On intersections with hypersurfaces and normal pseudo-flatness

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Abstract

We study the behaviour of the families of hypersurfaces that are applied in [4] to estimate the Łojasiewicz exponent and the exponent of separation at infinity.

1 Introduction

In the recent paper [4] the so-called index of contact is applied to estimate the exponent of separation at infinity and the Łojasiewicz exponent. This number is obtained in a procedure based on intersecting an analytic set Z with a hypersurface H that contains a fixed submanifold S, extracting the part of the cycle $H \cdot Z$ supported by S and counting its multiplicity $\nu((H \cdot Z)^S, c)$ (see [3], [4] for details). Similar procedures are applied in [10], [13], [6], [7] and [9]. The index of contact is also closely related to the extended index of intersection (see [13], [12], [1]).

The main aim of this note is to characterize the set of hypersurfaces for which $\nu((Z \cdot H)^S, c)$ is minimal (i.e. generic). According to [10], p. 91 no

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such characterization is known. We will prove that the hypersurfaces in question are to be smooth along a subset of $Z\cap S$ and to meet some algebraic cones along the tangent space to S.

The reader is expected to be familiar with the notion of proper intersection multiplicity of analytic sets (see [5]), its generalization given in [2], relative tangent cone (see [2]), basic properties of holomorphic cycles (also known as chains) (see [13], [11]) and the notion of normal pseudo-flatness (see [8]). Throughout the text we work over the field $\mathbf{C}$ of complex numbers.

2 Preliminaries

Let N be an n-dimensional manifold and let S be its s-dimensional closed submanifold.

For an analytic cycle $A=\sum_{j\in J}\alpha_j C_j$ in U, where U is an open subset of N, we define the *part of A supported by S* as $A^S:=\sum_{j\in J, C_j\subset S}\alpha_j C_j$. The *multiplicity of the cycle A* at the point $a\in U$ is defined to be $\nu(A,a):=\sum_{j\in J}\alpha_j\nu(C_j,a)$, where $\nu(C_j,a)$ denotes the multiplicity of the analytic set C_j at the point a. $|A|$ denotes the suport of A (i.e. $|A|:=\bigcup_{j\in J}C_j$). The tangent cone to C_j at a is denoted by $\mathrm{C}_a(C_j)$.

If all the irreducible sets C_j are of dimension p, we say that A is a p-cycle. A sequence $\{A_j\}_{j=1}^{\infty}$ of p-cycles in U converges to A iff it converges to a p-cycle A in the weak topology of currents.

If X, Y are analytic subsets of U that meet properly, C is an irreducible component of $X\cap Y$, then $i(X,Y;C)$ stands for the multiplicity of their intersection along C. The intersection product $\sum i(X,Y;C)\cdot C$, where the summation runs over all components of $X\cap Y$, is denoted by $X\cdot Y$.

Let X, Y be locally analytic subsets of N and let a be an isolated point of $X\cap Y$. By $i(X,Y;a)$ we denote the isolated intersection multiplicity defined in [2], and $\mathrm{C}_a(X,Y)$ stands for the relative tangent cone (see [2]).

The unit ball in $\mathbf{C}$ is denoted by E.

Let Z be a purely k-dimensional analytic subset of N such that S contains no component of Z. For an open $U\subset N$, and $a\in U$ we define the following families:

• $\mathcal{F}^1(U,Z):=\{H : H$ is a closed hypersurface in U, $\dim(H\cap Z)=k-1$, $U\cap S\subset H\}$,

• $\mathcal{F}^2_a(U,Z) := \{H \in \mathcal{F}^1(U,Z) : H_a \cap Z_a = (Z \cap S)_a\}$.

Let $H \in \mathcal{F}^1(U,Z)$. If $\{X_i^j\}_{j\in J_i}$ is the family of all $(k-i)$-dimensional irreducible components of $Z\cap S$, then $|(Z\cdot H)^S| = \bigcup_{j\in J_1} X_1^j$, and the following equality holds

$$\nu((Z\cdot H)^S, a) = \sum \nu(X_1^j, a)\cdot i(H, Z; X_1^j). \tag{1}$$

Definition 2.1 *Let $a \in \operatorname{reg}(Z\cap S)$, $\dim_a(Z\cap S) = k-1$. We define $\mathcal{T}_a(Z,S)$ to be the family of all $(n-k+1)$-dimensional submanifolds $P \subset N$ that are transversal to $Z\cap S$ at a, and meet Z properly.*

One can see that $\mathcal{T}_a(Z,S)$ is nonempty (by Rückert"s lemma) and every $P \in \mathcal{T}_a(Z,S)$ determines the 1-cycle $P\cdot Z = \sum \beta_j C_j$. Both $i((Z\cdot P), S; a) := \sum \beta_j i(C_j, S; a)$ and the cone $\mathrm{C}_a(Z,S)$ depend only on the germs Z_a, S_a, P_a. If $H \in \mathcal{F}^2_a(U,Z)$ and $a \in \operatorname{reg}(Z\cap S)\cap X_1^1$ then the following equality holds

$$i((P\cdot Z), H; a) = i(Z, H; X_1^1). \tag{2}$$

In the sequel we will need the following lemmas.

Lemma 2.2 *Let U be a neighbourhood of a, $H \in \mathcal{F}^2(U,Z)$, $a \in \operatorname{reg}(Z\cap H)$, $P \in \mathcal{T}_a(Z,S)$. Then the following conditions are equivalent:*

1. *$i((P\cdot Z), H; a) = i((P\cdot Z), S; a)$,*
2. *$a \in \operatorname{reg}(H)$ and $C_a(P\cap Z, S)\cap T_aH = T_aS$.*

Proof: Let V be a neighbourhood of the point a such that $P\cap V$ is a closed submanifold of V and $(S\cap Z)\cap V = X_1^1 \cap V$, where X_1^1 is the irreducible component of $Z\cap S$ to which a belongs.
Let $(P\cap V)\cdot(Z\cap V) = \sum_{j=1}^{j=m} \beta_j C_j$. By [2], Th. 4.4 the equality $i(C_j, H; a) = i(C_j, S; a)$ holds iff $a \in \operatorname{reg}(H)$ and $\bigcup_{j=1}^{j=m} \mathrm{C}_a(C_j, S)\cap T_aH = T_aS$. [2], Property 2.6 yields the equality $\bigcup_{j=1}^{j=m} \mathrm{C}_a(C_j,S) = \mathrm{C}_a(P\cap Z, S)$.
Hence, the inequality $i((P\cdot Z), H; a) \geq i((P\cdot Z), S; a)$ becomes an equality iff the condition 2 is satisfied. □

Lemma 2.3 *Let $N = E^n$, $S = E^s \times \{0\}^{n-s}$, $\{a_j\}_{j=1}^{\infty} \subset S$. Let C_j be locally analytic curves such that $a_j \in C_j$ and $(C_j)_{a_j} \cap S_{a_j} = \{a_j\}$. Then the set of hyperplanes $H \subset \mathbf{C}^{n-s}$ that satisfy the condition*

$$C_{a_j}(C_j, S)\cap \mathbf{C}^s \times H = \mathbf{C}^s \times \{0\}^{n-s} \tag{3}$$

for all j, is a dense subset of Grassman manifold $G_{n-s-1}(\mathbf{C}^{n-s})$.

Proof: By [2], Lemma 2.4 the equality $\mathrm{C}_{a_j}(C_j, S) = \mathbf{C}^s \times C_0(\pi(C_j \cap U_j))$, holds, where $\pi : \mathbf{C}^s \oplus \mathbf{C}^{n-s} \mapsto \mathbf{C}^{n-s}$ is the projection and U_j is a sufficiently small neighbourhood of a_j. Therefore, (3) is equivalent to the condition that H does not contain a finite family of lines. $\Box$

3 Intersections with hypersurfaces

Let N, S, Z be as in the previous section. We fix a point $c \in Z \cap S$.
• $g_1(Z, S)(c) := \min\{\nu((Z \cdot H)^S, c) : H \in \mathcal{F}^1(V, Z),\ c \in V\}$,
• $\mathrm{F}(S, Z) := \{a \in Z \mid \mathrm{h}(\mathrm{I}_{S,a}) = \mathrm{s}(\mathrm{I}_{S,a})\}$, where $\mathrm{h}(\mathrm{I}_{S,a})$ (resp. $\mathrm{s}(\mathrm{I}_{S,a})$ denotes the height (resp. the analytic spread) of $\mathrm{I}_{S,a}\,\mathcal{O}_{Z,a}$ (for more details about normal pseudo-flatness locus see [8])
In this section the following lemma will play crucial role.

Lemma 3.1 *Let $a \in \mathrm{reg}(Z \cap S) \cap \mathrm{F}(S, Z)$, $\dim_a(Z \cap S) = k - 1$ and let $P \in \mathcal{T}_a(Z, S)$. Then the equality $i((P \cdot Z), S; a) = g_1(Z, S)(a)$ holds.*

Proof: Lemma is an immediate consequence of [12], Th. 3.7. $\Box$

Let U be a neighbourhood of $c \in Z \cap S$. Our aim is to characterize the following family:

$$\mathcal{P}_c(U, Z) = \{H \in \mathcal{F}^1(U, Z) : \nu((Z \cdot H)^S, c) = g_1(Z, S)(c)\}.$$

We can assume that $\dim_c(Z \cap S) = k - 1$ (otherwise $\mathcal{P}_c(U, Z) = \mathcal{F}^1(U, Z)$).
For a given $H \in \mathcal{F}^1(U, Z)$ we define the following sets:
• $\mathrm{reg}_H(Z, S) := \{a \in \mathrm{reg}(Z \cap S) : \dim_a(Z \cap S) = k - 1,\ a \notin |Z \cdot H - (Z \cdot H)^S|$, the component of $Z \cap S$ to which a belongs passes through c, $\}$,
• $\mathrm{sreg}_H(Z, S) := \mathrm{reg}_H(Z, S) \cap \mathrm{F}(S, Z)$.

Now we are in position to prove the main result of this note:

Theorem 3.2 *Under the assumptions of this section, the following conditions are equivalent:*

1. *$H \in \mathcal{P}_c(U, Z)$,*
2. *for every point $a \in \mathrm{reg}_H(Z, S)$ and for every $P \in \mathcal{T}_a(Z, S)$ we have $a \in \mathrm{reg}(H)$ and the equality $T_aH \cap C_a(Z \cap P, S) = T_aS$ holds,*

3. *for each $(k-1)$-dimensional component X_j of $Z \cap S$ such that $c \in X_j$ there exists an $a_j \in \mathrm{sreg}_H(Z,S) \cap X_j$ and a submanifold $P_j \in \mathcal{T}_{a_j}(Z,S)$ such that $a_j \in \mathrm{reg}(H)$ and $T_{a_j}H \cap C_{a_j}(Z \cap P_j, S) = T_{a_j}S$.*

Proof: If $k=1$, then c is an isolated point of $Z \cap S$ and the theorem results immediately from [2], Th. 4.4. Therefore we can assume $k \geq 2$.
Let $\{X_j\}_{j=1}^{j=q}$ denote the family of all $(k-1)$-dimensional components of the set $Z \cap S$ that pass through the point c. Fix $H \in \mathcal{F}^1(U,Z)$ and consider the functions:
- $\Phi : \mathrm{sreg}_H(Z,S) \ni a \to i((Z \cdot R), S; a)$,
- $\Psi : \mathrm{sreg}_H(Z,S) \ni a \to \nu(Z \cdot H, a)$,

where $R \in \mathcal{T}_a(Z,S)$. Observe, that by Lemma 3.1, $\Phi(a)$ does not depend on the choice of $R \in \mathcal{T}_a(Z,S)$. According to Lemma 2.2 the equality $\Phi(a) = \Psi(a)$ holds iff $a \in \mathrm{reg}(H)$ and $T_a(H) \cap \mathrm{C}_a((P \cap Z), S) = T_aS$.
By Lemma 3.1, [12], Th. 4.1(e) and [12], Cor. 4.4 the integer $\Phi(a)$ equals the index of intersection of Z and S at a. Hence, [13], Th. 5.4 yields that Φ is locally constant and $\mathrm{sreg}_H(Z,S)$ is a Zariski open and dense subset of $\bigcup X_j$.
(1.$\Rightarrow$2.) Fix $a \in \mathrm{reg}_H(Z,S) \cap X_1$, $P \in \mathcal{T}_a(Z,S)$.
Assume that $a \in \mathrm{sreg}(Z,S)$. In this case we are to prove that $\Phi(a) = \Psi(a)$. Therefore, it suffices to find $b \in \mathrm{sreg}(Z,S) \cap X_1$ such that $\Phi(b) = \Psi(b)$.
Without loss of generality $N = E^n$, $S = E^s \times \{0\}^{n-s}$, $c = 0$. Choose a dense subset $\{b_j\}_{j=1}^{\infty}$ of $\mathrm{sreg}_H(Z,S)$ and a family $\{P_j\}_{j=1}^{\infty}$ such that $P_j \in \mathcal{T}_{a_j}(Z,S)$. By Lemma 2.3, there exists a linear space $L \in G_{n-s-1}(\mathbf{C}^{n-s})$ such that $\tilde{H} := E^s \times L$ meets Z properly and satisfies $\mathrm{C}_{b_j}(C_j, S) \cap \tilde{H} = T_{b_j}S$ for all j.

We can assume that $b_j \in X_j \cap \mathrm{reg}_{\tilde{H}}(Z,S)$ for $j = 1,\dots,q$. By Lemma 2.2 the equality $i(Z, \tilde{H}; b_j) = i((Z \cdot P_j), S; b_j)$ holds for $j = 1,\dots,q$. Consequently, $\nu(Z \cdot \tilde{H}, b_j) \leq \nu(Z \cdot H, b_j)$, and (see the equality (1)) we have

$$\nu((Z \cdot \tilde{H})^S, b_j) \leq \nu((Z \cdot H)^S, b_j).$$

By our assumption, the opposite inequality holds, which yields (by (2)) that $i((Z \cdot P_j), H; b_j)$ equals $i((Z \cdot P_j), S; b_j)$ for $j = 1,\dots,q$. Hence $\Phi(b_1) = \Psi(b_1)$.

To prove the implication (1.$\Rightarrow$2.) for an arbitrary $a \in \mathrm{reg}_H(Z,S)$ we can assume that $a = 0$, $P = \{0\}^{k-1} \times E^{n-k+1}$, $Z \cap S = E^{k-1} \times \{0\}^{n-k+1}$, and the projection $\pi : Z \to E^k \times \{0\}^{n-k}$ is proper. Observe, that $i((P \cdot Z), H; 0) = i(P, (Z \cdot H); 0) = i(Z \cdot H; X_1)$. Hence, by Lemma 2.2 it suffices to prove the inequality

$$i((P \cdot Z), S; 0) \geq i(Z, H; X_1).$$

We choose a sequence $\{a_j\}_{j=1}^{\infty} \subset \mathrm{sreg}_H(Z,S) \cap X_1$ such that $a_j \to 0$, and define $P_j := a_j + \{0\}^{k-1} \times E^{n-k+1}$. Observe that $P_j \in \mathcal{T}_{a_j}(Z,S)$ and $P_j \to P$, which yields $P_j \cdot Z \to P \cdot Z$ (by [13], Th. 3.6).
Choose a hypersurface $\tilde{H}$ such that $i(P \cdot Z, \tilde{H}; 0) = i(P \cdot Z, S; 0)$. We can assume that $\tilde{H}$ is an analytic subset of E^n which meets all the sets $|P_j \cdot Z|$ properly (by [14], Th. 2). Consequently, $(P_j \cdot Z) \cdot \tilde{H} \to (P \cdot Z) \cdot \tilde{H}$, which yields $i((P \cdot Z), \tilde{H}; 0) \geq i((P_j \cdot Z), \tilde{H}; a_j)$ for sufficiently large j (by [11], Def 2.4). Therefore

$$i((P \cdot Z), S; 0) \geq i((P_j \cdot Z), S; a_j).$$

Since $i((P_j \cdot Z), S; a_j) = i(Z, H; X_1)$ we obtain the inequality

$$i((P \cdot Z), S; 0) \geq i(Z, H; X_1).$$

(2.$\Rightarrow$3.) Obvious.
(3. $\Rightarrow$1.) By our assumption the equality $\Phi \equiv \Psi$ holds. We can assume that there exists an $\tilde{H} \in \mathcal{P}_c(U, Z)$. Let $b_j \in X_j \cap \mathrm{reg}_{\tilde{H}}(Z,S) \cap \mathrm{sreg}_H(Z,S)$. Then

$$i((P \cdot Z), \tilde{H}; b_j) \geq i((P \cdot Z), S; b_j) = i((P \cdot Z), H; b_j) = i(Z \cdot H; X_j),$$

which yields (see (1)) $\nu((\tilde{H} \cdot Z)^S, c) \geq \nu((H \cdot Z)^S, c)$. □

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References

[1] R. Achilles, M. Manaresi, *Multiplicities of a bigraded ring and intersection theory*, Math. Ann., **309** (1997) 573-591.

[2] R. Achilles, P. Tworzewski, T. Winiarski, *On improper isolated intersection in complex analytic geometry* , Ann. Polon. Math., **51** (1990) 21-36.

[3] E. Cygan, *Intersection theory and separation exponent in complex analytic geometry*, Ann. Polon. Math., **69** (1998) 287 - 299.

[4] E. Cygan, T. Krasiński, P. Tworzewski, *Separation at infinity and Lojasiewicz exponent of polynomial mappings*, Inventiones Math., **136** (1999) 75 - 87.

[5] R. N. Draper, *Intersection theory in analytic geometry*, Math.Ann., **180** (1969) 175-204.

[6] R. Gassler, T Gaffney *Segre numbers and hypersurface singularities*, Preprint http://xxx.lanl.gov/form/math.AG, alg-geom/9611002, 1996.

[7] R. Gassler: *Mixed Segre numbers and integral closures of ideals.* Preprint http://xxx.lanl.gov/form/math.AG, alg-geom/9703018, 1997.

[8] H. Hironaka, *Normal cones in analytic Whitney stratifications*, Publ IHES , **36** (1969) 127-138.

[9] J. Kollár, *Effective Nullstellensatz for arbitrary ideals*, Preprint http://xxx.lanl.gov/form/math.AG, alg-geom/9805091, 1998.

[10] D. B. Massey: *Lê cycles and hypersurface singularities.* Lecture Notes Math. 1615, Springer-Verlag Berlin Heidelberg 1995.

[11] S. Rams, *Convergence of holomorphic chains*, Ann. Polon. Math., **65** (1997) 227-234.

[12] S. Rams, *On the index of contact and multiplicities for bigraded rings*, IMUJ preprint 1999/26, http://www.im.uj.edu.pl/im/preprint, 1999.

[13] P. Tworzewski, *Intersection theory in complex analytic geometry*, Ann. Polon. Math., **62** (1995) 177-191.

[14] P. Tworzewski, T. Winiarski, *Continuity of intersection of analytic sets*, Ann. Polon. Math., **47** (1983) 387-396.

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