_{2,0}(\mathcal{O}) \subset H^1_0(\mathcal{O}) \quad (**)$$

follows from the trace theorem (see [17], p.48):

Theorem (trace theorem): Let $\mathcal{O}$ be a bounded set of class C^1. There exists a linear continuous operator $\gamma_0 \in \mathcal{L}(H^1(\mathcal{O}), L^2(\partial\mathcal{O}))$ such that:

$$\gamma_0 u = u|_{\partial\mathcal{O}}, \ \forall u \in C^1(\bar{\mathcal{O}}).$$

We have

$$u \in W^2_{2,0}(\mathcal{O}) \Longleftrightarrow \exists \{u_m\} \in C^2(\bar{\mathcal{O}}),\ u_m|_{\partial\mathcal{O}} = 0,\ u_m \to u \quad \text{in} \quad H^2(\mathcal{O}).$$

Since $\gamma_0 u_m = 0$, γ_0 is continuous in $H^1(\mathcal{O})$ and $u_m \to u$ in $H^1(\mathcal{O})$, we deduce that

$$\gamma_0 u = u|_{\partial\mathcal{O}} = 0.$$

In the case of bounded regions with smooth boundary $u \in H^1_0(\mathcal{O})$ is equivalent to the conditions $u \in H^1(\mathcal{O})$ and $u|_{\partial\mathcal{O}} = 0$, see [8], p.122. Therefore we conclude that $u \in H^1_0(\mathcal{O})$, which proves $(**)$.

The inclusions $(*)$ and $(**)$ give

$$W^2_{2,0}(\mathcal{O}) \subset H^1_0(\mathcal{O}) \cap H^2(\mathcal{O}).$$

Let us now take $u \in H^1_0(\mathcal{O}) \cap H^2(\mathcal{O})$. The operator A is symmetric on the space $H^1_0(\mathcal{O})$ and therefore is closable, see [9], p. 269. We have

$$Au = \bar{A}u = \bar{A}\sum_{j=1}^{\infty} u_j e_j(x) = \sum_{j=1}^{\infty} u_j \bar{A} e_j(x) = \sum_{j=1}^{\infty} u_j A e_j(x)$$

$$= \sum_{j=1}^{\infty} u_j \lambda_j e_j(x) \in L^2(\mathcal{O}).$$

Note that $\sum_{j=1}^{\infty} u_j \lambda_j e_j(x) \in L^2(\mathcal{O})$ if and only if $\sum_{j=1}^{\infty} u_j^2 \lambda_j^2 < \infty$, which implies $\sum_{j=1}^{\infty} u_j^2 (\lambda_0 - \lambda_j)^2$, and by Lemma 4 it gives $u \in W^2_{2,0}(\mathcal{O})$. Hence the inclusion

$$H^1_0(\mathcal{O}) \cap H^2(\mathcal{O}) \subset W^2_{2,0}(\mathcal{O})$$

follows. □

If the coefficients of the operator A are sufficiently smooth, then Theorem 2.7, p.211, from [16], ensures that the operator A generates an analytic semigroup of operators on the space $L^2(\mathcal{O})$. However our assumptions (C1)-(C5) do not guarantee the smoothness of the coefficients of A and therefore we present a separate proof of the fact that the operator A generates a C_0-semigroup.

Theorem 3 *The operator A with $\mathcal{D}(A) = H_0^1(\mathcal{O}) \cap H^2(\mathcal{O})$ generates a $\mathcal{C}_0$-semigroup $S(t)$ such that $\|S(t)\| \leq \exp(\lambda_1 t)$.*

In the proof of the above theorem we will make use of a more general result from [16] – Theorem 5.3, p.20.

Theorem A linear operator $A : \mathcal{D}(A) \subset X \to X$, where X is a Banach space, is the infinitesimal generator of a $\mathcal{C}_0$-semigroup $T(t)$ satisfying $\|T(t)\| \leq M\exp(\omega t)$, if and only if
(i) A is closed and $\mathcal{D}(A)$ is dense in X
(ii) The resolvent set $\varrho(A)$ of A contains the ray (δ, ∞) and $\|R(\lambda : A)^n\| \leq M/(\lambda - \delta)^n$, for $\lambda > \delta$, $n = 1, 2, \ldots$.

Proof of Theorem 3. We will first prove that condition (i) is satisfied. The space $H_0^1(\mathcal{O}) \cap H^2(\mathcal{O})$ is dense in $L^2(\mathcal{O})$. We have already remarked that the operator A is closable (see the proof of Proposition 2). Let us take $x_n \in \mathcal{D}(A)$ such that $x_n \to x$ and $Ax_n \to y$. We can write

$$x_n = \sum_{j=1}^{\infty} x_j^n e_j(x), \quad x = \sum_{j=1}^{\infty} x_j e_j(x) \quad \text{and} \quad y = \sum_{j=1}^{\infty} y_j e_j(x).$$

We compute

$$Ax_n = \bar{A}x_n = \bar{A}\sum_{j=1}^{\infty} x_j^n e_j(x) = \sum_{j=1}^{\infty} x_j^n \bar{A} e_j(x) = \sum_{j=1}^{\infty} x_j^n A e_j(x) =$$

$$= \sum_{j=1}^{\infty} x_j^n \lambda_j e_j(x) \ (n \to \infty)$$

Since $x_n \to x$ $(n \to \infty)$, we have that $x_j^n \to x_j$ $(n \to \infty)$ for each j. On the other hand since $Ax_n \to y$ $(n \to \infty)$, we deduce that $x_j^n \lambda_j \to y_j$ $(n \to \infty)$ for each j. Hence we obtain:

$$y_j = \lambda_j x_j$$

and

$$x = \sum_{j=1}^{\infty} \frac{y_j}{\lambda_j} e_j(x).$$

Therefore by Lemma 4 and Proposition 2, $x \in \mathcal{D}(A)$ and $y = Ax$, which implies that the operator A is closed.

To prove that (ii) holds let us fix an arbitrary constant $\lambda > \lambda_1$ (this implies $\lambda > \lambda_k$ for $k = 1, 2, \ldots$). We have

$$(\lambda I - A)u = \sum_{j=1}^{\infty}(\lambda - \lambda_j)u_j e_j(x)$$

and

$$(\lambda I - A)^{-1}v = \sum_{j=1}^{\infty} \frac{1}{\lambda - \lambda_j} v_j e_j(x).$$

The second operator is well defined for any $v \in L^2(\mathcal{O})$. We can write

$$R(\lambda : A)^n = \sum_{j=1}^{\infty} \left(\frac{1}{\lambda - \lambda_j}\right)^n v_j e_j(x).$$

We easily deduce that

$$\|R(\lambda : A)^n\| = \frac{1}{(\lambda - \lambda_1)^n}$$

and we conclude that condition (ii) is satisfied with $M = 1$ and $\delta = \lambda_1$. □

This semigroup is given by the following formula

$$S(t)f = \sum_{j=1}^{\infty} \exp(t\lambda_j) f_j e_j(x) \tag{13}$$

where $f_j =< f, e_j >$.

Corollary 2 *We have the following estimate:*

$$\lambda^{\det}(f) \leq \lambda_1.$$

Proof. It follows directly from Theorem 3.

Remark 1 In fact we will prove in Section 4 that this estimate cannot be improved.

The results from Section 2 guarantee that the deterministic equation (10) considered with the initial condition (11) and the Dirichlet boundary condition (12) can be stabilized by noise. More precisely, let us consider the following stochastic perturbation of the deterministic problem

$$dv(t) = Av(t)dt + \sigma \sum_{k=1}^{N} b_k v(t) d\beta_k(t), \tag{14}$$

where $b_k \neq 0$, $k = 1, \ldots, N$, with the initial condition

$$v(0, x, \omega) = f(x), \quad x \in \mathcal{O} \tag{15}$$

and the Dirichlet boundary condtion

$$v(t, x, \omega) = 0, \quad x \in \partial\mathcal{O} \tag{16}$$

Conditions (15) and (16) hold for a.a. $\omega \in \Omega$.

The estimate from Theorem 2 together with Theorem 3 give us the following conclusion

Corollary 3 *The stochastic system (14)-(16) is stable for σ satisfying*

$$\frac{1}{2}\sigma^2 \sum_{k=1}^{N} b_k^2 > \lambda_1.$$

In Section 4 we will prove that this result cannot be improved.

4. Computation of Lyapunov exponents.

For parabolic equations studied in Section 3, both deterministic and stochastic, we can obtain stronger results concerning their Lyapunov exponents, than those following from the general theory. In fact we can prove that these Lyapunov exponents exist as limits and in the stochastic case they are also non-random. Moreover, in both cases we can compute the Lyapunov exponents as functions of the eigenvalues of the operator A.

We consider the Dirichlet problem (10)-(12), where the domain $\mathcal{O}$ and the operator A are as in Section 3. Also as in the previous section the initial condition $f \in H_0^1(\mathcal{O}) \cap H^2(\mathcal{O})$ and is real valued.

If $f \in L^2(\mathcal{O})$, we can write

$$f = \sum_{j=1}^{\infty} f_j e_j, \quad \text{where} \quad f_j =< f, e_j > \tag{17}$$

It follows from (13) that the unique solution to the problem (10)-(12) is given by the following formula

$$u(t) = S(t)f = \sum_{j=1}^{\infty} \exp(t\lambda_j) f_j e_j \tag{18}$$

Theorem 4 *Let $f \neq 0$ and let j_0 be the smallest integer $j \geq 1$ in the expansion (17) of f such that $f_{j_0} \neq 0$.*

Then the Lyapunov exponent of (10) – (12) exists as a limit and is given by

$$\lambda^{\det}(f) = \lambda_{j_0}.$$

Therefore the top Lyapunov exponent is equal to λ_1.

Proof. On the one hand,

$$\frac{1}{t}\log \| \sum_{j=0}^{\infty} \exp(t\lambda_j) f_j e_j \| \leq \frac{1}{t}\log(\sum_{j=j_0}^{\infty} |\exp(t\lambda_{j_0}) f_j|^2)^{\frac{1}{2}} = \lambda_{j_0} + \frac{1}{t}\log\|f\|$$

while on the other hand,

$$\frac{1}{t}\log \| \sum_{j=0}^{\infty} \exp(t\lambda_j) f_j e_j \| \geq \frac{1}{t}\log|\exp(t\lambda_{j_0}) f_{j_0}| = \lambda_{j_0} + \frac{1}{t}\log|f_{j_0}|.$$

The existence of the limit and the equality follow. □

The formula (5) implies that the solution to the stochastic problem (14)-(16) is equal to

$$v(t) = \exp(\sum_{k=1}^{N} \sigma b_k \beta_k(t)) \exp[(-\frac{1}{2}\sigma^2 \sum_{k=1}^{N} b_k^2)t] u(t),$$

where $u(t)$ is the solution to the deterministic problem. It is now easy to prove the following theorem using the above formula and the fact that $\lim_{t\to\infty} \beta_k(t)/t = 0$ a.s. for $k = 1, 2, \ldots, N$.

Theorem 5 *The Lyapunov exponent of the system (14)-(16) almost surely exists as a limit, is non-random and the following formula holds:*

$$\lambda^{\mathrm{st}}(f,\omega) = \lambda^{\det}(f) - \frac{1}{2}\sigma^2 \sum_{k=1}^{N} b_k^2 = \lambda_{j_0} - \frac{1}{2}\sigma^2 \sum_{k=1}^{N} b_k^2 \quad a.s.$$

Therefore the top Lyapunov exponent is equal to $\lambda_1 - \frac{1}{2}\sigma^2 \sum_{k=1}^{N} b_k^2$.

Example Let us take

$$\mathcal{O} = (a,b), \quad \text{where} \quad a < b$$

and

$$A = \triangle + \alpha \mathrm{Id}, \quad \text{where } \alpha \text{ is a constant.}$$

Then the eigenvalues and the eigenfunctions of the operator A can be computed explicitely and are equal to, respectively:

$$\alpha_j = -\frac{j^2\pi^2}{(b-a)^2} + \alpha$$

and

$$e_j(x) = (\frac{2}{b-a})^{\frac{1}{2}} \sin(\frac{j\pi(x-a)}{(b-a)}),$$

$j = 1, 2, \ldots$, see [7] Example 1.2.3, p.10.

We deduce that the Lyapunov exponents of the deterministic system and the stochastic system are equal to, respectively:

$$\lambda^{\det}(f) = -\frac{j_0^2\pi^2}{(b-a)^2} + \alpha$$

and

$$\lambda^{\mathrm{st}}(f) = -\frac{j_0^2\pi^2}{(b-a)^2} + \alpha - \frac{1}{2}\sigma^2 \sum_{k=1}^{N} b_k^2.$$

We conclude that the stochastic system is stable if and only if its top Lyapunov exponent satisfies

$$\lambda = -\frac{\pi^2}{(b-a)^2} + \alpha - \frac{1}{2}\sigma^2 \sum_{k=1}^{N} b_k^2 < 0.$$

Ackonowledgements The author wishes to thank Professor M. Capiński for valuable guidance. The author is also indebted to E. Motyl and S. Peszat for helpful discussions.

References

[1] Arnold L.: Stochastic Differential Equations: Theory and Applications. (A Wiley - Interscience Publication) New York, Wiley (1974).

[2] Arnold L.: A new example of an unstable system being stabilized by random parameter noise. Inform. Comm. Math. Chem. 133-140 (1979).

[3] Arnold L., Crauel H. and Wihstutz V.: Stabilization of linear systems by noise. SIAM J. Control Optim. 21, 451-461 (1983).

[4] Arnold L. and Kloeden P.: Lyapunov exponents and rotation number of two-dimensional systems with telegraphic noise. SIAM J. Appl. Math. 49, 1242-1274 (1989).

[5] Caraballo T., Liu K. and Mao X.: On stabilization of partial differential equations by noise, manuscript (1999).

[6] Da Prato G. and Zabczyk J.: Stochastic Equations in Infinite Dimensions. (Encyclopedia of Mathematics and its Applications vol.44), Cambridge, Cambridge University Press (1992).

[7] Davies E. B.: Spectral Theory and Differential Operators. (Cambridge studies in advanced mathematics vol.42), Cambridge, Cambridge University Press (1996).

[8] Egorov Yu. V. and Shubin M. A.: Partial differential equations I. Foundations of the classical theory. (Encyclopaedia of Mathematical Sciences vol.30), Berlin, Springer-Verlag (1992).

[9] Kato T.: Perturbation Theory for Linear Operators (corrected printing of the second edition). (A Series of Comprehensive Studies in Mathematics vol. 132), Berlin Heildelberg New York, Springer-Verlag (1980).

[10] Kwiecińska A. A.: Stabilization of partial differential equations by noise. Stochastic Process. Appl. 79, 179-184 (1999).

[11] Ladyzhenskaya O. A.: The boundary value problems of mathematical physics (in Russian). Moscow, Nauka (1973). (English translation available – Springer-Verlag 1985).

[12] Leha G., Maslowski B. and Ritter G.: Stability of solutions to semilinear stochastic evolution equations. Stochastic Analysis and Applications, to appear.

[13] Liu K. and Truman A.: Lyapunov function approaches and asymptotic stability of stochastic evolution equations in Hilbert spaces - a survey of recent developments, manuscript.

[14] Pardoux E. and Wihstutz V.: Lyapunov exponents and rotation number of two-dimensional stochastic systems with small diffusion. SIAM J. Appl. Math. 48, 442-457 (1988).

[15] Pardoux E. and Wihstutz V.: Lyapunov exponents of linear stochastic systems with large diffusion term. Stochastic Process. Appl. 40, 289-308 (1992).

[16] Pazy A.: Semigroups of linear operators and applications to partial differential equations. (Applied Mathematical Sciences vol.44), New York Berlin, Springer-Verlag (1983).

[17] Temam R.: Infinite-dimensional dynamical systems in mechanics and physics (second edition). (Applied Mathematical Sciences vol.68) New York, Springer-Verlag (1997).

Institute of Mathematics, Jagiellonian University,
ul. Reymonta 4, 30-059 Kraków, Poland.
email: akwiecin@im.uj.edu.pl