RECTILINEARIZATION AND UNIFORMIZATION OF **k**-SUBANALYTIC SETS AND FUNCTIONS

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Abstract

We prove rectilinearization and uniformization theorems for **k**-subanalytic ($\mathbb{R}_{an}^{\mathbf{k}}$-definable) sets and functions using the Lion-Rolin formula. Parallel reasoning gives standard results for the subanalytic case.

1 Introduction.

The aim of this paper is to give a new proof of earlier known uniformization and rectilinearization theorems for subanalytic sets ([4], [1]) and to get generalizations of these theorems to a larger class of sets which we call **k**-subanalytic.

k-subanalytic sets are these which can be described (or defined) by compositions of globally subanalytic functions and power functions $|x|^\lambda$ with $\lambda \in \mathbf{k}$, where $\mathbf{k}$ is a given subfield of the field of real numbers $\mathbb{R}$.

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This class of sets was earlier studied by J.-Cl. Tougeron ([9]) and by L. van den Dries and C. Miller (for example [2, 6]).

In 1997 J.-M. Lion and J.-P. Rolin published a paper [5], where they gave an explicit formula for a globally subanalytic function using analytic functions and operations of division and taking roots. This theorem, called the preparation theorem for subanalytic functions is similar to the Weierstrass preparation theorem, and appeared in less explicit form in a paper of A. Parusiński [8].

We use the Lion-Rolin formula for **k**-subanalytic functions, which is an analog of the Parusiński-Lion-Rolin preparation theorem for subanalytic functions, and get uniformization and rectilinearization for **k**-subanalytic functions and sets. This gives a partial answer to a problem formulated by L. van den Dries and C. Miller in [3].

Parallel reasoning gives uniformization and rectilinearization theorems for the subanalytic case. In particular, we get a version of the characterization of subanalytic continuous functions obtained by E. Bierstone and P. Milman [1], and by A. Parusiński [7].

2 Notions.

2.1 General notions.

Let X be any set. Functions $f, g : X \to \mathbb{R}$ are called *equivalent* (notation: $f \sim g$) on X, when there exists a constant $M > 0$ such that for $x \in X$ $M^{-1} \cdot |f(x)| \leq |g(x)| \leq M \cdot |f(x)|$. A *unit* is a function equivalent to the constant function 1.

We assume that analytic manifolds are paracompact. If $Z \subset M$, where M is an analytic manifold, then reg Z is the set of these points of Z which are analytically regular (i.e. Z is an analytic manifold in a neighbourhood of any point of reg Z). By $\mathbb{P} = \mathbb{P}(\mathbb{R})$ we denote the 1-dimensional real projective space. A *box* is a cartesian product of (possibly degenerated) intervals.

2.2 Cylinders and cells.

Let X be any set and let $\mathcal{F}$ be a finite family of real functions on X. An *$\mathcal{F}$-set* is any subset of X of the form

$$E = \bigcup_{i=1}^{k} \bigcap_{j=1}^{l} E_{ij}, \qquad (k, l \in \mathbb{N})$$

where $E_{ij} = \{x \in X |\, g_{ij}(x) = 0\}$ or $E_{ij} = \{x \in X |\, g_{ij}(x) > 0\}$ or $E_{ij} = \{x \in X |\, g_{ij}(x) < 0\}$ and $g_{ij} \in \mathcal{F}$. Notice that $\mathcal{F}$-sets form a boolean algebra.

A set $C \subset X \times \mathbb{R}$ is called an *$\mathcal{F}$-cylinder* when it is of the form
(1) $C = \{(x, y) \in B \times \mathbb{R} |\, g_1(x) < y < g_2(x)\}$, where B is an $\mathcal{F}$-set in X and $g_1, g_2 \in \mathcal{F} \cup \{-\infty\}^X \cup \{+\infty\}^X$, with $g_1(x) < g_2(x)$ for every $x \in B$,
or
(2) $C = \{(x, y) \in B \times \mathbb{R} |\, y = g(x)\}$, where B is an $\mathcal{F}$-set in X and $g \in \mathcal{F}$.

Then B is called the *basis* of C and the functions g_1, g_2 (or the function g) are called the *generating functions* of the cylinder C.

Lemma 2.2.1 *Let $\mathcal{F}$ be an $\mathbb{R}$-algebra of functions on X. Then for every finite family $\mathcal{C}$ of $\mathcal{F}$-cylinders in $X \times \mathbb{R}$ there exists a finite decomposition $\mathcal{C}'$ of $X \times \mathbb{R}$ into $\mathcal{F}$-cylinders compatible with $\mathcal{C}$.*

The proof is easy and left to the reader. $\square$

Let, for every $n \in \mathbb{N}$, we have a family $\mathcal{A}_n$ of functions from $\mathbb{R}^n$ into $\mathbb{R}$. Denote $\mathcal{A} = \bigcup_{n=0}^{\infty} \mathcal{A}$.

An *$\mathcal{A}$-cell* (more exactly: *$\mathcal{A}_n$-cell*) in $\mathbb{R}^{n+1}$ is such an $\mathcal{A}_n$-cylinder K whose basis K' is an $\mathcal{A}_{n-1}$-cell in $\mathbb{R}^n$. ($\{0\}$ is $\mathcal{A}_{-1}$-cell in $\mathbb{R}^0$, where $\mathcal{A}_{-1} = \emptyset$.)

Every $\mathcal{A}$-cell $K \subset \mathbb{R}^{n+1}$ is associated with a sequence $K = K^{n+1}, K' = K^n, ..., K^0$ where K^j is an $\mathcal{A}_{j-1}$-cell in $\mathbb{R}^j$ $(j = n+1, ..., 0)$ and

$$(1) K^{j+1} = \{(x_1, ..., x_{j+1}) \in K^j \times \mathbb{R} | t_{j+1,1}(x_1, ..., x_j) < x_{j+1} < t_{j+1,2}(x_1, ..., x_j)\},$$

where $t_{j+1,1} < t_{j+1,2}$ on K^j,
or

$$(2) K^{j+1} = \{(x_1, ..., x_{j+1}) \in K^j \times \mathbb{R} | t_{j+1,1}(x_1, ..., x_j) = x_{j+1} = t_{j+1,2}(x_1, ..., x_j)\},$$

where $t_{j+1,1} = t_{j+1,2}$ on K^j,

with $t_{j+1,i} \in \mathcal{A}_j \cup \{-\infty\}^{\mathbb{R}^j} \cup \{+\infty\}^{\mathbb{R}^j} (i = 1, 2)$, and $t_{j+1,i} \equiv -\infty$ or $t_{j+1,i} \equiv +\infty$ can hold only in the case (1). The sequence $\{t_{j,i}\}_{j=1,\ldots,n+1}^{i=1,2}$ is called the *sequence of generating functions* of the $\mathcal{A}$-cell K.

We say that K is *full-dimensional*, when K' is full-dimensional and K is of the form (1). $\mathcal{A}$-cell $\{0\}$ is full-dimensional in $\mathbb{R}^0$.

We say that K is an *analytic* cell, when each of the functions $\{t_{j,i}|K^{j-1}\}_{j=1,\ldots,n+1}^{i=1,2}$ is a restriction of some analytic function in a neighbourhood of K^{j-1} (or, equivalently, its generating functions are analytic on respective bases as submanifolds).

If the family $\mathcal{A}$ is fixed, we will speak about cells and cyliders instead of $\mathcal{A}$-cells and $\mathcal{A}_n$-cylinders.

2.3 Subanalytic notions.

A subset Z of an analytic manifold M is called *semianalytic in* M, if for every $x \in M$ there exists an open neighbourhood U of a in M such that $Z \cap U$ is an $\mathcal{A}(U)$-set in U, where $\mathcal{A}(U)$ is the algebra of all analytic functions on U.

A subset $Z \subset \mathbb{R}^n$ is called *globally subanalytic in* $\mathbb{R}^n$ when it is an image $Z = \pi(A)$ of the natural projection π of some set $A \subset \mathbb{R}^{n+k}$ semianalytic in $\mathbb{P}^{n+k}$ ($\mathbb{R}$ is embedded in $\mathbb{P}$ in a standard way). A mapping $h : \mathbb{R}^n \supset X \to Y \subset \mathbb{R}^m$ is *globally subanalytic* if its graph is globally subanalytic in $\mathbb{R}^n \times \mathbb{R}^m$.

Denote

$$\mathcal{SAN}_n = \{f : \mathbb{R}^n \to \mathbb{R} |\, f \text{ is globally subanalytic}\}.$$

We will often assume that $\mathcal{SAN}_n \subset \mathcal{SAN}_{n+1}$, treating functions from $\mathcal{SAN}_n$ as independent on the last variable.

It follows from the Parusiński-Lion-Rolin preparation theorem that $\mathcal{SAN}_n$-sets in $\mathbb{R}^n$ are exactly globally subanalytic sets. A consequence of this is the Gabrielov theorem of the complement.

We will use the notion of a *blowing-up* in the most standard sense: any mappping of the form

$$R_{i,j} : \mathbb{R}^n \ni (x_1, ..., x_i, ..., x_j, ..., x_n) \mapsto (x_1, ..., x_i, ..., \overset{(j)}{x_j \cdot x_i}, ..., x_n) \in \mathbb{R}^n,$$

where $n \geq 2$, $1 \leq i, j \leq n$ and $i \neq j$.

A *power function* is any composition of mappings of the form

$$P : \mathbb{R}^n \ni (x_1, ..., x_i, ..., x_n) \mapsto (x_1, ..., x_i^\gamma, ..., x_n) \in \mathbb{R}^n,$$

where $\gamma \in \mathbb{N}^* = \mathbb{N} \setminus \{0\}$.

A *(subanalytic) normal crossing* is a function on an open, globally subanalytic subset of some $\mathbb{R}^n$ being a product of a monomial and an analytic and globally subanalytic unit. (We will drop the word "subanalytic".) A *local normal crossing* is a normal crossing with some analytic and globally subanalytic isomorphism substituted.

Remarks.

1) A composition of combinations of normal crossings is a combination of normal crossings.

2) A bounded quotient of normal crossings on a bounded open box in $\mathbb{R}^n$ is a normal crossing.

A *subanalytic modification* is any mapping $\Phi : U \to \mathbb{R}^n$, where U is open and globally subanalytic in $\mathbb{R}^n$, and Φ is a composition of blowings-up, power functions and analytic isomorphisms of open sets in $\mathbb{R}^n$ which are globally subanalytic. If $U = \mathbb{R}^n$, then we say that this modification is *global*. Subanalytic modifications are analytic and globally subanalytic mappings.

A *semialgebraic modification* is any mapping $\Phi : U \to \mathbb{R}^n$, where U is open and semialgebraic in $\mathbb{R}^n$, and Φ is a composition of blowings-up, power functions and semialgebraic Nash (analytico-algebraic) isomorphisms of open sets in $\mathbb{R}^n$. If $U = \mathbb{R}^n$, then the modification is *global*. Semialgebraic modifications are semialgebraic Nash mappings.

Lemma 2.3.1 (on boxes) *Every compact box (not necessarily full-dimensional) $K \subset \mathbb{R}^n_+$ is a union of images of subspaces $\mathbb{R}^{\dim K} \subset \mathbb{R}^n$ by a finite family of semialgebraic modifications $m_\mu : \mathbb{R}^n \to \mathbb{R}^n$ which are combinations of local normal crossings. Moreover, there exists a compact box $L \subset \mathbb{R}^{\dim K}$ such that $\bigcup_\mu m_\mu(L) = K$.*

Proof. As we can make cartesian products of m_μ-s, it suffices to consider the case $n = 1$. A singleton is the image of the trivial subspace of $\mathbb{R}$ by a translation. A nondegenerated interval $[a, b]$ is the union of $[a, a+\frac{3}{4}(b-a))$ and $(a+\frac{1}{4}(b-a), b]$ which are images of $\mathbb{R}$ by compositions of Nash isomorphisms, power functions, and linear isometries. (They are normal crossings in the variable $y = \frac{x}{\sqrt{1+x^2}}$.) The existence of a proper L is obvious. □

Let M be an analytic manifold of dimension n and let $x \in M, E \subset M$.

A germ E_x is called *subanalytic* when there exists a chart ϕ in a neighbourhood of x such that $\phi(E)_{\phi(x)}$ is a germ of a globally subanalytic set in $\mathbb{R}^n$. (It does not depend on the choice of a chart.)

A set E is called *subanalytic* if for every $x \in M$ its germ E_x is subanalytic. A mapping of analytic manifolds $f: M \to N$ is a *subanalytic mapping* if its graph is a subanalytic subset of $M \times N$. (A set $E \subset \mathbb{R}^n$ is globally subanalytic if and only if it is subanalytic in $\mathbb{P}^n$.)

A function $f: E \to \mathbb{R}$ has a subanalytic graph in $M \times \mathbb{P}$ if and only if for every $x \in M$ there exists a chart $\phi: U \to \phi(U) \subset \mathbb{R}^n$ in a neighbourhood of x such that $\phi \times id_{\mathbb{R}}$(graph f) is globally subanalytic in $\mathbb{R}^n \times \mathbb{R}$. Such functions will be called *globally subanalytic on* M.

A *subanalytic modification into* M is a subanalytic mapping $\pi : \mathbb{R}^n \supset V \to M$ such that $\overline{\operatorname{im} \pi}$ is contained in a domain of a chart ϕ and $\phi \circ \pi$ is a subanalytic modification.

2.4 k-subanalytic notions.

For every real number λ, we define a function

$$|\cdot|^\lambda : \mathbb{R} \ni x \mapsto \begin{cases} 1, & x = 0, \lambda = 0 \\ 0, & x = 0, \lambda \neq 0 \\ |x|^\lambda, & x \neq 0. \end{cases}$$

Notice that for $\lambda \geq 0$ the function $|\cdot|^\lambda$ is continuous, and if $\lambda > p \in \mathbb{N}$, then $|\cdot|^\lambda$ is of class C^p. If $x \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}^n$, then $|x|^\lambda$ denotes $|x_1|^{\lambda_1} \cdot \ldots \cdot |x_n|^{\lambda_n}$.

Let $\mathbf{k}$ be a fixed subfield of the real field $\mathbb{R}$. We put

$$\mathcal{PS}_n^{\mathbf{k}} = \{f : \mathbb{R}^n \to \mathbb{R} |\ f \text{ is a composition of functions from } \mathcal{SAN}$$

$$\text{and functions of the form } |x|^\lambda (\lambda \in \mathbf{k})\}.$$

We will often assume that $\mathcal{PS}_n^{\mathbf{k}} \subset \mathcal{PS}_{n+1}^{\mathbf{k}}$ treating functions from $\mathcal{PS}_n^{\mathbf{k}}$ as independent on the last variable.

$\mathbf{k}$-*subanalytic sets* in $\mathbb{R}^n$ are $\mathcal{PS}_n^{\mathbf{k}}$-sets. $\mathbf{k}$-*subanalytic functions* with domains in $\mathbb{R}^n$ are functions having $\mathbf{k}$-subanalytic graphs. ($\mathbb{Q}$-subanalytic sets and functions in $\mathbb{R}^n$ are exactly globally subanalytic sets and functions.)

By the $\mathbf{k}$-subanalytic version of the Lion-Rolin preparation theorem,

$$\mathcal{PS}_n^{\mathbf{k}} = \{f : \mathbb{R}^n \to \mathbb{R} |\ f \text{ is } \mathbf{k}\text{-subanalytic}\}.$$

A $\mathbf{k}$-*analytic function* is a function $V : U \to \mathbb{R}$ whose domain is a $\mathbf{k}$-subanalytic open set in $\mathbb{R}^n$, and being of the form $V(x) = \tilde{V}(\Theta(x))$, where

$\Theta(x) = (x_1, ..., x_n, |x_{\sigma(1)}|^{\lambda_1}, ..., |x_{\sigma(m)}|^{\lambda_m})$, $m \in \mathbb{N}$, $\sigma : \{1, ..., m\} \to \{1, ..., n\}$ is any sequence, $\lambda_i \in \mathbf{k}_+^* = \{\kappa \in \mathbf{k} | \ \kappa > 0\}$, for $i = 1, ..., m$ and $\tilde{V}$ is an analytic and globally subanalytic function in some neighbourhood of $\Theta(U) \subset \mathbb{R}^{n+m}$.

Remark. $\mathbf{k}$-analytic functions are continuous, analytic off $\{x \in \mathbb{R}^n | \ x_1 \cdot ... \cdot x_n = 0\}$, and if all $\lambda_i > p \in \mathbb{N}^*$ then they are of class C^p.

Example. The function $z(x, y) = ||x|^\pi + y|^\pi$ is a composition of $\mathbb{R}$-analytic functions which is not $\mathbb{R}$-analytic.

A $\mathbf{k}$-*analytic mapping* is a mapping $\Omega : \mathbb{R}^n \supset A \to B \subset \mathbb{R}^m (m, n \in \mathbb{N})$ whose combinational components are $\mathbf{k}$-analytic functions. $\mathbf{k}$-analytic mappings are continuous.

A $\mathbf{k}$-*analytic isomorphism* is a bijective mapping which is $\mathbf{k}$-analytic together with its inverse. $\mathbf{k}$-analytic isomorphisms are $\mathbf{k}$-subanalytic homeomorphisms of open sets.

A $\mathbf{k}$-*power function* is any composition of mappings of the form

$$P_i : \mathbb{R}^n \ni (x_1, ..., x_n) \mapsto (x_1, ..., |x_i|^\gamma, ..., x_n) \in \mathbb{R}^n,$$

where $\gamma \in \mathbf{k}_+^*$. $\mathbf{k}$-power function is a $\mathbf{k}$-analytic mapping.

A $\mathbf{k}$-*normal crossing* is a function defined on an open, $\mathbf{k}$-subanalytic subset of some $\mathbb{R}^n$ and being a product of a monomial with exponents from $\mathbf{k}_+ = \{\kappa \in \mathbf{k} | \ \kappa \geq 0\}$ and a $\mathbf{k}$-analytic unit. A *local* $\mathbf{k}$-*normal crossing* is a $\mathbf{k}$-normal crossing with some analytic and globally subanalytic isomorphism substituted.

Remarks.

1) A composition of combinations of $\mathbf{k}$-normal crossings is a combination of $\mathbf{k}$-normal crossings. A combination of $\mathbf{k}$-normal crossings and zero functions is a $\mathbf{k}$-analytic mapping. Substitution of a combination of $\mathbf{k}$-normal crossings and zero functions in a $\mathbf{k}$-analytic mapping is a $\mathbf{k}$-analytic mapping.

2) A bounded quotient of $\mathbf{k}$-normal crossings on a bounded open box is a $\mathbf{k}$-normal crossing.

3) The mapping $\mathbb{R}^{n+1} \ni (x, y) \mapsto (|x|, |x|^\alpha \cdot y) \in \mathbb{R}^{n+1}$ $(y \in \mathbb{R}, x \in \mathbb{R}^n, \alpha \in \mathbf{k}_+^n)$ is a composition of blowings-up and $\mathbf{k}$-power functions.

4) The mapping $\mathbb{R}^{1+1} \ni (x, y) \mapsto (|x \cdot y|^\alpha, y) \in \mathbb{R}^{1+1}$ $(y \in \mathbb{R}, x \in \mathbb{R}, \alpha \in \mathbf{k}_+^*)$ is a composition of a blowing-up and a $\mathbf{k}$-power function.

A $\mathbf{k}$-*subanalytic modification* is a mapping $\Phi : U \to \mathbb{R}^n$, where U is open and $\mathbf{k}$-subanalytic, and Φ is a composition of blowings-up, $\mathbf{k}$-power functions and $\mathbf{k}$-analytic isomorphisms. If $U = \mathbb{R}^n$, then the modification is *global*. $\mathbf{k}$-subanalytic modifications are $\mathbf{k}$-subanalytic continuous mappings.

Let M be an analytic manifold of dimension n and let $x \in M, E \subset M$.

A germ E_x is called **k**-*subanalytic* if there exists a chart ϕ in a neighbourhood of x such that $\phi(E)_{\phi(x)}$ is a germ of a **k**-subanalytic set in $\mathbb{R}^n$. (It does not depend on the choice of a chart.)

A set E is called *locally* **k**-*subanalytic* if for every $x \in M$ its germ E_x is **k**-subanalytic. A mapping of analytic manifolds $f : M \to N$ is called *locally* **k**-*subanalytic* if its graph is locally **k**-subanalytic in $M \times N$.

A function $f : E \to \mathbb{R}$ has a locally **k**-subanalytic graph in $M \times \mathbb{P}$ if and only if for every $x \in M$ there exists a chart $\phi : U \to \phi(U) \subset \mathbb{R}^n$ in a neighbourhood of x such that $\phi \times id_{\mathbb{R}}$(graph f) is **k**-subanalytic in $\mathbb{R}^n \times \mathbb{R}$. Such functions will be called **k**-*subanalytic on* M.

A **k**-*subanalytic modification into* M is a locally **k**-subanalytic mapping $\pi : \mathbb{R}^n \supset V \to M$ such that $\overline{\mathrm{im}\ \pi}$ is contained in the domain of some chart ϕ where $\phi \circ \pi$ is a **k**-subanalytic modification.

3 k-subanalytic rectilinearization.

3.1 Cells with reducers.

Let $E \subset \mathbb{R}^n_x \times \mathbb{R}^1_y$ be a $\mathcal{PS}^{\mathbf{k}}_{n+1}$-set.

A *reducing system* R *on* E is a system $R = (\theta, \{\phi_1, ..., \phi_s\}, a, b, \{\lambda_i\}_{i=1}^r, \{\tilde{\lambda_k}\}_{k=1}^q)$, where $s, q, r \in \mathbb{N}^*$, $\theta, \phi_1, ..., \phi_s, a, b \in \mathcal{PS}^{\mathbf{k}}_n$, $\lambda_i, \tilde{\lambda_k} \in \mathbf{k}^*_+$ and for $(x, y) \in E$ the following conditions hold:

$$y - \theta(x) \neq 0, \text{ functions } y - \theta(x), a(x), b(x) \text{ are of constant sign},$$

$$\text{if } \theta(x) \not\equiv 0, \text{ then } y \sim \theta(x),$$

$$\text{the mapping } \psi(x,y) = (\phi_1(x), ..., \phi_s(x), |a(x)\cdot(y-\theta(x))|^{\lambda_1}, ..., |a(x)\cdot(y-\theta(x))|^{\lambda_r},$$

$$|\frac{b(x)}{y-\theta(x)}|^{\tilde{\lambda}_1}, ..., |\frac{b(x)}{y-\theta(x)}|^{\tilde{\lambda}_q}) \text{ is bounded.}$$

The mapping $\psi(x, y)$ is called the *reducing morphism of* R.

A finite family of functions $\{f_\nu\}_\nu \subset \mathcal{PS}^{\mathbf{k}}_{n+1}$ is *R-reducible on* E, when every function f_ν has on E the form (called *R-reduction*)

$$(\star) \qquad f_\nu(x,y) = |y - \theta(x)|^{\gamma_\nu} \cdot A_\nu(x) \cdot V_\nu(\psi(x,y)),$$

where $\gamma_\nu \in \mathbf{k}$, $A_\nu \in \mathcal{PS}_n^{\mathbf{k}}$, V_ν is an analytic, nonvanishing function of constant sign on some neighbourhood of the set $\overline{\psi(E)} \subset \mathbb{R}^{s+r+q}$ and for every ν, $A_\nu(x) = 0$ or $A_\nu(x) \neq 0$ holds on E. (Thus also $f_\nu = 0$ or $f_\nu \neq 0$ on E.)

A *reducer* $\mathcal{R}$ *on* K, where K is a $\mathcal{PS}^{\mathbf{k}}$-cell in $\mathbb{R}^{n+1}$, is a sequence $\mathcal{R} = (R^1, ..., R^{n+1})$, where R^j is a reducing systemem on $K^j \subset \mathbb{R}^j$.

For a finite family of functions $\{f_\nu\}_\nu \subset \mathcal{PS}_{n+1}^{\mathbf{k}}$ and a reducer $\mathcal{R}$ on K, where

$$R^j = (\theta_j, \{\phi_{j,1}, ..., \phi_{j,s_j}\}, a_j, b_j, \{\lambda_{j,i}\}_{i=1}^{r_j}, \{\tilde{\lambda}_{j,k}\}_{k=1}^{q_j}), j = 1, ..., n+1$$

we say that it is $\mathcal{R}$-*reducible on* K, when is R^{n+1}-reducible on K:

$$f_\nu(x, y) = |y - \theta_{n+1}(x)|^{\gamma_\nu} \cdot A_\nu(x) \cdot$$

$$\cdot V_\nu(\phi_{n+1,1}(x), ..., \phi_{n+1,s_{n+1}}(x), |a_{n+1}(x) \cdot (y - \theta_{n+1}(x))|^{\lambda_{n+1,1}}, ...,$$

$$|a_{n+1}(x) \cdot (y - \theta_{n+1}(x))|^{\lambda_{n+1,r}}, |\frac{b_{n+1}(x)}{y - \theta_{n+1}(x)}|^{\tilde{\lambda}_{n+1,1}}, ..., |\frac{b_{n+1}(x)}{y - \theta_{n+1}(x)}|^{\tilde{\lambda}_{n+1,q}}),$$

and the family $\{\theta_{n+1}, \phi_{n+1,1}, ..., \phi_{n+1,s_{n+1}}, a_{n+1}, b_{n+1}\} \cup \{A_\nu\}_\nu$ is $(R^1, ..., R^n)$-reducible on K^n. (Constant functions on $\mathbb{R}^0$ are reducible.)

Remark. If $f \in \mathcal{PS}_{n+1}^{\mathbf{k}}$, $\lambda \in \mathbf{k}$, and f is $\mathcal{R}$-reducible on K, then $|f|^\lambda$ is $\mathcal{R}$-reducible on K.

A *cell with a reducer in* $\mathbb{R}^{n+1}$ is a pair $(K, \mathcal{R})$, where K is a cell in $\mathbb{R}^{n+1}$, a $\mathcal{R}$ is reducer on K such that for $j = 1, ..., n$ the family $\{t_{j+1,1}, t_{j+1,2}\}$ is $(R^1, ..., R^j)$-reducible on K^j. We say that a finite family $\{f_\nu\}_\nu \subset \mathcal{PS}_{n+1}^{\mathbf{k}}$ is *reducible on* $(K, \mathcal{R})$, when it is $\mathcal{R}$-reducible on K.

The following proposition gives a kind of "partial desingularization".

Proposition 3.1.1 *For a finite family* $\{f_\nu\}_\nu \subset \mathcal{PS}_{n+1}^{\mathbf{k}}$ *reducible on a cell with a reducer* $(K, \mathcal{R})$ *in* $\mathbb{R}^{n+1}$*, there exist: an open neighbourhood* U *of the cell* K *and a* $\mathbf{k}$*-subanalytic and analytic isomorphism* $\Omega : U \ni z \mapsto \tilde{z} \in \tilde{U}$ *of open sets in* $\mathbb{R}^{n+1}$ *such that* $\tilde{U} \subset (0, +\infty)^{n+1}$ *and for every* ν

$$\left(f_\nu \circ \Omega^{-1} | \, \Omega(K)\right)(\tilde{z}_1, ..., \tilde{z}_{n+1}) = |\tilde{z}|^{\gamma_\nu} \cdot h_\nu(\tilde{z}),$$

where $\gamma_\nu \in \mathbf{k}^{n+1}$*,* h_ν *are zeroes or analytic and* $\mathbf{k}$*-subanalytic units on* $\tilde{U}$*. Thus* $\{f_\nu | \, K\}$ *are restrictions of analytic functions on* U*.*

Proof. We will use induction on n. On K we have reductions with a common reducing system:

$$f_\nu(x,y) = |y-\theta(x)|^{\gamma_\nu} \cdot A_\nu(x) \cdot V_\nu(\psi(x,y)),$$

where

$$\psi(x,y) = \Big(\phi_1(x), ..., \phi_s(x), |a(x)\cdot(y-\theta(x))|^{\lambda_1}, ..., |a(x)\cdot(y-\theta(x))|^{\lambda_r},$$

$$|\frac{b(x)}{y-\theta(x)}|^{\tilde{\lambda}_1}, ..., |\frac{b(x)}{y-\theta(x)}|^{\tilde{\lambda}_q}\Big),$$

$\gamma_\nu \in \mathbf{k}$, θ, A_ν, ϕ_j, $a, b \in \mathcal{PS}_n^{\mathbf{k}}$, V_ν are analytic units on some common open neighbourhood W of the set $\overline{\psi(K)} \subset \mathbb{R}^{s+r+q}$. We can assume that they are $\mathbf{k}$-subanalytic.

Let $\{g_\mu\}_\mu = \{\theta, \phi_1, ..., \phi_s, a, b\} \cup \{A_\nu\}_\nu$. There exist: an open neighbourhood U' of the cell K' and an analytic and $\mathbf{k}$-subanalytic isomorphism $\Omega' : U' \to \tilde{U}' \subset (0,+\infty)^n$ such that for every μ

$$\left(g_\mu \circ (\Omega')^{-1}|\, \Omega'(K')\right)(\tilde{x}) = |\tilde{x}|^{\delta_\mu} k_\mu(\tilde{x}) \text{ , where } \delta_\mu \in \mathbf{k}^n$$

and k_μ is a zero or an analytic and $\mathbf{k}$-subanalytic unit on $\tilde{U}'$. (For $n=0$ it is obvious; for $n>0$ we use the induction assumption for K' and the family $\{g_\mu\}_\mu$.)

Let

$$\tilde{g}_\mu : \tilde{U}' \ni \tilde{x} \mapsto |\tilde{x}|^{\delta_\mu} k_\mu(\tilde{x}) \in \mathbb{R}$$

and

$$Z = \{(x,y) \in U' \times \mathbb{R}|\, \operatorname{sgn}(y-\theta(x)) = \operatorname{sgn}(\frac{1}{2}(t_1+t_2)(x) - \theta(x))\}.$$

Moreover, let $\tilde{Z} = \{(\tilde{x}, y)|(x,y) \in Z\}$,

$$\tau : \tilde{Z} \ni (\tilde{x}, y) \mapsto (\tilde{x}, \tilde{y}) = (\tilde{x}, |y - \tilde{\theta}(\tilde{x})|) \in \tilde{U}' \times (0,+\infty),$$

$$\tilde{\psi}(\tilde{x},\tilde{y}) = \Big(\tilde{\phi}_1(\tilde{x}), ..., \tilde{\phi}_s(\tilde{x}), |\tilde{a}(\tilde{x})\cdot\tilde{y}|^{\lambda_1}, ...,$$

$$|\tilde{a}(\tilde{x})\cdot\tilde{y}|^{\lambda_r}, |\frac{\tilde{b}(\tilde{x})}{\tilde{y}}|^{\tilde{\lambda}_1}, ..., |\frac{\tilde{b}(\tilde{x})}{\tilde{y}}|^{\tilde{\lambda}_q}\Big).$$

Notice that $\tilde{\tau}$ is an analytic and $\mathbf{k}$-subanalytic isomorphism, and $\tilde{\psi}$ is an analytic and $\mathbf{k}$-subanalytic mapping on $\tilde{U}' \times (0, +\infty)$.

Let us set $\tilde{U} = \tilde{\psi}^{-1}(W)$,

$$U = (\tau \circ (\Omega' \times id_{\mathbb{R}}))^{-1}(\tilde{U}) \text{ and } \Omega = \tau \circ (\Omega' \times id_{\mathbb{R}})|\, U.$$

Notice that $\psi|_K = \tilde{\psi} \circ \Omega|_K$. Let $\tilde{A}_\nu(\tilde{x}) = |\tilde{x}|^{\omega_\nu} h_\nu(\tilde{x})$.

Thus every function

$$\left(f_\nu \circ \Omega^{-1}|\, \Omega(K)\right)(\tilde{x}, \tilde{y}) = |\tilde{y}|^{\gamma_\nu} |\tilde{x}|^{\omega_\nu} h_\nu(\tilde{x})(V_\nu \circ \tilde{\psi})(\tilde{x}, \tilde{y}),$$

is a restriction of an analytic and $\mathbf{k}$-subanalytic functiom on $\tilde{U} \subset (0, +\infty)^{n+1}$ of desired form. $\square$

Corollary 3.1.2 *If $(K, \mathcal{R})$ is a $\mathcal{PS}^{\mathbf{k}}$-cell with a reducer, then K is an analytic cell.*

The preparation theorem for $\mathbf{k}$-subanalytic functions can be stated as follows:

Theorem 3.1.3 (Lion-Rolin [5]) *For a finite family of functions $\{f_\nu\}_\nu \subset \mathcal{PS}^{\mathbf{k}}_{n+1}$, there exists a finite family $\{(C_\alpha, R_\alpha)\}_\alpha$, where:*

1) $\{C_\alpha\}_\alpha$ is a decomposition of $\mathbb{R}^{n+1}$ into $\mathcal{PS}^{\mathbf{k}}_n$-cylinders (we can require that cylinders to be compatible with a given $\mathcal{PS}^{\mathbf{k}}_{n+1}$- set adding the characteristing function of this set to the family of functions).

2) R_α is a reducing system on C_α.

3) The family $\{f_\nu\}_\nu$ is R_α-reducible on C_α.

Corollary 3.1.4 *Let E be $\mathcal{PS}^{\mathbf{k}}_{n+1}$-set, and $\{f_\nu\}_\nu$ a finite family of functions from $\mathcal{PS}^{\mathbf{k}}_{n+1}$. Then there exists a finite family of cells with reducers $\{(K_\alpha, \mathcal{R}_\alpha)\}_\alpha$ such that $\{K_\alpha\}_\alpha$ form a decomposition of E and the family $\{f_\nu\}_\nu$ is reducible on every $(K_\alpha, \mathcal{R}_\alpha)$.*

If the family $\{f_\nu\}_\nu$ is already R-reducible on E with some reducing system R, then we can additionally get $R = R^{n+1}_\alpha$ for every α.

Additionally, for a given finite family of $\mathcal{PS}^{\mathbf{k}}_{n+1}$-sets, we can get compatibility of cells with this family of sets.

Remark. Assume that $\mathcal{E}$ is a decomposition of the set E into a finite family of $\mathcal{PS}^{\mathbf{k}}_{n+1}$-sets. If Corollary 3.1.4 holds for every element of $\mathcal{E}$ (the family of functions remains unchanged), then it holds also for E.

The proof of Corollary 3.1.4 is left to the reader. □

Corollary 3.1.5 *For every finite family of $\mathcal{PS}^{\mathbf{k}}_{n}$-sets there exists a decomposition of $\mathbb{R}^n$ into a finite family of analytic $\mathcal{PS}^{\mathbf{k}}_{n-1}$-cells which are compatible with given sets.*

Remark. Thus the family (Tarski system) of $\mathcal{PS}^{\mathbf{k}}$-sets admits analytic cell decomposition and (finite) analytic stratifications (see also the proofs of D.12, D.16, and D.18 in [3]).

3.2 Lemmas.

Let p be any natural number.

We say that $(E, \{f_\nu\}_\nu)$ is a (D^P)-pair, where E is a compact $\mathcal{PS}^{\mathbf{k}}_{n+1}$-set, and $\{f_\nu\}_\nu \subset \mathcal{PS}^{\mathbf{k}}_{n+1}$ is a finite family of bounded functions on E, if:

there exists a finite family of pairs (π_α, I_α), where $\pi_\alpha : N_\alpha \to \mathbb{R}^{n+1}$ (N_α open in $\mathbb{R}^{n+1}$) is a $\mathbf{k}$-subanalytic modification of class C^p, I_α is a compact box in $\mathbb{R}^{n+1}_+ \cap N_\alpha$, such that $\bigcup_\alpha \pi_\alpha(I_\alpha) = E$ and for all α, ν the function $f_\nu \circ \pi_\alpha |$ reg I_α is a restriction of a function $\tilde{f}_{\nu,\alpha}$, which is zero or a $\mathbf{k}$-normal crossing of class C^p on N_α.

Remarks.

a) Boxes I_α do not have to be full-dimensional.

b) If each $(E_i, \{f_\nu\}_\nu)$ is a (D^p)-pair $(i = 1, ..., k.)$ and $E = \bigcup_{i=1}^k E_i$, then $(E, \{f_\nu\}_\nu)$ is a (D^p)-pair.

c) By Remark 2 from 2.4, it suffices that functions $\tilde{f}_{\nu,\alpha}$ be quotients of $\mathbf{k}$-normal crossings or zeroes.

d) If f_{ν_0} is continuous on E, then, for every α, $\tilde{f}_{\nu_0,\alpha}|I_\alpha = f_{\nu_0} \circ \pi_\alpha|I_\alpha$.

Lemma 3.2.1 *If $n = 0$, then every pair $(E, \{f_\nu\}_\nu)$ is a (D^p)-pair.*

Proof. By the Lion-Rolin preparation theorem and Remark b), we can assume that E is a point or a compact interval and $\{f_\nu\}_\nu$ is reducible on reg E. For $y \in$ reg E, where $E = [t_1, t_2]$,

$$f_\nu(y) = |y - \theta|^{\gamma_\nu} \cdot A_\nu \cdot V_\nu(|y - \theta|),$$

where $\gamma_\nu \in \mathbf{k}, \lambda_i, \tilde{\lambda}_j \in \mathbf{k}_+^*, a, b, A_\nu \in \mathbb{R}, V_\nu$ is a $\mathbf{k}$-analytic unit on a $\mathbf{k}$-subanalytic neighbourhood of a set $\overline{\psi(E)} \subset \mathbb{R}$ and $\psi : \mathbb{R} \ni y \mapsto |y - \theta| \in \mathbb{R}$. Moreover, $\theta \notin \operatorname{reg} E$.

Let

$$\pi : \mathbb{R} \ni y_1 \mapsto \epsilon \cdot |y_1|^l + \theta \in \mathbb{R},$$

where $\epsilon = \operatorname{sgn}\left(\frac{1}{2}(t_1 + t_2) - \theta\right)$ and l is some number from $2\mathbb{Z}$ which will be defined later. The mapping π is a composition of a $\mathbf{k}$-power function and an affine isomorphism.

Let $I = \tilde{\psi}(E)$, with $\tilde{\psi}(y) = |\psi(y)|^{\frac{1}{l}}$, and let us take an open $\mathbf{k}$-subanalytic $W \subset \mathbb{R}$ such that every $\tilde{V}_\nu$ is defined on W and $I \subset W$.

We can take the family $\{((\pi|W), I)\}$:

$\pi(I) = E$ and $f_\nu \circ \pi|_{\operatorname{reg} I}$ is a restriction of a function

$$y_1 \mapsto |y_1|^{l \cdot \gamma_\nu} \cdot A_\nu \cdot \tilde{V}_\nu(|y_1|^l),$$

which is zero or a $\mathbf{k}$-normal crossing of class C^p on W (see Remark c)) for sufficiently large l. The mapping $\pi|W$ is analytic (even polynomial). $\square$

Lemma 3.2.2 *If E is fat (i.e. $E = \overline{\operatorname{int} E}$), then $(E, \{f_\nu\}_\nu)$ is a (D^p)-pair with full-dimensional boxes I_α.*

Proof. We use induction on n. The case $n = 0$ was proved in Lemma 3.2.1. Let $n > 0$ and let us assume that the Lemma holds for $n - 1$.

By Corollary 3.1.4 and Remark b), we can assume that E is a closure of a full-dimensional cell on which the family $\{f_\nu\}_\nu$ is reducible with

$$R = (\theta(x), \{\phi_1(x), ..., \phi_s(x)\}, a(x), b(x), \{\lambda_i\}_{i=1}^r, \{\tilde{\lambda}_k\}_{k=1}^q),$$

that is, it has an R-reduction on E of the form

$$f_\nu(x, y) = |y - \theta(x)|^{\gamma_\nu} A_\nu(x) V_\nu(\psi(x, y)), \text{ where}$$

$$\psi(x, y) = (\phi_1(x), ..., \phi_s(x), |a(x) \cdot (y - \theta(x))|^{\lambda_1}, ..., |a(x) \cdot (y - \theta(x))|^{\lambda_r},$$
$$|\frac{b(x)}{y - \theta(x)}|^{\tilde{\lambda}_1}, ..., |\frac{b(x)}{y - \theta(x)}|^{\tilde{\lambda}_q}).$$

Extracting some bounded functions as new functions $\phi_i(x)$, we can assume that

$$0 \neq a(x) = \frac{1}{\max\{|t_1(x) - \theta(x)|, |t_2(x) - \theta(x)|\}}$$

$$b(x) = \min\{|t_1(x) - \theta(x)|, |t_2(x) - \theta(x)|\},$$

where t_1 and t_2 are functions generating the cylinder K. (We can assume that the number of functions $\phi_i(x)$ remains unchanged.)

By Corollary 3.1.4, there exists a finite family of cells with reducers $\{(K_\xi, \mathcal{R}_\xi)\}_{\xi\in\Xi}$ compatible with sets

$$F_\nu = \{(x,y) \in \mathbb{R}^{n+1} : \ |A_\nu(x)| < 1\}$$

such that $\{K_\xi\}_{\xi\in\Xi}$ is a decomposition of E and for every $\xi \in \Xi$ family $\{f_\nu\}_\nu$ is reducible on $(K_\xi, \mathcal{R}_\xi)$, where $R_\xi^{n+1} = R$ and functions $A_\nu(x)$ remain unchanged.

By Remark b), it suffices to prove the Lemma for the closures of full-dimensional cells $\overline{K_\xi}$. Fix such ξ (we will drop this index).

Let

$$A'_\nu(x) = \begin{cases} A_\nu(x), & K \subset F_\nu \\ \frac{1}{A_\nu(x)}, & K \cap F_\nu = \emptyset. \end{cases}$$

Denote by $\{g_\mu\}_\mu$ the family

$$\{a'(x), b(x), b'(x), \theta(x), \phi_1(x), ..., \phi_s(x)\} \cup \{A'_\nu(x)\}_\nu$$

of bounded $\mathcal{PS}_n^{\mathbf{k}}$-functions on the basis K' of the cell K, where $a'(x) = \frac{1}{a(x)}$ and $b'(x) = a'(x) - b(x)$.

We use the induction assumption for $(\overline{K'}, \{g_\mu\}_\mu)$.

There exists a finite family of pairs (Φ_β, J_β), where $\Phi_\beta : M_\beta \to \mathbb{R}^n$ (M_β open in $\mathbb{R}^n$) is a $\mathbf{k}$-subanalytic modification of class C^p, a J_β is a full-dimensional box in $\mathbb{R}^n_+ \cap M_\beta$, such that $\bigcup_\beta \Phi_\beta(J_\beta) = \overline{K'}$ and for all β, μ the function $g_\mu \circ \Phi_\beta|$ reg J_β is a restriction of a function $\tilde{g}_{\mu,\beta}$, which is zero or a $\mathbf{k}$-normal crossing of class C^p on M_β.

The mappings $\Psi_\beta = \Phi_\beta \times id_{\mathbb{R}}$ are $\mathbf{k}$-subanalytic modifications of class C^p.

Let $L_\beta = \{(u,y) \in$ int $J_\beta \times \mathbb{R}|\ (t_1 \circ \Phi_\beta)(u) < y < (t_2 \circ \Phi_\beta)(u)\}$. Then $\bigcup_\beta \Psi_\beta(\overline{L_\beta}) = \overline{K}$, so it suffices to prove the Lemma for each pair $(\overline{L_\beta}, \{f_\nu \circ \Psi_\beta\}_\nu)$. Fix β (we will drop this index).

Let $J = [\zeta_1, \eta_1] \times ... \times [\zeta_n, \eta_n]$ and $\mathcal{J} = \{j | \zeta_j = 0\}$, $\mathbf{k}_+^{\mathcal{J}} = \{\gamma \in \mathbf{k}_+^n |\ \gamma_j \neq 0 \Rightarrow j \in \mathcal{J}\}$. Set

$$\tilde{b}(u) = t_1(u)|u|^{c_1},$$

$$\tilde{a'}(u) = t_2(u)|u|^{c_2},$$
$$\tilde{b'}(u) = t_3(u)|u|^{c_3},$$

where $c_1, c_2, c_3 \in \mathbf{k}_+^{\mathcal{J}}$, functions t_1, t_2, t_3 are $\mathbf{k}$-analytic positive units on M, where t_1 can also be zero (in this case $c_1 = c_2$).

Set

$$C^1 = \{(u, y_1) \in \text{ int } J \times \mathbb{R}_+ | \; \tilde{b}(u) < y_1 < \tilde{a'}(u)\}$$
$$\text{and } \pi : M \times \mathbb{R} \ni (u, y_1) \mapsto (u, \epsilon \cdot y_1 + \tilde{\theta}(u)) \in M \times \mathbb{R},$$

where $\epsilon = \text{ sgn } (\frac{1}{2}(t_1(x) + t_2(x)) - \theta(x))$ for $x \in K$ (does not depend on x).

Then $\pi(\overline{C^1}) = \overline{L}$ and $\hat{f}_\nu(u, y_1) = (f_\nu \circ \Psi \circ \pi | \, C^1)(u, y_1) =$

$$= |y_1|^{\gamma_\nu} \tilde{A}_\nu(u) V_\nu(\tilde{\phi}_1(u), ..., \tilde{\phi}_s(u), |\frac{1}{\tilde{a'}}(x) \cdot (y - \tilde{\theta}(x))|^{\lambda_1}, ..., |\frac{1}{\tilde{a'}}(x) \cdot (y - \tilde{\theta}(x))|^{\lambda_r},$$

$$|\frac{\tilde{b}(x)}{y - \tilde{\theta}(x)}|^{\tilde{\lambda}_1}, ..., |\frac{\tilde{b}(x)}{y - \tilde{\theta}(x)}|^{\tilde{\lambda}_q}),$$

where

$$\tilde{A}_\nu = \begin{cases} \tilde{A'}_\nu & \text{when } A'_\nu = A_\nu \\ \frac{1}{\tilde{A}'_\nu} & \text{when } A'_\nu = \frac{1}{A_\nu} \end{cases} .$$

By the assumption, $\tilde{A}_\nu(u) = |u|^{\delta_\nu} h_\nu(u)$ z $\delta_\nu \in \mathbf{k}^{\mathcal{J}}$ and $h_\nu = 0$ or h_ν are $\mathbf{k}$-analytic units on M.

Extracting the units $\frac{1}{\tilde{a'}}, \tilde{b}$ as new functions $\tilde{\phi}_i$ and joining h_ν with V_ν, we get

$$\hat{f}_\nu(u, y_1) = |y_1|^{\gamma_\nu} |u|^{\delta_\nu} \hat{V}_\nu(u, \frac{y_1}{|u|^{c_2}}, \frac{|u|^{c_1}}{y_1}),$$

where $\hat{V}_\nu$ are zeroes or $\mathbf{k}$-analytic units.

By boudedness od the function $\frac{\tilde{b}(u)}{\tilde{a'}(u)}$ on J, $c_2 \le c_1$.

Let

$$C^2 = \{(u, y_2) \in \text{ int } J \times \mathbb{R} | \; t_1(u) \; |u|^{c_1 - c_2} < y_2 < t_2(u)\}$$

and

$$R_1 : \mathbb{R}^{n+1} \ni (u, y_2) \mapsto (u, y_2 \cdot |u|^{c_2}) \in \mathbb{R}^{n+1}$$

(a composition of blows-up and **k**-power functions). Then $R_1(\overline{C^2}) = \overline{C^1}$ and

$$(\hat{f}_\nu \circ R_1|\, C^2)(u, y_2) = |y_2|^{\gamma_\nu} |u|^{\omega_\nu} \hat{V}_\nu(u, y_2, \frac{|u|^{c_1-c_2}}{y_2}),$$

where $\omega_\nu = \delta_\nu + \gamma_\nu \cdot c_2$.

Case 1: $\tilde{b}(u) \neq 0,\ c_1 = c_2$.

Then y_2 is a unit on C^2 and $t_2 - t_1 = t_3 |u|^{c_3-c_2}$ $(c_3 \geq c_2)$.

Let us introduce the mappings

$$T : M \times \mathbb{R} \ni (u, y_3) \mapsto (u, y_3 \cdot t_3(u) + t_1(u)) \in M \times \mathbb{R}$$

$$R_2 : \mathbb{R}^{n+1} \ni (u, y_4) \mapsto (u, y_4 \cdot |u|^{c_3-c_2}) \in \mathbb{R}^{n+1}$$

and cells

$$C^3 = \{(u, y_3) \in \text{ int } J \times \mathbb{R}|\ 0 < y_3 < |u|^{c_3-c_2}\}$$

$$C^4 = \{(u, y_4) \in \text{ int } J \times \mathbb{R}|\ 0 < y_4 < 1\}.$$

Then $T(\overline{C^3}) = \overline{C^2}$, $R_2(\overline{C^4}) = \overline{C^3}$ and

$$(\hat{f}_\nu \circ R_1 \circ T \circ R_2|\, C^4)(u, y_4) = |u|^{\omega_\nu} |y_2(u, y_4)|^{\gamma_\nu} \hat{V}_\nu(u, y_2(u, y_4), \frac{1}{y_2}(u, y_4)).$$

Each of functions $(\hat{V}_\nu(u, y_2, \frac{1}{y_2}))(u, y_4)$ is zero or a **k**-analytic unit on a common open **k**-subanalytic neighbourhood $\hat{W}$ of a set $\overline{C^4}$. We substitute a **k**-power function P with sufficiently large exponents and set $N_\alpha = P^{-1}((M \times \mathbb{R}) \cap \hat{W})$, $\pi_\alpha = \Psi \circ \pi \circ R_1 \circ T \circ R_2 \circ P|\, N_\alpha$ and $I_\alpha = P^{-1}(\overline{C^4}) \cap \mathbb{R}^{n+1}_+$ Both the **k**-subanalytic modification π_α, and the desingularized functions are of class C^p.

Case 2: $\tilde{b}(u) = 0$.

Then $\hat{V}_\nu$ does not depend on the last variable. Applying the mapping

$$T' : M \times \mathbb{R} \ni (u, y_3) \mapsto (u, y_3 \cdot t_2(u)) \in M \times \mathbb{R}$$

we get

$$T'(\overline{C^3}) = \overline{C^2} \text{ and } (\hat{f}_\nu \circ R_1 \circ T')(u, y_3) = |y_3|^{\gamma_\nu} |u|^{\omega_\nu} |t_2(u)|^{\gamma_\nu} \hat{\hat{V}}_\nu(u, y_3)$$

for $(u, y_3) \in C^3 = (\text{ int } J) \times (0, 1)$, where $\hat{\hat{V}}_\nu$ are zeroes or **k**-analityc units on a common open **k**-subanalytic neighbourhood $\hat{W}$ of a set $\overline{C^3}$. We substitute a

$\mathbf{k}$-power function P with sufficiently large exponents and set $I_\alpha = P^{-1}(\overline{C^3}) \cap \mathbb{R}_+^{n+1}$, $N_\alpha = P^{-1}((M \times \mathbb{R}) \cap \hat{W})$, and $\pi_\alpha = \Psi \circ \pi \circ R_1 \circ T' \circ P|\, N_\alpha$. Both the $\mathbf{k}$-subanalytic modification π_α, and the desingularized functions are of class C^p.

Case 3: $c_1 \neq c_2$ *and* $\tilde{b}(u) \neq 0$.

We put

$$C^3 = \{(u, y_3) \in \text{ int } J \times \mathbb{R}: \ |u|^{c_1 - c_2} < y_3 < \frac{t_2}{t_1}(u)\} \text{ and}$$

$$T'': M \times \mathbb{R} \ni (u, y_3) \mapsto (u, y_3 \cdot t_1(u)) \in M \times \mathbb{R}.$$

Then

$$T''(\overline{C^3}) = \overline{C^2} \text{ and } (\hat{f}_\nu \circ R_1 \circ T''|\, C^3)(u, y_3) = |y_3|^{\gamma_\nu} |u|^{\omega_\nu} \hat{\hat{V}}_\nu(u, y_3, \frac{|u|^{c_1 - c_2}}{y_3}).$$

Making, if necessary, a decomposition of the cell C^3, we can assume that $|\eta|^{c_1 - c_2} < \frac{t_2}{t_1}(u)$ on C^3 or $\mathcal{J} = \emptyset$, reducing the situation to Case 1.

Let $0 = \kappa_0 < \kappa_1 < \ldots < \kappa_k = c_1 - c_2$, where $\kappa_j \in \mathbf{k}_+^{\mathcal{J}}$ and $\kappa_j - \kappa_{j-1} = \rho_j \cdot e_{i(j)}$, $\rho_j \in \mathbf{k}_+^*$. We set

$$C_0^3 = \{(u, y_3) \in \text{ int } J \times \mathbb{R}: \ |\eta|^{\kappa_k} < y_3 < \frac{t_2}{t_1}(u)\}$$

$$C_j^3 = \{(u, y_3) \in \text{ int } J \times \mathbb{R}: \ |\eta|^{\kappa_k - \kappa_j} |u|^{\kappa_j} < y_3 < |\eta|^{\kappa_k - \kappa_{j-1}} |u|^{\kappa_{j-1}}, j = 1, \ldots, k.$$

It suffices to prove the Lemma for each pair $(\overline{C_j^3}, \{\hat{f}_\nu \circ R_1 \circ T''\}_\nu)$, $j = 0, \ldots, k$.

For $j = 0$, Case 1 holds.

For $j = 1, \ldots, k$, we put

$$C_j^4 = \{(u, y_4) \in \text{ int } J \times \mathbb{R}: \ |\eta|^{\kappa_k - \kappa_j} |u_{i(j)}|^{\rho_j} < y_4 < |\eta|^{\kappa_k - \kappa_j} |\eta_{i(j)}|^{\rho_j}\}$$

$$R_2: \mathbb{R}^{n+1} \ni (u, y_4) \mapsto (u, y_4 \cdot |u|^{\kappa_{j-1}}) \in \mathbb{R}^{n+1}.$$

Then $R^2(\overline{C_j^4}) = \overline{C_j^3}$ and

$$(\hat{f}_\nu \circ R_1 \circ T'' \circ R_2|\, C_j^4)(u, y_4) = |y_4|^{\gamma_\nu} |u|^{\omega'_\nu} H_\nu(u, y_4, \frac{|u_{i(j)}|^{\rho_j}}{y_4}),$$

where $\omega_\nu' = \omega_\nu + \gamma_\nu \cdot |u|^{\kappa_j - 1}$, H_ν is zero or a **k**-analytic unit in a neighbourhood W of the set $\overline{\hat{\psi}(C_j^4)}$, where $\hat{\psi}(u, y_4) = (u, y_4, \frac{|u_{i(j)}|^{\rho_j}}{y_4})$. Moreover let

$$C_j^5 = \{(u, y_5) \in \text{ int } J \times \mathbb{R} : \ |u_{i(j)}|^{\rho_j} < y_5 < |\eta_{i(j)}|^{\rho_j}\}$$

$$T''' : \mathbb{R}^{n+1} \ni (u, y_5) \mapsto (u, y_5 \cdot |\eta|^{\kappa_k - \kappa_j}) \in \mathbb{R}^{n+1}.$$

Then $T'''(\overline{C_j^5}) = \overline{C_j^4}$. We can assume that $i(j) = n$. Set $J' = [\zeta_1, \eta_1] \times ... \times [\zeta_{n-1}, \eta_{n-1}]$ and

$$\hat{C}_j^5 = \{(\hat{u}, y_5) \in \text{ int } J' \times \mathbb{R}^2 |\ 0 < \hat{u}_n < 1, 0 < y_5 < \eta_n\}$$

$$\hat{R} : \mathbb{R}^{n+1} \ni (\hat{u}, y_5) \mapsto (\hat{u}_1, ..., \hat{u}_{n-1}, |\hat{u}_n \cdot y_5|^{\frac{1}{\rho_j}}, y_5) \in \mathbb{R}^{n+1}.$$

Then $\hat{R}(\overline{\hat{C}_j^5}) = \overline{C_j^5}$ and

$$(\hat{f}_\nu \circ R_1 \circ T'' \circ R_2 \circ T''' \circ \hat{R} | \hat{C}_j^5)(\hat{u}, y_5) = |y_5|^{l_\nu'} |\hat{u}|^{\omega_\nu'} H_\nu^\star(\hat{u}_1, ..., \hat{u}_{n-1}, |\hat{u}_n \cdot y_5|^{\frac{1}{\rho_j}}, y_5, \hat{u}_n),$$

where $l_\nu' = \gamma_\nu + (\omega_\nu')_n$. Thus the above functions are of the form $|\hat{z}|^{\sigma_\nu} \hat{H}_\nu(\hat{z})$, where $\hat{z} = (\hat{u}, y_5)$, $\sigma_\nu \in \mathbf{k}^{n+1}$, and every $\hat{H}_\nu$ is zero or a **k**-analytic unit in a common open **k**-subanalytic neighbourhood $\hat{W}$ of the set $\overline{\hat{C}_j^5}$. We substitute a **k**-power function P with sufficiently large exponents and set $I_\alpha = P^{-1}(\overline{\hat{C}_j^5}) \cap \mathbb{R}_+^{n+1}$, $N_\alpha = P^{-1}(\hat{W} \cap (\hat{R})^{-1}(M \times \mathbb{R}))$ and $\pi_\alpha = \hat{f}_\nu \circ R_1 \circ T'' \circ R_2 \circ T''' \circ \hat{R} \circ P | N_\alpha$. Both the **k**-subanalytic modification π_α, and the desingularized functions are of class C^p. □

Lemma 3.2.3 *For a bounded cylinder $C \subset \mathbb{R}^{n+1}$ with basis $B \subset \mathbb{R}^n$ and generating functions $t_1, t_2 \in \mathcal{PS}_n^{\mathbf{k}}$, if the pair $(\overline{B}, \{t_1, t_2\})$ is a (D^p)-pair, then for any finite family $\{f_\nu\}_\nu \subset \mathcal{PS}_{n+1}^{\mathbf{k}}$ of functions bounded on $\overline{C}$, the pair $(\overline{C}, \{f_\nu\}_\nu)$ is a (D^p)-pair.*

Proof. We can assume that $\{t_1, t_2\}$ is reducible on $(B, \mathcal{R})$ with some reducer $\mathcal{R}$. By the assumption, there exists a finite family of pairs (Φ_β, J_β), where $\Phi_\beta : M_\beta \to \mathbb{R}^n$ is a **k**-subanalytic modification of class C^p, J_β is a box in $\mathbb{R}_+^n \cap M_\beta$, such that $\bigcup_\beta \Phi_\beta(J_\beta) = \overline{B}$ and for each β the functions $t_i \circ \Phi_\beta |$ reg J_β $(i = 1, 2)$ are restrictions of functions $\tilde{t}_{i,\beta}$ that are zeroes or **k**-normal crossings of class C^p on M_β.

Let $\Psi_\beta = \Phi_\beta \times id_{\mathbb{R}}$.

The box J_β is isomorphic to a fat box $\tilde{J}_\beta$ in $\mathbb{R}^k$ $(0 \le k \le n)$ through the mapping $L_\beta : \mathbb{R}^k \supset \tilde{J}_\beta \to J_\beta \subset \mathbb{R}^n$. Let $L_{\beta,1} = L_\beta \times id_{\mathbb{R}}$. We can assume that $J_\beta = \{b\} \times \tilde{J}_\beta$, with $b \in \mathbb{R}_+^{n-k}$.

Case 1. C is of the form $\{(x,y) \in B \times \mathbb{R} |\, t_1(x) < y < t_2(x)\}$.

The proof of Case 1 is similar to the proof of Case 2.

Case 2. C is of the form $\{(x,y) \in B \times \mathbb{R} |\, t_1(x) = y = t_2(x)\}$.

We define

$$C_\beta = \{(u,y) \in \text{reg } J_\beta \times \mathbb{R} |\, (t_1 \circ \Phi_\beta)(u) = y\}$$

$$\tilde{C}_\beta = \text{ int } \tilde{J}_\beta \subset \mathbb{R}^k.$$

Then $\bigcup_\beta \Psi_\beta \circ L_{\beta,1} \circ (id_{\mathbb{R}^k}, \tilde{t}_{1,\beta})(\tilde{J}_\beta) = \overline{C}$.

We apply Lemma 3.2.2 do each pair $(\tilde{J}_\beta, \{f_\nu \circ \Psi_\beta \circ L_{\beta,1} \circ (id_{\mathbb{R}^k}, \tilde{t})\}_\nu)$, where $\tilde{t}(v) = \tilde{t}_{1,\beta}(b,v)$.

There exists a finite family of pairs $(\hat{\pi}_{\hat{\alpha}}, \hat{I}_{\hat{\alpha}})$, where $\hat{\pi}_{\hat{\alpha}} : \hat{N}_{\hat{\alpha}} \to \mathbb{R}^k$ is a **k**-subanalytic modification of class C^p, $\hat{I}_{\hat{\alpha}}$ is a fat box in $\mathbb{R}_+^k \cap \hat{N}_{\hat{\alpha}}$, such that $\bigcup_{\hat{\alpha}} \hat{\pi}_{\hat{\alpha}}(\hat{I}_{\hat{\alpha}}) = \tilde{J}_\beta$ and for every $\hat{\alpha}$ and ν the function $f_\nu \circ \Psi_\beta \circ L_{\beta,1} \circ (id_{\mathbb{R}^k}, \tilde{t}) \circ \hat{\pi}_{\hat{\alpha}}|$ int $\hat{I}_{\hat{\alpha}}$ is a restriction of the function $\tilde{f}_{\nu,\hat{\alpha}}$, which is zero or a **k**-normal crossing of class C^p on $\hat{N}_{\hat{\alpha}}$.

Let $N_\alpha = \mathbb{R}^{n-k} \times \hat{N}_{\hat{\alpha}} \times \mathbb{R}$, $I_\alpha = \{b\} \times \hat{I}_{\hat{\alpha}} \times \{0\}$. The mapping $\pi_\alpha = \Psi_\beta \circ \Gamma \circ (id_{\mathbb{R}^{n-k}} \times \hat{\pi}_{\hat{\alpha}} \times id_{\mathbb{R}}) : N_\alpha \to \mathbb{R}^{n+1}$, where $\Gamma(x_1, ..., x_{n+1}) = (x_1, ..., x_n, x_{n+1} + \tilde{t}(x_{n-k+1}, ..., x_n))$, is a **k**-subanalytic modification of class C^p.

Then $\bigcup_\alpha \pi_\alpha(I_\alpha) = \overline{C}$ and each of functions

$$f_\nu \circ \pi_\alpha | \text{ reg } I_\alpha = f_\nu \circ \Psi_\beta \circ L_{\beta,1} \circ (id_{\mathbb{R}^k}, \tilde{t}) \circ \hat{\pi}_{\hat{\alpha}} \circ \rho' | \text{ reg } I_\alpha,$$

where $\rho' : \mathbb{R}^{n+1} \ni (w,v,y) \mapsto v \in \mathbb{R}^k$, is zero or a restriction of a **k**-normal crossing on N_α. This proves that $(\overline{C}, \{f_\nu\}_\nu)$ is a (D^p)-pair. □

Corollary 3.2.4 *Each pair $(E, \{f_\nu\}_\nu)$ is a (D^p)-pair.*

Proof. If E is the closure of an analytic cell, then it suffices to use inductively Lemma 3.2.3. For any E, we use Remark b) and Corollary 3.1.4.

□

Remark. We can additionally obtain π_α as combinations of **k**-normal crossings. Indeed, it suffices to use the fact that $(I_\alpha, \{f_\nu \circ \pi_\alpha\} \cup \{\pi_{\alpha,1}, ..., \pi_{\alpha,n+1}\})$ is a (D^p)-pair for all α. In this case we can treat all functions as globally defined extending them by zero.

3.3 Rectilinearization and uniformization theorems.

Theorem 3.3.1 (on rectilinearization in $\mathbb{R}^n$) *Let $\{E_\nu\}_\nu$ be a finite family of $\mathbf{k}$-subanalytic bounded sets in $\mathbb{R}^n$ and let $f_\nu : E_\nu \to \mathbb{R}$ be a family of $\mathbf{k}$-subanalytic continuous functions. Then there exists a finite family of mappings $\Phi_\alpha : \mathbb{R}^{l(\alpha)} \to \mathbb{R}^n$ such that:*

a) every Φ_α is a composition of an inclusion $\mathbb{R}^{l(\alpha)} \to \mathbb{R}^n$ and global $\mathbf{k}$-subanalytic modification of class C^p, which is a combination of local $\mathbf{k}$-normal crossings,

b) there are compact boxes $L_\alpha \subset \mathbb{R}^{l(\alpha)}$ such that $\bigcup_\alpha \Phi_\alpha(L_\alpha)$ is a neighbourhood of a set $\bigcup_\nu \overline{E_\nu}$ and each of $\overline{E_\nu}$ is a union of some $\Phi_\alpha(L_\alpha)$

c) each of sets $\Phi_\alpha^{-1}(E_\nu)$ is a boolean combination of hyperplanes $H_i^l = \{x \in \mathbb{R}^l : x_i = 0\}$,

d) if $\Phi_\alpha^{-1}(E_\nu)$ is dense in $\mathbb{R}^{l(\alpha)}$, then every $f_\nu \circ \Phi_\alpha : \Phi_\alpha^{-1}(E_\nu) \to \mathbb{R}$ is a restriction of a function $f_\nu^\star : \mathbb{R}^{l(\alpha)} \to \mathbb{P}$, which is zero, a local $\mathbf{k}$-normal crossing of class C^p or an inverse of a local $\mathbf{k}$-normal crossing of class C^p.

Proof. Let F be a compact and $\mathbf{k}$-subanalytic neighbourhood of a set $\bigcup_\nu \overline{E_\nu}$. Each of sets E_ν is a boolean combination of $\mathbf{k}$-subanalytic compact sets E_ν^j, $j = 1, ..., r_\nu$. On F we consider a family of functions $d_\nu^j = \mathrm{dist}(\cdot, E_\nu^j)$.

By Remark b) from 3.2, we can assume that for every ν: (a) $|f_\nu| \leq 1$ on E_ν or (b) $|f_\nu| \leq 1$ on E_ν. Let

$$\hat{f}_\nu(x) = \begin{cases} f_\nu(x) & \text{for } x \in E_\nu, \text{ when (a)}, \\ \frac{1}{f_\nu(x)} & \text{for } x \in E_\nu, \text{ when (b) and not (a)}, \\ 0 & \text{when } x \notin E_\nu. \end{cases}$$

By Corollary 3.1.5 and the following Remark, there exists a stratification $\{S_\xi\}_\xi$ of $\mathbb{R}^n$ on analytic $\mathbf{k}$-subanalytic cells compatible with the family of sets $\{F\} \cup \{E_\nu^j\}$. Each pair $(\overline{S_\xi}, \{\hat{f}_\nu\} \cup \{d_\nu^j\})$ with S_ξ being maximal cells for some E_ν or for F [1], is a (D^p)-pair.

We can assume that $F = \overline{S_\xi}$ and drop the index ξ. There exists a finite family of pairs (ϕ_β, I_β), where $\phi_\beta : M_\beta \to \mathbb{R}^n$ is a $\mathbf{k}$-subanalytic modification of class C^p and a combination of $\mathbf{k}$-normal crossings, I_β is a compact box in $M_\beta \cap \mathbb{R}^n_+$ such that $\bigcup_\beta \phi_\beta(I_\beta) = \overline{S}$, $\bigcup_\beta \phi_\beta(\mathrm{reg}\ I_\beta) \subset S$ and for any ν, β the function $\hat{f}_\nu \circ \phi_\beta | \mathrm{reg}\ I_\beta$ is a restriction of a function $\tilde{f}_{\nu,\beta} : M_\beta \to \mathbb{R}$, which is zero or a $\mathbf{k}$-normal crossing of class C^p.

[1] i.e. open in some E_ν or in F

If $S \not\subset E_\nu$, then, for every β, $\phi_\beta^{-1}(E_\nu) \cap \operatorname{reg} I_\beta = \emptyset$.

If $S \subset E_\nu$, then by the continuity of functions $\hat{f}_\nu$ on E_ν,

$$\hat{f}_\nu \circ \phi_\beta = \tilde{f}_{\nu,\beta} \text{ on } \phi_\beta^{-1}(E_\nu) \cap I_\beta.$$

Moreover,

$$d_\nu^j \circ \phi_\beta = \tilde{d}_{\nu,\beta}^j \text{ on } I_\beta.$$

Thus each of sets $Z_{j,\nu,\beta} = (\phi_\beta|I_\beta)^{-1}(E_\nu^j) = (d_\nu^j \circ \phi_\beta|I_\beta)^{-1}(0)$ is a union of closed walls of codimension at most one in I_β.

For each box I_β, we apply Lemma 2.3.1: there is a finite family of semialgebraic modifications $m_{\mu,\beta} : \mathbb{R}^n \to \mathbb{R}^n$ being combinations of local $\mathbf{k}$-normal crossings and a compact box $L_\beta \subset \mathbb{R}^{\dim I_\beta}$ such that $\bigcup_\mu m_{\mu,\beta}(\mathbb{R}^{\dim I_\beta}) = \bigcup_\mu m_{\mu,\beta}(L_\beta) = I_\beta$.

Let $\alpha = (\beta, \mu)$, $L_\alpha = L_\beta$, $l(\alpha) = \dim I_\beta$ and $\Phi_\alpha = \phi_\beta \circ m_{\mu,\beta} \circ i_{l(\alpha)} : \mathbb{R}^{l(\alpha)} \to \mathbb{R}^n$.

The mapping Φ_α is a composition of an inclusion $i_{l(\alpha)}$ and a $\mathbf{k}$-subanalytic modification $\phi_\beta \circ m_{\mu,\beta}$ of class C^p, which is a composition local $\mathbf{k}$-normal crossings. Now, $\bigcup_\alpha \Phi_\alpha(L_\alpha) = F$ and $\Phi_\alpha^{-1}(E_\nu^j) = i_{l(\alpha)}^{-1} \circ m_{\mu,\beta}^{-1}(Z_{j,\nu,\beta})$ is a union of hyperplanes H_i^l or the whole space $\mathbb{R}^{l(\alpha)}$. Thus $\Phi_\alpha^{-1}(E_\nu)$ is a boolean combination of such hyperplanes.

If $\Phi_\alpha^{-1}(E_\nu)$ is dense, then $\hat{f}_\nu \circ \Phi_\alpha : \Phi_\alpha^{-1}(E_\nu) \to \mathbb{R}$ is a restriction of a function $\tilde{f}_{\nu,\beta} \circ m_{\mu,\beta} \circ i_{l(\alpha)} : \mathbb{R}^{l(\alpha)} \to \mathbb{R}$, which is a local $\mathbf{k}$-normal crossing of class C^p or zero. □

The above version of the rectilinearization theorem for sets bounded in $\mathbb{R}^n$ can be moved to manifolds:

Theorem 3.3.2 (rectilinearization theorem) *Let $\{E_\nu\}_\nu$ be a finite family of local $\mathbf{k}$-subanalytic sets on an analytyic manifold M of dimension n and let $f_\nu : E_\nu \to \mathbb{R}$ be a family of continuous and $\mathbf{k}$-subanalytic functions in M. Then there exists a locally finite family of mappings $\Phi_\alpha : \mathbb{R}^{l(\alpha)} \to M$ such that:*

a) every Φ_α is a composition of an inclusion $\mathbb{R}^{l(\alpha)} \to \mathbb{R}^n$ and a global $\mathbf{k}$-subanalytic modification of class C^p into M,

b) there are compact boxes $L_\alpha \subset \mathbb{R}^{l(\alpha)}$ such that $\bigcup_\alpha \Phi_\alpha(L_\alpha) = M$ and each of $\overline{E_\nu}$ is a union of some $\Phi_\alpha(L_\alpha)$,

c) each of sets $\Phi_\alpha^{-1}(E_\nu)$ is a boolean combination of hyperplanes $H_i^l = \{x \in \mathbb{R}^l : x_i = 0\}$, *d) if $\Phi_\alpha^{-1}(E_\nu)$ is dense in $\mathbb{R}^{l(\alpha)}$, then every $f_\nu \circ \Phi_\alpha : \Phi_\alpha^{-1}(E_\nu) \to \mathbb{R}$ is a restriction of the zero function, a local* $\mathbf{k}$*-normal crossing of class C^p or an inverse of a local* $\mathbf{k}$*-normal crossing of class C^p.* □

Corollary 3.3.3 (uniformization theorem) *Let E be a closed and locally* $\mathbf{k}$*-subanalytic set of dimension k on an analytic manifold M. Then there exists an analytic manifold N of dimension k and a proper mapping $f : N \to M$ which is locally* $\mathbf{k}$*-subanalytic of class C^p and such that $f(N) = E$.*

Proof. We substitute the projections of k-dimensional toruses onto the boxes $L_\alpha \subset \mathbb{R}^{l(\alpha)}$ in the mappings Φ_α from the rectilinearization theorem.

□

Remark. We can mix uniformization with rectilinearization of functions.

3.4 Subanalytic case.

In the subanalytic case, we can make parallel reasonings and obtain versions of classical desingularization theorems. It is due to the fact that rational exponents appearing in the Parusiński-Lion-Rolin formula always have a common divisor.

Theorem 3.4.1 (rectilinearization theorem) (cf. Desingularization II in [4], thm. 0.2 in [1], thm. 2.6 and thm. 2.7 in [7]) *Let $\{E_\nu\}_\nu$ be a finite family sets of subanalytic sets on an analytic manifold M of dimension n and let $f_\nu : E_\nu \to \mathbb{R}$ be a family of continuous, globally subanalytic functions on M. Then there exists a locally finite family of mappings $\Phi_\alpha : \mathbb{R}^{l(\alpha)} \to M$ such that:*

a) every Φ_α is a composition of an inclusion $\mathbb{R}^{l(\alpha)} \to \mathbb{R}^n$ and a global subanalytic modification into M,

b) there exist compact boxes $L_\alpha \subset \mathbb{R}^{l(\alpha)}$ such that $\bigcup_\alpha \Phi_\alpha(L_\alpha) = M$ and each of $\overline{E_\nu}$ is a union of some $\Phi_\alpha(L_\alpha)$,

c) each of sets $\Phi_\alpha^{-1}(E_\nu)$ is a boolean combination of hyperplanes $H_i^l = \{x \in \mathbb{R}^l : x_i = 0\}$,

d) if $\Phi_\alpha^{-1}(E_\nu)$ is dense in $\mathbb{R}^{l(\alpha)}$, then every $f_\nu \circ \Phi_\alpha : \Phi_\alpha^{-1}(E_\nu) \to \mathbb{R}$ is a restriction of a function $f_\nu^\star : \mathbb{R}^{l(\alpha)} \to \mathbb{P}$, which is zero, a local normal crossing or an inverse of a local normal crossing. □

Corollary 3.4.2 (uniformization theorem) (cf. Desingularization I in [4], thm. 0.1 in [1]) *Let E be a closed and subanalytic set of dimension k in an analytic manifold M. Then there exist: an analytic manifold N of dimension k and an analytic proper mapping $f : N \to M$ such that $f(N) = E$.* $\square$

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