On the index of contact and multiplicities for bigraded rings

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IMUJ PREPRINT **1999/26** KRAKÓW

Abstract

We compute the index of contact $p(Z, S)(c)$ of a set Z with a submanifold S at a point $c \in \operatorname{reg}(Z \cap S)$ at which Z is normal pseudo-flat along $Z \cap S$ (Thm 4.1, Cor. 4.4).

1 Introduction

In [5] so-called index of contact of a set Z with a submanifold S is applied to estimate the Łojasiewicz exponent of polynomial mappings and study a regular separation condition for projective algebraic sets (see also [10]). The main aim of this note is to prove that the index of contact and the index of intersection (introduced in [14]) coincide along a Zariski open subset of each component of $Z \cap S$. The latter invariant is the Samuel multiplicity of the associated graded ring; hence, it can be computed by various computer algebra systems (see [3], [2] for details).

In analytic intersection theory, the intersection multiplicity along a proper component is defined in two steps. Having defined the intersection multiplicity at an isolated point ([6], Def. 3.4) Draper passes to the case of (proper) component of dimension greater than zero by cutting by a transversal hyperplane of complementary dimension at a regular point ([6], Def. 4.5). In this note, we prove that a similar procedure can be applied to compute the index of contact at a point $c \in \operatorname{reg}(Z \cap S)$ at which Z is normal pseudo-flat along $Z \cap S$ (Thm 4.1(d), (e)). We compute the extended index of intersection of Z and S ([14], Def. 4.2) at such points and prove (Cor. 4.4, Cor. 4.5) that all the components of the complement of the normal pseudo-flateness locus (see [1] for details) appear as embedded components with positive multiplicities in so-called intersection product ([14], Def. 6.3).

[6], Def. 3.4 is generalized in [4] where the isolated intersection multiplicity of locally analytic sets X, Y at a point a is defined by intersecting some extensions of the diagonal $\Delta \subset N^2$ with the set $X \times Y$. We prove here that the index of contact can be computed in a similar way (Thm 4.1(b), (c)). We give also a precise description of the family of the extensions of S that can be applied to compute $p(Z,S)(c)$ in this way (Cor. 4.2).

2 Preliminaries

Let N be an n-dimensional (complex) manifold and let S be its s-dimensional closed submanifold. For an analytic cycle $A = \sum_{j \in J} \alpha_j C_j$ in V, where V is an open subset of N, we define the *part of A supported by S* as $A^S := \sum_{j \in J, C_j \subset S} \alpha_j C_j$. The multiplicity of the cycle A at the point $a \in V$ is defined as $\nu(A,a) := \sum_{j \in J} \alpha_j \nu(C_j, a)$, where $\nu(C_j, a)$ denotes the multiplicity of the analytic set C_j at the point a. Its *support* $\bigcup_{j \in J} C_j$ is denoted by $|A|$.

We fix a purely k-dimensional analytic set $Z \subset N$ and a point $c \in Z \cap S$. We put $d := p - (k + s - n)$, where $p := \dim_c(Z \cap S)$.

$\mathcal{F}_c^1(Z,S)$ denotes the family of locally analytic sets $R \subset N$ of pure dimension $n + p - k$ which meet Z properly and satisfy the condition $S_c \subset R_c$. Let $\{X_j\}_{j=1}^{j=q}$ be the family of all p-dimensional components of $Z \cap S$. One can see that for every $R \in \mathcal{F}_c^1(Z,S)$ the following equality holds

$$\nu((Z \cdot R)^S, c) = \sum_{j=1}^{j=q} \nu(X_j, c) \cdot i(Z, R; X_j), \tag{1}$$

where $i(Z, R; X_j)$ is the multiplicity of intersection of Z and R along X_j.

We introduce the following notation:

$$\begin{array}{lcl} \mathcal{F}_c^2(Z,S) & := & \{R \in \mathcal{F}_c^1(U,Z) : Z_c \cap R_c = (Z \cap S)_c\}, \\ \mu_1(Z,S)(c) & := & \inf\{\nu((Z \cdot R)^S, c) : R \in \mathcal{F}_c^1(Z,S)\}, \\ \mu_2(Z,S)(c) & := & \inf\{\nu(Z \cdot R, c) : R \in \mathcal{F}_c^2(Z,S)\}. \end{array}$$

If $c \in \mathrm{reg}(Z \cap S)$, we define the family $\mathcal{T}_c(Z,S)$ as the set of all $(n-p)$-dimensional submanifolds $P \subset N$ that are transversal to $Z \cap S$ at the point c, and meet Z properly (by Rückert's lemma $\mathcal{T}_c(Z,S) \neq \emptyset$). Every $P \in \mathcal{T}_c(Z,S)$ determines the cycle $P \cdot Z$ in a neighbourhood of c, where " $\cdot$ " denotes the intersection product ([6], p. 190). Since c is an isolated point of $Z \cap P \cap S$, the multiplicity of isolated intersection $i((Z \cdot P), S; c)$ ([4], Def. 5.1) and the relative tangent cone $\mathrm{C}_c(Z,S)$ (see [4], p. 22) are well-defined.

Let $\mathbf{G}_d(\mathbf{C}^{n-s})$ denote the Grassmann manifold of d-dimensional subspaces of $\mathbf{C}^{n-s}$ and let $E \subset \mathbf{C}$ stand for the unit ball.

In section 4 we will need the following lemmas:

Lemma 2.1 *Let $N = E^n$, $S = E^s \times \{0\}^{n-s}$, $a \in \mathrm{reg}_p(Z \cap S)$, $P \in \mathcal{T}(Z,S)$. Then the set of $L \in \mathbf{G}_d(\mathbf{C}^{n-s})$ which satisfy the condition*

$$(\mathbf{C}^s \times L) \cap C_a(Z \cap P, S) = \mathbf{C}^s \times \{0\}^{n-s} \tag{2}$$

is an open and dense subset of $\mathbf{G}_d(\mathbf{C}^{n-s})$.

Proof: We can assume that $d > 0$. Let $\pi : \mathbf{C}^s \oplus \mathbf{C}^{n-s} \mapsto \mathbf{C}^{n-s}$ stand for the projection. By [4], Lemma 2.4 the equality $\mathrm{C}_a(Z \cap P, S) = \mathbf{C}^s \times \mathrm{C}_0(\pi(Z \cap P \cap U))$ holds, where U is a suficiently small neighbourhood of a and $\mathrm{C}_0(\,\cdot\,)$ denotes the tangent cone at the point 0. $\square$

Lemma 2.2 *Let $c \in \mathrm{reg}(Z \cap S)$, $P \in \mathcal{T}_c(Z,S)$. If $R \in \mathcal{F}_c^2(Z,S)$, the germ R_c is smooth and satisfies the condition $C_c(P \cap Z, S) \cap T_cR = T_cS$, then the following equality holds*

$$\nu(Z \cdot R, c) = \mu_2(Z,S)(c) = i((P \cdot Z), S; c).$$

Proof: Fix $\tilde{R} \in \mathcal{F}_c^2(Z,S)$ and observe that the sets $\tilde{R}$, P, Z meet properly at the point c. Since P, Z and $\tilde{R}$, Z also meet properly, we have

$$i((P \cdot Z), \tilde{R}; c) = i(P, (Z \cdot \tilde{R}); c) = \nu(Z \cdot \tilde{R}, c).$$

Let $P \cdot Z = \sum_{j \in J} \alpha_j Y_j$. By [4], Prop. 5.3 the following holds

$$\nu(Z \cdot \tilde{R}, c) = \sum_{j \in J} \alpha_j \cdot i(Y_j, \tilde{R}; c) \geq \sum_{j \in J} \alpha_j \cdot i(Y_j, S; c) = i((P \cdot Z), S; c), \tag{3}$$

which yields $\mu_2(Z,S)(c) \geq i((P \cdot Z), S; c)$. Apply [4], Prop. 2.6 and Thm 4.4 to get the equality $\nu(Z \cdot R, c) = i((P \cdot Z), S; c)$. $\square$

3 Index of intersection and index of contact

In this section we outline the construction of the index of intersection and the index of contact (for more details see [14], [5]). We maintain the notation and the assumptions of the previous section.

Let U be an open subset of N.
$\mathcal{B}(U) := \{H \subset U : H$ is a closed hypersurface in U, $H \supset (S \cap U)\}$.
$\mathcal{E}(U, Z)$ denotes the family of systems $(R, H_1, \ldots, H_d)$, where R is a locally analytic set and each $H_i \in \mathcal{B}(U)$, which satisfy the condition:

$$(\mathrm{r}_q(U,R)) \qquad \dim[((U \setminus S) \cap (Z \cap R)) \cap H_1 \cap \ldots \cap H_q] \leq \max\{p - q, -1\}$$

for $q = 1, \ldots, d$.

The family $\mathcal{H}(U, Z)$ is defined as the set of $\mathcal{H} = (H_1, \ldots, H_{n-s})$, where each $H_i \in \mathcal{B}(U)$, such that for every $a \in U \cap S$ the equality $\bigcap_{j=1}^{n-s} T_a H_j = T_a S$ holds and $\mathcal{H}$ satisfies $(\mathrm{r}_q(U, Z))$ for $q = 1, \ldots, n-s$ (see [14], p. 184).
For every $\mathcal{H} \in \mathcal{H}(U, Z)$ one can define inductively an analytic cycle $Z \cdot \mathcal{H} = Z_0^S + \cdots + Z_{n-s}^S$ in $S \cap U$ ([14], Algorithm 4.1):

$$\begin{aligned} Z_0 &:= Z \cap U, & Z_0^S &:= \text{part of } Z_0 \text{ supported by } S, \\ Z_j &:= (Z_{j-1} - Z_{j-1}^S) \cdot H_j, & Z_j^S &:= \text{part of } Z_j \text{ supported by } S, \end{aligned}$$

where $j = 1, \ldots, n-s$ and "$\cdot$" denotes the intersection product ([6], p. 190). The cycle $Z \cdot \mathcal{H}$ has a unique decomposition $Z \cdot \mathcal{H} = T_{(s)} + T_{(s-1)} + \cdots + T_{(0)}$, where $T_{(j)}$ is a j-cycle (i.e. a formal combination of analytic sets of pure dimension j). One defines (see [14], [5])

$$\begin{array}{rcl} p(Z,S)(c) & := & \min\{\nu(Z \cdot \mathcal{H}, c) \mid \mathcal{H} \in \mathcal{H}(V, Z), V \ni c\}, \\ \tilde{g}(Z,S)(c) & := & \min_{\mathrm{lex}}\{\tilde{\nu}(Z \cdot \mathcal{H}, c) \mid \mathcal{H} \in \mathcal{H}(V, Z), V \ni c\}, \\ g(Z,S)(c) & := & \text{the sum of the coordinates of } \tilde{g}(Z,S)(c), \end{array}$$

where "lex" denotes the lexicographical ordering in $\mathbf{N}^{s+1}$ and $\tilde{\nu}(Z \cdot \mathcal{H}, c) := (\nu(T_{(s)}, c), \ldots, \nu(T_{(0)}, c))$.
The integer $p(Z,S)(c)$ (resp. $\tilde{g}(Z,S)(c)$, $g(Z,S)(c)$) is called *the index of contact* (resp. *the extended index of intersection*, *the index of intersection*) of Z with the submanifold S at the point c.

Observe that if $R \in \mathcal{F}_c^1(Z, S)$ and $(R, H_1, \ldots, H_d) \in \mathcal{E}(U, Z)$, we can apply the first d steps of [14], Algorithm 4.1 to the cycle $R \cdot Z$, the submanifold

S and the system $(H_1, \dots, H_d)$. The result of this procedure is denoted by $Z \cdot (R, H_1, \dots, H_d)$.

For a system $(Y_1, \dots, Y_r)$ of subsets of N we define $(Y_1, \dots, Y_r)|_U := (Y_1 \cap U, \dots, Y_r \cap U)$.

In the section 4 we will need the following lemma:

Lemma 3.1 *Let $N = E^n$, $S = E^s \times \{0\}^{n-s}$, $c = 0$, $\dim_0(Z \cap S) = p$. Fix a system $(H_1, \dots, H_d)$ of elements of $\mathcal{B}(U)$, where $U \subset N$ is a neighbourhood of the point 0. Then the set of $L \in \mathbf{G}_d(\mathbf{C}^{n-s})$ which satisfy the condition*

$(\star)$ *there exists a neighbourhood $\tilde{U}$ of 0 such that $(E^s \times L, H_1, \dots, H_d)|_{\tilde{U}}$ belongs to $\mathcal{E}(\tilde{U}, Z)$ and $\tilde{\nu}(Z \cdot (E^s \times L, H_1, \dots, H_d)|_{\tilde{U}}, 0) = \tilde{g}(Z, S)(0)$,*

is an open subset of $\mathbf{G}_d(\mathbf{C}^{n-s})$.

Proof: Since the topology of $\mathbf{G}_d(\mathbf{C}^{n-s})$ coincides with the topology induced by the weak topology of currents, it suffices to prove that the set defined by $(\star)$ is open in the latter.

Suppose that there exists a sequence $\{L^j\}_{j=0}^{\infty} \subset \mathbf{G}_d(\mathbf{C}^{m-s})$ such that $L^j \to L^0$ and L^j satisfies $(\star)$ only for $j = 0$. We can assume that $E^s \times L^j$ and Z meet properly (see e.g. [3], proof of Cor. 3).

Let $Z_0(j, N)$ denote the cycle $(E^s \times L^j) \cdot Z$. If $\tilde{U}$ is a neighbourhood of the point 0 such that $(H_1, \dots, H_d)|_{\tilde{U}}$ satisfies $(\mathrm{r}_q(\tilde{U}, E^s \times L^j))$ for $q = 1, \dots, l$, then $Z_l(j, \tilde{U})$, $Z_l^S(j, \tilde{U})$ stand for the cycles obtained in the lth step of [14], Algorithm 4.1 applied to this system, the cycle $Z_0(j, N)$ and S.

According to [14], Thm 3.6 the cycles $Z_0(j, N)$ converge to $Z_0(0, N)$. By [14], Thm 3.4 there exists a neighbourhood $\tilde{U}_0$ of the point 0 such, that

$$(Z_0(j, N) - Z_0^S(j, N)) \cap \tilde{U}_0 \to (Z_0(0, N) - Z_0^S(0, N)) \cap \tilde{U}_0$$

and $\nu(Z_0^S(j, N), 0) = \nu(Z_0^S(0, N), 0)$ for almost all j.
Since $\nu((Z_0(0, N) - Z_0^S(0, N))^{H_1}, 0) = 0$ we apply [14], Thm 3.4 and find a neighbourhood $U_0 \subset \tilde{U}_0$ of the point 0 and a j_0 such, that $H_1 \cap U_0$ meets $|Z_0(j, N) - Z_0^S(j, N)| \cap U_0$ properly for $j > j_0$.
By [14], Thm 3.6 and Thm 3.4 the cycles $Z_1(j, U_0)$ converge to $Z_1(0, U_0)$ and $\nu(Z_1^S(j, U_0), 0) = \nu(Z_1^S(0, U_0), 0)$ for almost all j.
Repeating this reasoning we find an integer j_d and a neighbourhood U_{d-1} such that for $j > j_d$ the system $(E^s \times L^j, H_1, \dots, H_d)|_{U_{d-1}}$ belongs to the family $\mathcal{E}(U_{d-1}, Z)$ and the following equality holds

$$\tilde{\nu}(Z \cdot (E^s \times L^j, H_1, \dots, H_d)|_{U_{d-1}}, 0) = \tilde{g}(Z, S)(0),$$

which contradicts our assumption about L^j. $\square$

Let h(I) (resp. s(I)) denote the height (resp. the analytic spread) of an ideal I. For the proof of Cor. 4.4, we will need the following version of [2], Cor. 2.4(iii) in the case of an equidimensional catenary Noetherian local ring:

Lemma 3.2 *Let (A, m) be an equidimensional and catenary Noetherian local ring and let I be a proper ideal of A. Then $h(I) = s(I)$ if and only if precisely one of the generalized Samuel multiplicities is non-zero.*

Proof: We put k := dim A, $q :=$ dim A/I, $r :=$ s(I). Let $c_j := c_j(I, A) :=$ $\mathrm{G}_m(\mathrm{G}_I(A))$ be the generalized Samuel multiplicities ([2], Def. 2.2).
Since the ring $G := \mathrm{G}_I(A)$ is catenary and equidimensional (by [7], B 6.6), the equality $\dim G/J + \dim G_J = \dim G$ holds for all $J \in \mathrm{Spec}(G)$. Therefore $c_{k-r} \neq 0$ by [2], Prop. 2.3(ii). According to [2], Prop. 2.3(iii) we have $c_q \neq 0$, which yields the equality $r = k - q$.
If h(I) = s(I), then $c_{k-r} \neq 0$ by [2], Cor. 2.4(iii). $\square$

4 Main theorem and its applications

We maintain the notation and the assumptions of previous sections.

sreg(Z, S) denotes the set of points $a \in \mathrm{reg}(Z \cap S)$ which satisfy the condition: $\tilde{g}(Z, S)(a) = \min_{lex}\{\tilde{g}(Z, S)(x) : x \in C\}$, where C is the irreducible component of $Z \cap S$ to which a belongs.
$\mathrm{sreg}_p(Z \cap S) := \{a \in \mathrm{sreg}(Z, S) | \dim_a(Z \cap S) = p\}$.
Observe that sreg(Z, S) is open and dense in $Z \cap S$ (see [14], Thm 5.4).

In this section, we compute $p(Z, S)(c)$, where $c \in \mathrm{sreg}(Z, S)$ and give a precise algebraic description of sreg(Z, S).

Theorem 4.1 *Let $c \in \mathrm{sreg}_p(Z, S)$ and $P \in \mathcal{T}_c(Z, S)$. Then:*
(a) $\tilde{g}(Z, S)(c) = (0, \ldots, 0, \mu_2(Z, S)(c), 0, \ldots, 0)$, *(where $(s-p)$ zeroes precede the integer $\mu_2(Z, S)(c)$),*
(b) $g(Z, S)(c) = \mu_1(Z, S)(c)$,
(c) $g(Z, S)(c) = \mu_2(Z, S)(c)$,
(d) $g(Z, S)(c) = p(Z, S)(c)$,
(e) $g(Z, S)(c) = i((P \cdot Z), S; c)$,
(f) *if $R \in \mathcal{F}^2_c(Z, S)$ and $\nu(Z \cdot R, c) = \mu_2(Z, S)(c)$, then the germ R_c is smooth and satisfies the condition $\mathrm{C}_c(P \cap Z, S) \cap T_cR = T_cS$,*

Proof: Choose a neighbourhood U of the point c such that $U \cap Z \cap S = X$, where $X \subset \operatorname{sreg}(Z,S)$ is connected, a dense subset $\{a_i\}_{i=0}^{\infty}$ of X and a sequence $\{P_i\}_{i=0}^{\infty}$ of elements of the families $\mathcal{T}_{a_i}(Z,S)$, where $P_i \in \mathcal{T}_{a_i}(Z,S)$, such that $a_0 = c$, $P_0 = P$. We can assume that $U = E^n$, $S \cap U = E^s \times \{0\}^{n-s}$, $c = 0$ and there exists $L \in \mathbf{G}_d(\mathbf{C}^{n-s})$ and a system of smooth hypersurafces $(H_1, \ldots, H_d)$ satisfying the conditions:

$$(\mathbf{C}^s \times L) \cap \bigcap_{j=1}^{j=d} T_x H_j = T_x S \quad \text{for every } x \in U \cap S,$$

$$(E^s \times L, H_1, \ldots, H_d) \in \mathcal{E}(U, Z), \tag{4}$$

$$\tilde{\nu}(Z \cdot (E^s \times L, H_1, \ldots, H_d), 0) = \tilde{g}(Z,S)(0). \tag{5}$$

By Lemmas 2.1, 3.1 we can choose an L which satisfies the condition

$$(\mathbf{C}^s \times L) \cap \mathrm{C}_{a_i}(P_i \cap Z, S) = \mathbf{C}^s \times \{0\}^{n-s} \tag{6}$$

for every i. There exists a point a_{i_0} such that $E^s \times L \in \mathcal{F}^2_{a_{i_0}}(Z,S)$. Then, by Lemma 2.2 the equality $\nu((E^s \times L) \cdot Z, a_{i_0}) = \mu_2(Z,S)(a_{i_0})$ holds.

L satisfies (4); hence, we get $\tilde{g}(Z,S)(a_{i_0}) \leq_{lex} (0, \ldots, \mu_2(Z,S)(a_{i_0}), \ldots, 0)$ (one can see that there exists a system $\mathcal{H} \in \mathcal{H}(U,Z)$ such that the equality $\tilde{\nu}(Z \cdot \mathcal{H}, c) = (0, \ldots, \mu_2(Z,S)(a_{i_0}), \ldots, 0)$ holds).
By (1), p. 2 and (5) the cycle $\mu_2(Z,S)(a_{i_0}) \cdot X$ is the p-dimensional part of $Z \cdot (E^s \times L, H_1, \ldots, H_d)$; consequently, we have

$$(0, \ldots, \mu_2(Z,S)(a_{i_0}), \ldots, 0) \leq_{lex} \tilde{g}(Z,S)(0).$$

Since the points 0 and a_{i_0} belong to the same component of $\operatorname{sreg}(Z,S)$, we have $\tilde{g}(Z,S)(0) = \tilde{g}(Z,S)(a_{i_0})$. Consequently, (a) holds for $c = a_{i_0}$ and only the $(s-p+1)$th component of $\tilde{g}(Z,S)(0)$ is non-zero, which implies that $(E^s \times L) \in \mathcal{F}^2_0(Z,S)$ (by (5)).
(a) Since $(E^s \times L)$ satisfies (6) for $i = 0$, Lemma 2.2 yields the equality

$$\nu((Z \cdot (E^s \times L))^S, 0) = \mu_2(Z,S)(0) = i((P \cdot Z), S; 0)\ . \tag{7}$$

(c) is an immediate consequence of (a).
(b) We can assume that there exists an analytic set $R \in \mathcal{F}^1_0(Z,S)$ such that $(Z \cdot R)^S = \mu_1(Z,S)(0) \cdot X$. Fix a point $a \in X \setminus |(Z \cdot R) - (Z \cdot R)^S|$. Then $R \in \mathcal{F}^2_a(Z,S)$, which gives the inequality

$$\mu_1(Z,S)(0) = \nu(Z \cdot R, a) \geq \mu_2(Z,S)(a).$$

By (c) we have $\mu_2(Z,S)(a) = g(Z,S)(a) = g(Z,S)(0) = \mu_2(Z,S)(0)$, which concludes the proof of (b).
(d) results from (b) and the inequalities $g(Z,S)(0) \geq p(Z,S)(0) \geq \mu_1(Z,S)(0)$.
(e) is an immediate consequence of (c) and (7).
(f) If $R \in \mathcal{F}_0^2(Z,S)$ satisfies the assumptions of (f), then by (7) we have $i((P\cdot Z),R;0) = \nu(Z\cdot R,0) = i((P\cdot Z),S;0)$. [4], Thm 4.4 yields (f). □

We obtain the following corollaries:

Corollary 4.2 *If $c \in \mathrm{sreg}_p(Z,S)$ and $\mathcal{H} := (H_1,\ldots,H_{m-s}) \in \mathcal{H}(U,Z)$, where U is a neighbourhod of c, then the following conditions are equivalent*
1. $\tilde{\nu}(\mathcal{H}\cdot Z,c) = \tilde{g}(Z,S)(c)$,
2. *there exists a neighbourhood $\tilde{U}$ of c and a submanifold $P \in \mathcal{T}_c(Z,S)$ such that $(H_1\cap\ldots\cap H_{k-p})\cap\tilde{U} \in \mathcal{F}_c^2(Z,S)$ and the following equality holds*

$$C_c(P\cap Z,S)\cap T_c(H_1\cap\ldots\cap H_{k-p}) = T_cS,$$

3. *there exists a $\tilde{U}$ such that $(H_1\cap\ldots\cap H_{k-p})\cap\tilde{U} \in \mathcal{F}_c^2(Z,S)$ and for every $P \in \mathcal{T}_c(Z,S)$ the following equality holds*

$$C_c(P\cap Z,S)\cap T_c(H_1\cap\ldots\cap H_{k-p}) = T_cS.$$

Proof: $(1 \Rightarrow 3)$. Choose a $\tilde{U}$ such that $\bigcap_{j=1}^{j=k-p} H_j\cap\tilde{U}$ meets Z properly. By Thm 4.1(a) we have $\bigcap_{j=1}^{j=k-p} H_j\cap\tilde{U} \in \mathcal{F}_c^2(Z,S)$. Thm 4.1.(f) yields 3.
$(3 \Rightarrow 2)$. Obvious.
$(2 \Rightarrow 1)$. By Lemma 2.2 we have $\nu((H_1\cap\ldots\cap H_{k-p}\cap\tilde{U})\cdot Z,c) = \mu_2(Z,S)(c)$. Thm 4.1(a) gives 1. □

Corollary 4.3 *Let $c \in reg(Z\cap S)$. If only one coordinate of $\tilde{g}(Z,S)(c)$ is non-zero, then $c \in \mathrm{sreg}(Z,S)$.*

Proof: Let $c \in \mathrm{sreg}_p(Z,S)$ and let C denote the irreducible component of $Z\cap S$ to which c belongs. Choose a dense subset $\{a_i\}_{i=0}^{\infty} \subset \mathrm{sreg}(Z,S)\cap C$. By Corollary 4.2 and Lemmas 2.1, 3.1 there exists $\mathcal{H} := (H_1,\ldots,H_{m-s}) \in \mathcal{H}(U,Z)$, where U is a neighbourhod of the point c, and an $a_i \in U\cap C$ such that $\tilde{\nu}(Z\cdot\mathcal{H},c) = \tilde{g}(Z,S)(c)$ and $\tilde{\nu}(Z\cdot\mathcal{H},a_i) = \tilde{g}(Z,S)(a_i)$. Since $(H_1\cap\ldots\cap H_{k-p}) \in \mathcal{F}_c^2(Z,S)\cap\mathcal{F}_{a_i}^2(Z,S)$ and the points c, a_i belong to C, we have $\tilde{g}(Z,S)(c) = \tilde{g}(Z,S)(a_i)$. □

Let $\mathrm{F}_0(S,Z)$ denote the normal-pseudoflatness locus (see [1] for details) $\mathrm{F}_0(S,Z) := \{a \in Z \mid \mathrm{h}(\mathrm{I}_{S,a}) = \mathrm{s}(\mathrm{I}_{S,a})\}$, where $\mathrm{I}_{S,a}$ is the ideal $\mathrm{I}_{S,a}\,\mathcal{O}_{Z,a}$.

Corollary 4.4 *The following equality holds*

$$\mathrm{reg}(Z \cap S) \cap \mathrm{F}_0(S, Z) = \mathrm{sreg}(Z, S).$$

Proof: By Thm. 4.1 and Cor. 4.3 we have that $c \in \mathrm{sreg}(Z, S)$ iff only one coordinate of $\tilde{g}(Z, S)(c)$ is non-zero. Lemma 3.2 and [3], Thm 1 (see also [12], §4) conclude the proof. □

For a cycle $A = \sum_{j \in J} \alpha_j C_j$ we define its *geometric part* A_{g} as the cycle $\sum_{j \in I} \alpha_j C_j$, where $\{C_j\}_{j \in I}$ is the family of all irreducible components of $|A|$. The cycle A is *positive* if $\alpha_j > 0$ for all j.
Let $\mathrm{D}(Z, S)$ denote the cycle defined by the analytically constructible function $g(Z, S)$ (see [14], Thm 5.5 and Prop. 2.1.3).

Corollary 4.5 *For every $c \in S$ the following inequality holds*

$$g(Z, S)(c) \geq \nu(\mathrm{D}(Z, S)_{\mathrm{g}}, c).$$

Proof: Let $Z \cap S = \bigcup_{i \in J_j, i=0,\ldots,s} X_{ij}$, where $\{X_{ij} \mid i \in J_j\}$ is the family of all j-dimensional components of $Z \cap S$. Fix $c \in S$. We can assume that there exists $\mathcal{H} = (H_1, \ldots, H_{m-s}) \in \mathcal{H}(N, Z)$ such that $\tilde{\nu}(\mathcal{H} \cdot Z, c) = \tilde{g}(Z, S)(c)$. Let $\mathcal{H} \cdot Z = \sum \alpha_{ij} X_{ij}$. Since $\mathcal{H} \cdot Z$ is positive, we have

$$\nu(\mathcal{H} \cdot Z, c) \geq \sum_{i \in J_j, j=0,\ldots,s} \alpha_{ij} \cdot \nu(X_{ij}, c).$$

By Thm 4.1 we have $\nu(\mathrm{D}(Z, S)_{\mathrm{g}}, c) = \sum_{i,j} \mu_2(Z, S)(x_{ij}) \cdot \nu(X_{ij}, c)$, where $x_{ij} \in X_{ij} \cap \mathrm{sreg}(Z, S)$ satisfies $(Z \cap \bigcap_{q=1}^{q=k-j} H_q)_{x_{ij}} = (X_{ij})_{x_{ij}}$. Observe that $\alpha_{ij} \geq \mu_2(Z, S)(x_{ij})$. □

As in [1] we denote $\mathcal{G}_1(S, Z) := \{a \in Z \mid \mathrm{s}(\mathrm{I}_{S,a}) - \mathrm{h}(\mathrm{I}_{S,a}) \geq 1\}$.

Corollary 4.6 *The cycle $(\mathrm{D}(Z, S) - \mathrm{D}(Z, S)_{\mathrm{g}})_{\mathrm{g}}$ is positive. Its support coincides with the set $\mathcal{G}_1(S, Z)$.*

Acknowledgements. I would like to express my gratitude to Prof. Rüdiger Achilles for suggesting improvements of Cor. 4.4 and to Prof. Piotr Tworzewski for helpful conversations.

References

[1] R. Achilles and M. Manaresi: Analytic deviation of ideals and intersection theory of analytic spaces. *Manuscripta Math.* **80** (1993), 291–308.

[2] R. Achilles and M. Manaresi: Multiplicities of a bigraded ring and intersection theory. *Math. Ann.* **309** (1997), 573–591.

[3] R. Achilles and S. Rams: Intersection numbers, Segre numbers and generalized Samuel multiplicities. Preprint, 1999.

[4] R. Achilles, P. Tworzewski, T. Winiarski, On improper isolated intersection in complex analytic geometry, *Ann. Polon. Math.* **51** (1990), 21-36.

[5] E. Cygan, T. Krasiński, P. Tworzewski: Separation of algebraic sets and the Łojasiewicz exponent of polynomial mappings. *Invent. Math.* **136** (1999), 75–87.

[6] R. Draper: Intersection theory in analytic geometry. *Math. Ann.* **180** (1969), 175–204.

[7] W. Fulton: Intersection theory. Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, Bd. 2, Springer-Verlag, Berlin-Heidelberg-New York, 1984.

[8] T. Gaffney and R. Gassler: Segre numbers and hypersurface singularities. Preprint http://xxx.lanl.gov/form/math.AG, alg-geom/9611002, 1996.

[9] R. Gassler: Mixed Segre numbers and integral closures of ideals. Preprint http://xxx.lanl.gov/form/math.AG, alg-geom/9703018, 1997.

[10] J. Kollár: Effective Nullstellensatz for arbitrary ideals. Preprint http://xxx.lanl.gov/form/math.AG, alg-geom/9805091, 1998.

[11] D. B. Massey: *Lê cycles and hypersurface singularities.* Lecture Notes Math. 1615, Springer-Verlag Berlin Heidelberg, 1995.

[12] K. Nowak: An algebraic approach to multiplicities for analytic improper intersections. IMUJ Preprint 1998/07, http://www.im.uj.edu.pl/ im/ preprint, 1998.

[13] S. Rams: On intersection product of analytic cycles. IMUJ Preprint 1998/01, http://www.im.uj.edu.pl/im/preprint, 1998.

[14] P. Tworzewski: Intersection theory in complex analytic geometry. *Ann. Polon. Math.* **62** (1995), 177–191.

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