On the structure of the Conley index

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Abstract. We present a recent improvement of the homotopy Conley index, the so-called Conley index over a base. We apply it to endow the homology and cohomology Conley indices with the structure of module over the cohomology ring of the phase space, and indicate how that structure provides information on continuation of cohomology classes of isolated invariant sets.

Introduction. Let ϕ be a flow on a space X and let S be an isolated invariant set for ϕ. The homotopy Conley index $h(\phi, S)$ is the homotopy type of the pointed space $(B/B^-, [B^-])$, where B is an arbitrary isolating block for S with the exit set B^-. The definition of the index was announced in the Charles Conley paper [C1], however proofs of its correctness and properties appeared later in [C2]. A proof of correctness of a derived invariant, the cohomology Conley index, was presented slightly earlier in [Ch]. An improvement of the homotopy Conley index was proposed in [MRS1] (see also [MRS2]). Let $\omega : X \to Z$ be a continuous map, where Z be a Hausdorff space. The Conley index of S over the base Z, denoted $h_\omega(\phi, S)$, is the fiberwise deforming homotopy type of the adjunction of B and Z via the map $\omega|_{B^-}$. If Z is a one-point space, the index coincides with $h(\phi, S)$. Theorem 1 asserts the correctness of this definition. Here we present a geometric idea of its proof; the details can be found in [MRS1]. We recollect also basic four properties of the index as well as results on comparison of indices over various bases.

Using the index over the base X we introduce the structure of left $H^*(X)$-module for the cohomology Conley index $CH^*(\phi, S)$; see Theorem 2. The construction is based on properties of the cup-product. We use the cap-product in order to introduce in Theorem 3 a similar structure of right $H^*(X)$-module for the homology Conley index $CH(\phi, S)$.

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In Corollary 1, we indicate how to apply the module structure of the cohomology index to the problem of preservation of nontrivial Alexander-Spanier cohomology classes of isolated invariant sets under continuation. The result is based on Proposition 2 which is a cohomology version of the Ważewski Theorem and seems to represent a separate interest. It asserts that if B is an isolating block for S and $u \cup \bar{H}^*(B, B^-) \neq 0$ for an Alexander-Spanier class $u \in \bar{H}^*(B)$ then $u|_S \neq 0$. There are not so much results on the existence of such nontrivial classes in S for dimensions ≥ 1; compare [Ch, P] and also [S, Problem 2].

We provide an example of lack of continuity between two repelling periodic orbits on the plane having two points removed. The example indicates the role of the module structure of the cohomology Conley index.

Notation. We fix some topological terminology and notation which will be useful in the sequel. In this section X denotes an arbitrary topological space. Let $A \subset X$. Assume that A is nonempty. Then X/A is defined as the quotient space obtained by identifying all points of A. The point in X/A obtained by collapsing of A is denoted by $[A]$. In order to define X/A in the case $A = \emptyset$ we fix a point $*$ outside of X, define $X/\emptyset := X \cup \{*\}$, and put $[\emptyset] := *$.

The pair of spaces (X, A), with A closed, is called *cofiberd* if $X \times 0 \cup A \times [0, 1]$ is a retract of $X \times [0, 1]$. It follows that the quotient map induces an isomorphism $H(X, A) \cong H(X/A, [A])$ for every (co-)homology functor H.

Let H^* be a cohomology functor. If $u \in H^*(X, A)$ and $i : (X', A') \hookrightarrow (X, A)$ is the inclusion, we write $u|_{(X', A')}$ instead of $H^*(i)u$ or i^*. In the sequel we will consider cohomology functors over a fixed ring-with-unit R with the cup-product structure, for example the singular cohomology over R or the Alexander-Spanier cohomology over R. The latter is denoted here by $\bar{H}^*$. The unit in $H^0(X)$ is denoted by 1 or 1_X.

Isolated invariant sets. In the sequel X denotes a metrizable locally compact space and $\phi : X \times \mathbb{R} \to X$ is a flow on X, i.e a continuous map satisfying $\phi(x, 0) = x$ and $\phi(\phi(x, s), t) = \phi(x, s + t)$ for every $x \in X$ and $s, t \in \mathbb{R}$. We write $\phi_t(x) := \phi(x, t)$. X is called the *phase space* for ϕ.

For a subset $W \subset X$ define

$$
\begin{aligned}
\operatorname{Inv} W &:= \{x \in W : \forall t \in \mathbb{R} : \phi_t(x) \in W\}, \\
W^+ &:= \{x \in W : \exists \{\epsilon_n\}, 0 < \epsilon_n \to 0 : \phi_{-\epsilon_n}(x) \notin W\}, \\
W^- &:= \{x \in W : \exists \{\epsilon_n\}, 0 < \epsilon_n \to 0 : \phi_{\epsilon_n}(x) \notin W\}.
\end{aligned}
$$

$\operatorname{Inv} W$ is thus the maximal invarant set contained in W; W^+ and W^- are called the *entry set* and the *exit set*, respectively. In this note we will state results for pairs (W, W^-) consisting of a given set W and its exit subset W^-. By the time reversibility of the flow, the same results hold for (W, W^+).

A compact set $S \subset X$ is called an *isolated invariant set* provided $S = \operatorname{Inv} U$ for some its neighborhood U. Such an U is called an *isolating neighborhood.* Let S and T be isolated invariant sets for the flows ϕ and ψ, respectively, on the phase space X. We call S and T *related by continuation* (and write $(\phi, S) \simeq (\psi, T)$) provided there exists a flow

Φ on $X \times [0,1]$, an isolated invariant set S^* for Φ, and flows ϕ^λ on X, $\lambda \in [0,1]$, such that $\Phi_t(x,\lambda) = (\phi_t^\lambda(x), \lambda)$, $\phi^0 = \phi$, $\phi^1 = \psi$, $S = \{x : (x,0) \in S^*\}$, and $T = \{x : (x,1) \in S^*\}$.

Let ϕ be a flow on X and let ψ be a flow on Y. We define the flow $\phi\#\psi$ on $X \times Y$ by $(\phi\#\psi)_t(x,y) := (\phi_t(x), \psi_t(y))$. If S and T are isolated invariant sets for ϕ and ψ, respectively, then $S \times T$ is an isolated invariant set for $\phi\#\psi$.

Isolating blocks. Let B be a compact subset of X. B is called an *isolating block* provieded $\operatorname{Inv} B$ is contained in the interior of B and both B^+ and B^- are compact. The latter condition is equivalent to the continuity of the maps

$$\begin{aligned} \sigma_B^+ :&\quad B \ni x \to \sup\{t \geq 0 : \phi(x,[-t,0]) \subset B\} \in [0,\infty], \\ \sigma_B^- :&\quad B \ni x \to \sup\{t \geq 0 : \phi(x,[0,t]) \subset B\} \in [0,\infty] \end{aligned}$$

by the Ważewski Theorem. Let $S := \operatorname{Inv} B$. It follows that S is an isolated invariant set and B is its isolating neighborhood. Obviously,

$$B^\pm = (\sigma_B^\pm)^{-1}(0), \quad S = (\sigma_B^+)^{-1}(\infty) \cap (\sigma_B^-)^{-1}(\infty).$$

For an isolating block B we define

$$A := \{x \in B : \sigma_B^+(x) + \sigma_B^-(x) = \infty\}.$$

It follows that $A^\pm = A \cap B^\pm$. By the tautness of $\bar{H}^*$ and the continuity of $\sigma_B^\pm$ it is easy

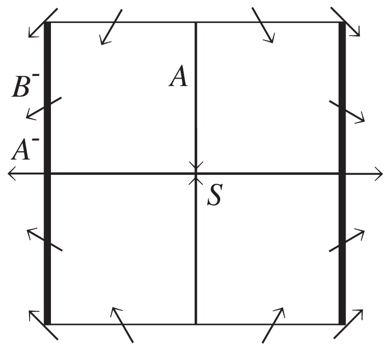

Figure 1: An isolating block B.

to prove

Proposition 1 ([Ch, Lem. 4.3, Prop. 4.6]) *The inclusions $(A, A^-) \hookrightarrow (B, B^-)$ and $S \hookrightarrow A$ induce isomorphisms*

$$\bar{H}^*(B, B^-) \cong \bar{H}^*(A, A^-), \quad \bar{H}^*(A) \cong \bar{H}^*(S).$$

As a consequence we obtain

Proposition 2 *Let $u \in \bar{H}^*(B)$. Assume, that there exists an $v \in \bar{H}^*(B, B^-)$ such that $u \cup v \neq 0$. Then $u|_S \neq 0$.*

Proof: It follows by the naturality of the cup-product that

$$u|_A \cup v|_{(A,A^-)} = (u \cup v)|_{(A,A^-)} \neq 0$$

because $(A, A^-) \hookrightarrow (B, B^-)$ induces an isomorphism. Thus $u|_A \neq 0$. Since $S \hookrightarrow A$ induces an isomorphism, we get

$$u|_S = (u|_A)|_S \neq 0. \qquad \square$$

The above result can be regard as a generalization of a weaker version of the Ważewski Theorem: if $\bar{H}^*(B, B^-) \neq 0$ then $S \neq \emptyset$. Indeed, $1_B \cup \bar{H}^*(B, B^-) \neq 0$, hence $1_S \in \bar{H}^0(S)$ is nonzero and thus S cannot be empty.

Proposition 3 ([R, Th. I.3.7]) (B, B^-) *is a cofiberd pair.*

The classical Conley index. The crucial role in the theory of isolated invariant sets plays the following result: for every neighborhood U of an isolated invariant set S there exists an isolating block B such that $B \subset U$ and $S = \operatorname{Inv} B$. There is another main result of the theory: for every two isolating blocks B_1 and B_2 such that $\operatorname{Inv} B_1 = \operatorname{Inv} B_2$ the homotopy types of the pointed spaces $(B_1/B_1^-, [B_1^-])$ and $(B_2/B_2^-, [B_2^-])$ coincide. Those results (proved in [Ch, C2]) lead to the definition of the *Conley index* $h(\phi, S)$: it is the homotopy type of $(B/B^-, [B^-])$ for a block B such that $S = \operatorname{Inv} B$. The properties of the Conley index will be listed later, after presenting a more accurate invariant, the Conley index over a base.

It follows by Proposition 3 that $H(B, B^-)$ is an invariant of the set S, where H denotes an arbitrary (co-)homology theory. We call it the *H-Conley index* and denote $CH(\phi, S)$. In the forthcoming sections we will show that usually $CH(\phi, S)$ can be endowed with a richer structure then just the inner structure of $H(B, B^-)$.

Fiberwise pointed spaces. Let Z be a fixed topological space. A triple (U, r, s) (or shortly: just U) is called a *fiberwise pointed space over* Z if U is a topological space, $r : U \to Z$ and $s : Z \to U$ are continuous maps, and $r \circ s = \mathrm{id}_Z$ (compare [J, p. 41–42]). Z is called the *base space*, to r as the *projection*, and to s as the *section* of U. It follows that $s(Z)$ is a retract (hence a closed subset) of U, $s \circ r : U \to s(Z)$ is a retraction, and s and the restriction of r are mutually inverse homeomorphisms between Z and $s(Z)$. If $*$ is a fixed one-point space then every pointed topological space can be regarded, in an obvious way, as a fiberwise pointed space over $*$.

Let (U', r', s') be another fiberwise pointed space over Z. We define $(U, r, s) \vee (U', r', s')$, the *fiberwise pointed wedge*, as the triple $(U \vee U', r_\vee, s_\vee)$ such that $U \vee U'$ is the disjoint union of U and U' with $s(z)$ and $s'(z)$ identified, i.e. $U \vee U' := U \times 0 \cup U' \times 1/ \sim_\vee$, where $\sim_\vee$ denotes the equivalence relation generated by $(s(z), 0) \sim_\vee (s'(z), 1)$ for any $z \in Z$. The maps $r_\vee, s_\vee$ are given by

$$r_\vee([(x,i)]) := \begin{cases} r(x) & \text{if } x \in U, \\ r'(x) & \text{if } x \in U'. \end{cases}$$

and $s_\vee(z) := [(s(z),0)] = [(s'(z),0)]$. It is easy to check that $U \vee U'$ is a fiberwise pointed space over Z and that this agrees with the usual definition of wedge of pointed spaces in the case Z is equal to $*$.

Let (U,r,s) and (U',r',s') be two fiberwise pointed spaces over Z and Z', respectively. In the cartesian product $U \times U'$ define an equivalence relation $\sim_\wedge$ by $(u,u') \sim_\wedge (v,v')$ if $(u,u') = (v,v')$ or $u = v \in s(Z)$, $r'(u') = r'(v')$ or $r(u) = r(v)$, $u' = v' \in s'(Z')$. Put

$$U \wedge U' := U \times U' / \sim_\wedge$$

and denote by $u{\wedge}u'$ the equivalence class of (u,u'). Define maps $r_\wedge : U \wedge U' \to Z \times Z'$ and $s_\wedge : Z \times Z' \to U \wedge U'$ by

$$r_\wedge(u{\wedge}u') := (r(u), r'(u')), \quad s_\wedge(z,z') := s(z){\wedge}s'(z').$$

The triple $(U \wedge U', r_\wedge, s_\wedge)$ is a fiberwise pointed space over $Z \times Z'$ and is called the *fiberwise pointed smash product* of U and U'. Again, if both Z and Z' are equal to the one-pointed space $*$ than this notion agrees with the usual smash product.

Fiberwise deforming homotopy type. Let (U,r,s) and (U',r',s') be two fiberwise pointed spaces over Z. They have the same *fiberwise deforming homotopy type over* Z if there exist continuous maps $f : U \to U'$ and $f' : U' \to U$ satisfying

$$f \circ s = s', \quad f' \circ s' = s, \tag{1}$$

$$r' \circ f \simeq r \ \mathrm{rel}\, s(Z), \quad r \circ f' \simeq r' \ \mathrm{rel}\, s'(Z), \tag{2}$$

$$f' \circ f \simeq \mathrm{id}_U \ \mathrm{rel}\, s(Z), \quad f \circ f' \simeq \mathrm{id}_{U'} \ \mathrm{rel}\, s'(Z). \tag{3}$$

The above definition was introduced in [MRS1]. (The book [J] deals with a more natural concept of *fiberwise pointed homotopy type*, where it is assumed additionally that $r' \circ f = r$ and $r \circ f' = r'$. For our purposes related to the theory of isolated invariant sets, such a notion is too restrictive.) The class of spaces having the same fiberwise deforming homotopy type over Z as (U,r,s) is denoted by $[U,r,s]_Z$ (or shortly: $[U]_Z$).

On can define

$$[U,r,s]_Z \vee [U',r',s']_Z := [(U,r,s) \vee (U',r',s')]_Z,$$

the wedge of fiberwise deforming homotopy types. The type

$$\overline{0}_Z := [Z, \mathrm{id}_Z, \mathrm{id}_Z]_Z,$$

called the *trivial* fiberwise deforming homotopy type over Z, is the neutral element for the wedge:

$$[U]_Z \vee \overline{0}_Z = \overline{0}_Z \vee [U]_Z = [U]_Z.$$

On the other hand, one cannot provide a general definition of the smash product of fiberwise deforming homotopy types. Such a product can be defined under some restriction on considered spaces. We call (U,r,s) a *fiberwise well-pointed space over* Z provided it is fiberwise pointed and $(U, s(Z))$ is a cofiberd pair (or equivalently: s is a cofibration). For fiberwise well-pointed spaces (U, r, s) over Z and (U', r', s') over Z' one can define

$$[U, r, s]_Z \wedge [U', r', s']_{Z'} := [(U, r, s) \wedge (U', r', s')]_{Z \times Z'}.$$

The Conley index over a base. As before, let X be the phase space of a flow ϕ. Assume, that Z is a Hausdorff space. Let $\omega : X \to Z$ be a fixed continuous map. For an isolating block B let $U_\omega(B)$ denote the adjunction $B \cup_{\omega|_{B^-}} Z$, i.e.

$$U_\omega(B) := Z \times 0 \cup B \times 1 / \sim,$$

where $\sim$ denotes the minimal equivalence relation such that $(x, 1) \sim (\omega(x), 0)$ for every $x \in B^-$. Let $[y, i]$ denote the equivalence class of an element $(y, i) \in Z \times 0 \cup B \times 1$ with respect to $\sim$. Define $r : U_\omega(B) \to Z$ and $s : Z \to U_\omega(B)$ by

$$r([x, 1]) := \omega(x), \quad r([z, 0]) := z$$
$$s(z) := [z, 0].$$

It is easy to verify that the maps r, s are correctly defined and continuous, and Proposition 3 implies that the triple $(U_\omega(B), r, s)$ is a fiberwise well-pointed space over Z.

Let S be an isolated invariant set. The following result enables an improvement of the Conley index:

Theorem 1 ([MRS1, Th. 2.1]) *The fiberwise deforming homotopy type of $U_\omega(B)$ does not depend on the choice of an isolating block B such that $S = \operatorname{Inv} B$.*

A proof of the above result can be found in [MRS1]. We explain its basic idea on a simple example. Let B_1 and B_2 be isolating blocks for S. Without loss of generality on can assume that $B_1 \subset B_2$. One should find maps f and f' between $U_\omega(B_1)$ and $U_\omega(B_2)$ satisfying (1), (2), and (3). In our example, let $X := \mathbb{R}$, $\omega = \mathrm{id}_X$, the flow ϕ is generated by the equation $\dot{x} = x$, hence $S := \{0\}$, $B_1 := [-1, 1]$, and $B_2 := [-2, 2]$. We have $B_1^- = \{-1, 1\}$ and $B_2^- = \{-2, 2\}$, hence both the spaces $U_\omega(B_1)$ and $U_\omega(B_2)$ are homeomorphic to a line with a closed segment attached to the line at its two ends. In Figure 2, we present them simultaneously, together with a schematic view of possible maps f and f' (both the maps restricted to the line are identities). In the proof of the general case similar maps act along orbits of ϕ.

As an immediate consequence of Theorem 1 we get correctness of the definition of the *Conley index over Z* (with respect to the map ω) $h_\omega(\phi, S)$:

$$h_\omega(\phi, S) := [U_\omega(B)]_Z$$

for any isolating block B such that $S = \operatorname{Inv} B$. In particular, if Z is a one-point space $*$ (hence $\omega : X \to *$ is unique) then $h_\omega(\phi, S)$ coincides with the classical Conley index $h(\phi, S)$. Actually, the index over a base preserves all the basic properties of the classical one, compare [MRS1, Th. 2.2–5]. We list them below (we assume that Z is Hausdorff and $\omega : X \to Z$ is continuous.)

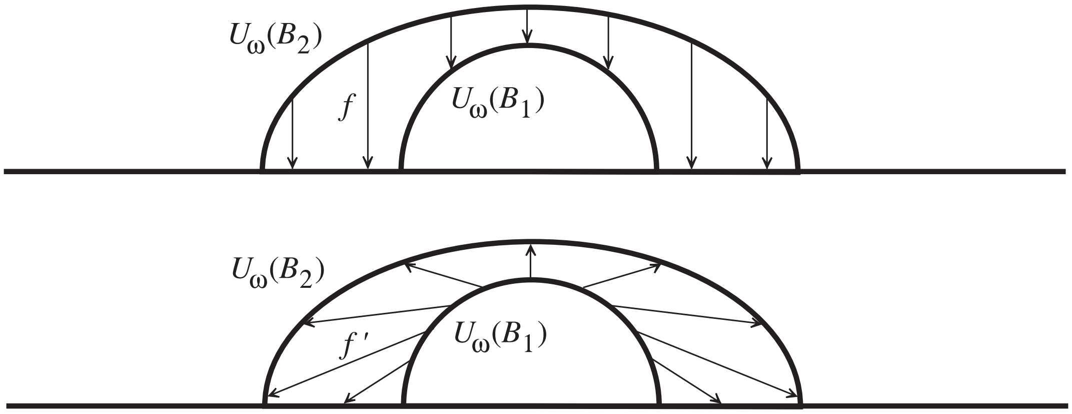

Figure 2: The maps $f : U_\omega(B_2) \to U_\omega(B_1)$ and $f' : U_\omega(B_1) \to U_\omega(B_2)$.

(I) $h_\omega(\phi, \emptyset) = \overline{0}_Z$.

(II) If S and S' are disjoint isolated invariant sets for ϕ then

$$h_\omega(\phi, S \cup S') = h_\omega(\phi, S) \vee h_\omega(\phi, S').$$

(III) Let S and S' be isolated invariant sets; S for a flow ϕ on X and S' for ϕ' on X'. Let $\omega' : X' \to Z'$ be continuous and let Z' be Hausdorff. Then

$$h_{\omega \times \omega'}(\phi \# \phi', S \times S') = h_\omega(\phi, S) \wedge h_{\omega'}(\phi', S').$$

(IV) If $(\phi, S) \simeq (\phi', S')$ then $h_\omega(\phi, S) = h_\omega(\phi', S')$.

An important question is to compare indices over different bases. The following results give an answer (compare [MRS1, Th. 7.2,3]):

(A) If $\omega_0, \omega_1 : X \to Z$ are continuous and $\omega_0|_N \simeq \omega_1|_N : N \to Z$ for some neighborhood N of S then $h_{\omega_0}(\phi, S) = h_{\omega_1}(\phi, S)$.

(B) If $\zeta : Z \to Z'$ is continuous, Z' is Hausdorff, and $h_\omega(\phi, S) = h_\omega(\phi, S')$ then $h_{\zeta \circ \omega}(\phi, S) = h_{\zeta \circ \omega}(\phi, S')$.

It follows immediately, that the index over a base provides maximum information in the case $Z = X$ and $\omega = \mathrm{id}_X$ or, more generally, if $\omega : X \to Z$ is a homotopical equivalence. On the other hand, the minimum information is given by the classical index, i.e. in the case Z is one-point. If X is contractible then the index over any base provides no more information then the classical index. If S has a neighborhood which is contractible in X to x_0 and $z_0 := \omega(x_0)$ then $h_\omega(\phi, S)$ is equal to $[(B/B^-, [B^-]) \vee (Z, z_0), r, s]_Z$, where B is an isolating block for S, $\vee$ denotes the wedge of pointed spaces, r restricted to B/B^- is the constant map at z_0, and s is the inclusion.

The module structure of the homology and cohomology Conley indices. We consider an isolated invariant set S for a flow ϕ on X. By remarks in the previous section we restrict to the case $\omega = \mathrm{id}_X$. Let H be a homology or cohomology functor and let B be an isolating block such that $S = \mathrm{Inv}\, B$. Since (B, B^-) is a cofiberd pair, the inclusion $(B, B^-) \hookrightarrow (U_{\mathrm{id}}(B), s(X))$ induces an isomorphism by the excision theorem, hence $CH(\phi, S) \cong H(U_{\mathrm{id}}(B), s(X))$. Since $s \circ r$ is a retraction,

$$H(U_{\mathrm{id}}(B)) \cong H(X) \oplus CH(\phi, S),$$

which means that $H(U_{\mathrm{id}}(B))$ and $H(U_{\mathrm{id}}(B), s(X))$, considered separately, provide no more information in comparison to the H-Conley index.

Assume now that H^* is equal to the singular cohomologies or Alexander-Spanier cohomologies. In this case the cup-product provides on $H^*(U_{\mathrm{id}}(B), s(X))$ the structure of a left module over $H^*(U_{\mathrm{id}}(B))$. Using the Conley index over X we will show that this structure generates the structure of a module over $H^*(X)$ on $CH^*(\phi, S)$, which is invariant under continuation.

Let $u \in H^*(X)$ and let $v \in H^*(U_{\mathrm{id}}(B), s(X))$. We define

$$u \cdot v := r^*(u) \cup v \in H^*(U_{\mathrm{id}}(B), s(X)).$$

It follows by properties of the cup-product that $1 \cdot v = v$ and $(u \cup u') \cdot v = u \cdot (u' \cdot v)$, hence we have the required structure of left module. We should show that this structure does not depend on the choice of an isolating block B such that $\mathrm{Inv}\, B$ is related to S by continuation.

Let ϕ' be another flow on X and let $(\phi, S) \simeq (\phi', S')$. By the property (IV),

$$[U_{\mathrm{id}}(B), r, s]_X = [U_{\mathrm{id}}(B'), r', s']_X,$$

where B and B' are isolating blocks for, respectively, S and S', and r, s, r', s' are the canonical maps providing the structure of fiberwise pointed space. It follows that there are mutually inverse homotopic equivalences

$$f : (U_{\mathrm{id}}(B), s(X)) \to (U_{\mathrm{id}}(B'), s'(X)), \quad f' : (U_{\mathrm{id}}(B'), s'(X)) \to (U_{\mathrm{id}}(B), s(X))$$

such that $f^* \circ r'^* = r^*$ and $f'^* \circ r^* = r'^*$. If $u \in H^*(X)$ and $v \in H^*(U_{\mathrm{id}}(B'), s'(X))$ then

$$f^*(u \cdot v) = f^*(r'^*(u) \cup v) = r^*(u) \cup f^*(v) = u \cdot f^*(v).$$

Thus f^* and f'^* are isomorphisms of $H^*(X)$-modules. As a conclusion we receive the following

Theorem 2 *The external mutliplication $\cdot$ provides the H^*-Conley index $CH^*(\phi, S)$ with the structure of left module over $H^*(X)$, which is invariant under continuation.* $\square$

As a consequence of Proposition 2 we obtain

Corollary 1 *If $u \in \bar{H}^*(X)$ is such that $u \cdot C\bar{H}^*(\phi, S) \neq 0$ then $u|_{S'} \neq 0$ for every isolated invariant set S' which is related to S by continuation.* $\square$

As a particular case we obtain a well known consequence of the continuation property of the index: if $C\bar{H}^*(\phi, S) \neq 0$ then $S' \neq \emptyset$ for every S' related to S by continuation. Indeed, $1 \cdot C\bar{H}^*(\phi, S)$ is also nonzero, hence $1_{S'} \neq 0$ and thus $S' \neq \emptyset$.

Actually, one can define a module structure also to some homology Conley indices. Let H denote now the corresponding homologies to H^* (i.e. singular or, respectively, homologies with compact supports based on Alexander-Spanier cochains as in [M]). For $u \in H^*(X)$ and $\xi \in H(U_{\mathrm{id}}(B), s(X))$ define

$$\xi \cdot u := \xi \cap r^*(u) \in H(U_{\mathrm{id}}(B), s(X)).$$

Properties of the cap-product imply that $\xi \cdot 1 = \xi$ and $\xi \cdot (u \cup u') = (\xi \cdot u) \cdot u'$, hence $\cdot$ generates the structure of right $H^*(X)$-module on $H(U_{\mathrm{id}}(B), s(X))$. By a similar argument as above one can prove

Theorem 3 *The external multiplication $\cdot$ provides the H-Conley index $CH(\phi, S)$ with the structure of right module over $H^*(X)$, which is invariant under continuation.* $\square$

Examples. In the following example we consider the flow ϕ on $\mathbb{R}^2 \setminus 0$ genarated by the equation $\dot{\theta} = 1$, $\dot{r} = r - 1$ in the polar coordinates. The set $S = \{(\theta, r) : r = 1\}$ is a repelling periodic orbit and the annulus B on Figure 3 is an isolating block for S; B^- is equal to the boundary of B. Let $\mathbb{Z}$ be the ring of coefficients of cohomologies. We have $H^1(\mathbb{R}^2 \setminus 0) \cong H^1(B, B^-) \cong \mathbb{Z}$. We interpret geometrically cohomology classes by supports of its representing cocycles. In Figure 3, the support of a cocycle of a class u

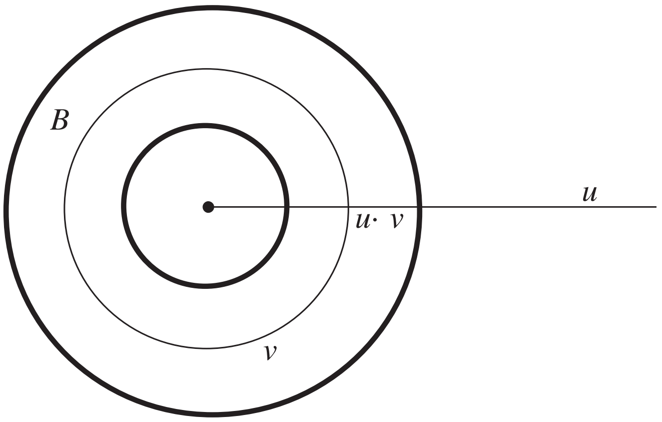

Figure 3: The supports of cocycles representing the classes u, v, and $u \cdot v$.

generating $H^1(\mathbb{R}^2 \setminus 0)$ is equal to the half-line starting at the origin and the support of a cocycle of a generator $v \in H^1(B, B^-)$ is equal to the central circle in B. Their product $u \cdot v \in H^2(B, B^-)$ is nontrivial (actually, it is also a generator) and the support of its cocycle is equal to the point of intersection of the support representing u and v.

Let ξ be a generator of $H_2(B, B^-) \cong \mathbb{Z}$. The support of its representing cycle is equal to the whole set B. It follows that the support of a cycle representing $\xi \cdot u \in H_1(B, B^-)$ is equal to the segment on the half-line u joining both the circles which form B^-. The class $\xi \cdot u$ generates $H_1(B, B^-)$, hence it is also nonzero.

Let us consider flows ϕ, ψ on $\mathbb{R}^2 \setminus \{p, q\}$, where $p \neq q$. Assume that S is a repelling periodic orbit for ϕ and T is a repelling periodic orbit for ψ as shown in Figure 4. Let

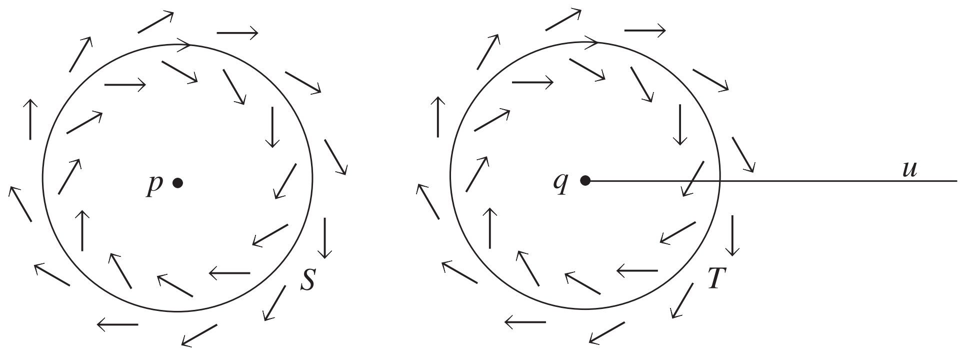

Figure 4: The repelling periodic orbits S and T.

$u \in H^1(\mathbb{R}^2 \setminus \{p, q\})$ be the cohomology class represented by the half-line in the figure. It follows by the previous example that $u \cdot CH^*(\phi, S) = 0$ and $u \cdot CH^*(\psi, T) \neq 0$, which means that (ϕ, S) and (ψ, T) cannot be related by continuation.

In the above example the classical Conley indices of S and T are obviously the same. Moreover, one can choose isolating blocks B_S and B_T for S and T, respectively, such that $U_{\mathrm{id}}(B_S)$ is homeomorphic to $U_{\mathrm{id}}(B_T)$, hence we have taken advantage of their structure of fiberwise pointed space in order to distinguish them, and, in consequence, to prove the required lack of continuation.

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