REGULARITY OF THE PLURICOMPLEX GREEN FUNCTION WITH SEVERAL POLES

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ABSTRACT. We show that if Ω is a $C^{2,1}$ smooth, strictly pseudoconvex domain in $\mathbb{C}^n$ then the pluricomplex Green function for Ω with several fixed poles and positive weights is $C^{1,1}$.

1. INTRODUCTION

If Ω is a bounded domain in $\mathbb{C}^n$, $p^1, \dots, p^k \in \Omega$ are distinct and $\mu_1, \dots, \mu_k > 0$ then the corresponding pluricomplex Green function is given by

$$g = \sup \mathcal{B},$$

where

$$\mathcal{B} = \{v \in PSH(\Omega) : v < 0, \limsup_{z \to p^i}(u(z) - \mu_i \log|z - p^i|) < \infty,\ i = 1, \dots, k\}.$$

One can show that $g \in \mathcal{B}$, g is a maximal plurisubharmonic (psh) function in $\Omega \setminus \{p^1, \dots, p^k\}$ and

$$Mg = \frac{\pi^n}{n!2^n} \sum_i \mu_i \delta_{p^i}$$

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(see [Le]), where M is the complex Monge-Ampère operator. For smooth u

$$Mu = \det(\frac{\partial^2 u}{\partial z_i \partial z_{\overline{j}}})$$

and by [De] Mu can be well defined as a nonegative Borel measure if $u \in PSH(\Omega)$ and u is locally bounded near $\partial\Omega$.

In this paper we want to show the following regularity result:

Theorem 1.1. *Assume that Ω is $C^{2,1}$ smooth and strictly pseudoconvex. Then $g \in C^{1,1}(\Omega \setminus \{p^1, \dots, p^k\})$ and*

$$|\nabla^2 g(z)| \leq \frac{C}{\min_i |z - p^i|^2}, \quad z \in \Omega \setminus \{p^1, \dots, p^k\},$$

where C is a constant depending only on $\Omega, p^1, \dots p^k, \mu_1, \dots, \mu_k$.

One can treat it as a regularity result for the complex Monge-Ampère operator and indeed this the main tool in the proof. The obtained regularity is the best possible: as shown in [Co] and [EZ], the Green function for a ball with two poles and equal weights is not C^2.

The case $k = 1$ was treated in [Gu] and [Bł3]. In [Gu] the $C^{1,\alpha}$ regularity for $\alpha < 1$ was claimed. However, the proof contained an error (inequality (3.6) on p. 697 in [Gu] is false). Then in [Bł3], using some results from [Gu] and a method similar to the one used in [BT1] involving holomorphic automorphisms of a ball, the $C^{1,1}$ regularity was shown. Afterwards, in the correction to [Gu], a different method was used to show the $C^{1,\alpha}$ regularity.

Here we adapt the methods from [Gu] and [Bł3] for $k \geq 1$. This yields also a slightly different proof for $k = 1$, as instead of Lemma from [Bł3] we use a holomorphic mapping

$$z \longmapsto z + \frac{(z_1 - p_1^1) \dots (z_1 - p_1^k)}{(a_1 - p_1^1) \dots (a_1 - p_1^k)} h$$

(in appriopriate variables given by Lemma 3.2 below) which for $a \notin \{p^1, \dots, p^k\}$ and small $h \in \mathbb{C}^n$ fixes p^i and maps a to $a + h$.

To get a priori estimate for the second derivative on the boundary we follow the method from [CKNS] (used also in [Gu]). In fact, all quantitative estimates necessary to obtain the constant from Theorem 1.1 are included here. We only make use of existence result - Theorem 1.1 from [Gu] (it would even be enough to use Theorem 1 from [CKNS]).

By the way, we are also able to show the following regularity of g:

Theorem 1.2. *If Ω is hyperconvex then g is continuous as a function defined on the set*

$$(1.1) \qquad \{(z, p^1, \dots, p^k, \mu_1, \dots, \mu_k) \in \overline{\Omega} \times \Omega^k \times (\mathbb{R}_+)^k : z \neq p^i \neq p^j \quad \textit{if } i \neq j\},$$

where for $z \in \partial\Omega$ we set $g := 0$.

(Recall that Ω is called hyperconvex if there exists $\psi \in PSH(\Omega)$ with $\psi < 0$ and $\lim\limits_{z \to \partial\Omega} \psi(z) = 0$.)

Theorem 1.3. *Assume that*

$$\limsup_{z\to\partial\Omega}\frac{|g(z)|}{\operatorname{dist}(z,\partial\Omega)}<\infty.$$

Then

$$|\nabla g(z)|\le\frac{C}{\min_i|z-p^i|},\quad z\in\Omega\setminus\{p^1,\dots,p^k\},$$

where C is a constant depending only on Ω, $p^1,\dots,p^k$, $\mu_1,\dots,\mu_k$.

Notation. If $z=(z_1,\dots,z_n)\in\mathbb{C}^n$ then $x_i=\operatorname{Re}z_i$, $y_i=\operatorname{Im}z_i$. If $\zeta\in\mathbb{C}^n$, $|\zeta|=1$ then by $\partial^m_\zeta u(z)$ we will denote the m-th derivative of u in direction ζ at z. For the partial derivatives we will use the notation

$$u_{x_i}=\frac{\partial u}{\partial x_i},\quad u_{y_i}=\frac{\partial u}{\partial y_i},\quad u_i=\frac{\partial u}{\partial z_i},\quad u_{\bar{i}}=\frac{\partial u}{\partial\bar{z}_i}.$$

If we write

$$|\nabla u|\le f\quad\text{in an open } D\subset\mathbb{C}^n,$$

where f is locally bounded, nonnegative in D, then we mean that u is locally Lipschitz and the inequality holds almost everywhere ($|\nabla u|$ makes then sense by the Rademacher theorem). If we write $dd^cu\ge dd^c|z|^2$, in fact it means exactly that $u-|z|^2$ is psh. When proving existence of a constant depending only on given quantities, by $C_1,C_2,\dots$ we will denote positive constants depending only on those quantities and call them *under control*.

1. Basic estimates

Given a bounded domain Ω in $\mathbb{C}^n$, distinct poles $p^1,\dots,p^k\in\Omega$ and weights $\mu_1,\dots,\mu_k>0$ fix positive R,r,m and M so that for $i,j=1,\dots,k$

$$\Omega\subset B(p^i,R),$$

$$\overline{B}(p^i,r)\subset\Omega\ \text{ and }\ \overline{B}(p^i,r)\cap\overline{B}(p^j,r)=\emptyset,$$

$$m\le\mu_i\le M.$$

One can easily check the following estimates for g:

$$\sum_i\mu_i\log\frac{|z-p^i|}{R}\le g(z)<0,\quad z\in\Omega,$$

$$\mu_i\log\frac{|z-p^i|}{R}-(k-1)M\log(R/r)\le g(z)\le\mu_i\log\frac{|z-p^i|}{r},\quad z\in\overline{B}(p^i,r).$$

For ε with $0<\varepsilon<r$ define

$$\Omega^\varepsilon:=\Omega\setminus\bigcup_i\overline{B}(p^i,\varepsilon)$$

and

$$g^\varepsilon := \sup\{v \in PSH(\Omega) : v < 0, v|_{\overline{B}(p^i,\varepsilon)} \le \mu_i \log(\varepsilon/r),\ i = 1, \dots, k, \}.$$

One can easily check that

$$g^\varepsilon(z) \le \mu_i \log \frac{\max\{|z-p^i|\varepsilon\}}{r}, \quad z \in \overline{B}(p^i, r), \tag{2.1}$$

$$g^\varepsilon \in PSH(\Omega),$$

$$g \le g^\varepsilon \le \frac{\log(r/\varepsilon)}{\log(R/\varepsilon) + (k-1)\frac{M}{m}\log(R/r)} g \ \text{ in } \Omega^\varepsilon, \tag{2.2}$$

$g^\varepsilon \downarrow g^0 := g$ as $\varepsilon \downarrow 0$ and the convergence is locally uniform in $\Omega \setminus \{p^1, \dots, p^k\}$.

Proposition 2.1. *Assume that Ω is C^∞ smooth and strictly pseudoconvex. Then there exists r_0 depending only on k, r, R, m and M, $0 < r_0 \le r$, such that for ε with $0 < \varepsilon < r_0$ we can find $v \in PSH(\Omega) \cap C^\infty(\overline{\Omega})$ with $dd^c v \ge dd^c|z|^2$ in Ω, $v = 0$ on $\partial\Omega$ and for $i = 1, \dots, k$*

$$\mu_i \log(\varepsilon/r) \le v(z) \le \mu_i \log \frac{|z-p^i|}{r} \quad \textit{if} \ \ \varepsilon \le |z-p^i| \le r.$$

Proof. Set

$$w(z) := \sum_i \mu_i \log \frac{|z-p^i|}{R} + |z-p^1|^2 - R^2,$$

so that $w < 0$ on $\overline{\Omega}$, $dd^c v \ge dd^c|z|^2$ and $w < \mu_i \log(\varepsilon/r)$ on $\partial B(p^i, \varepsilon)$. On the other hand, for $z \in \partial B(p^i, r)$ we have

$$w(z) \ge kM \log(r/R) + r^2 - R^2 > \mu_i \log(\varepsilon/r) + |z-p^i|^2 - \varepsilon^2,$$

provided that ε is such that

$$m \log(\varepsilon/r) - \varepsilon^2 < kM \log(r/R) - R^2.$$

Similarly as in [Bł2], let $\chi : \mathbb{R} \to \mathbb{R}$ be C^∞ smooth and such that

$$\begin{aligned} \chi(t) &= 0, \quad t \le -1, \\ \chi(t) &= t, \quad t \ge 1, \\ 0 \le \chi'(t) &\le 1, \quad t \in \mathbb{R}, \\ \chi''(t) &\ge 0, \quad t \in \mathbb{R}. \end{aligned}$$

For $x, y \in \mathbb{R}$ set

$$f_j(x,y) := x + \frac{1}{j}\chi(j(y-x)),$$

so that

$$f_j(x,y) = \max\{x,y\} \quad \text{if} \quad |x-y| \geq \frac{1}{j}.$$

If u, v are psh functions with $dd^c u, dd^c v \geq dd^c|z|^2$, then

$$dd^c f_j(u,v) \geq (1-\chi'(j(v-u)))dd^c u + \chi'(j(v-u))dd^c v \geq dd^c|z|^2.$$

Let ψ be a defining function for Ω. If we choose j, A sufficiently big then the function

$$v(z) = \begin{cases} f_j(w(z), \mu_i \log(\varepsilon/r) + |z-p^i|^2 - \varepsilon^2), & z \in \bigcup_i \overline{B}(p^i, r), \\ f_j(w(z), A\psi(z)) & z \in \overline{\Omega} \setminus \bigcup_i \overline{B}(p^i, r) \end{cases}$$

has all the required properties. □

Note that if $k=1$ then we may choose $r_0 = r$ in Proposition 2.1.

Proof of Theorem 1.2. By (2.2) $g^\varepsilon \to g$ locally uniformly on the set (1.1) as $\varepsilon \to 0$. It is thus enough to show that for a fixed small ε g^ε is continuous as a function defined on

$$\overline{\Omega} \times \{(p^1, \ldots, p^k) \in \Omega^k : \mathrm{dist}(p^i, \partial\Omega) > \varepsilon, |p^i - p^j| > 2\varepsilon \ \text{ if } \ i \neq j\} \times (\mathbb{R}_+)^k.$$

Let $p^{i,j} \to p^i, \mu_{i,j} \to \mu_i$ as $j \to \infty$, $i = 1, \ldots, k$, and

$$g_j^\varepsilon := \sup\{v \in PSH(\Omega) : v < 0, v|_{\overline{B}(p^{i,j},\varepsilon)} \leq \mu_{i,j} \log(\varepsilon/r)\}.$$

Note that if $0 < \varepsilon < r_0$ and j is big enough then by Proposition 2.1 applied to a ball containing Ω we have $\lim\limits_{z \to \partial B(p^i,\varepsilon)} g_j^\varepsilon(z) = \mu_i \log(\varepsilon/r)$. Moreover, $\lim\limits_{z \to \partial\Omega} g_j^\varepsilon(z) = 0$, since Ω is hyperconvex. Therefore, by a result from [Wa] (see also [Bł1], Theorem 1.5), g_j^ε is continuous on $\overline{\Omega}$.

To finish the proof it is enough to show that $g_j^\varepsilon \to g^\varepsilon$ uniformly as $j \to \infty$ in $\overline{\Omega}$. Fix $c > 0$. For $z \in \overline{B}(p^i, \varepsilon)$ and j big enough by (2.1) we have

$$g_j^\varepsilon(z) \leq \mu_{i,j} \log \frac{\max\{|z-p^{i,j}|, \varepsilon\}}{r} \leq \mu_{i,j} \log \frac{\varepsilon + |p^i - p^{i,j}|}{r} \leq \mu_i \log(\varepsilon/r) + c,$$

whereas for $z \in \overline{B}(p^{i,j}, \varepsilon)$

$$g^\varepsilon(z) \leq \mu_i \log \frac{\max\{|z-p^i|, \varepsilon\}}{r} \leq \mu_i \log \frac{\varepsilon + |p^i - p^{i,j}|}{r} \leq \mu_{i,j} \log(\varepsilon/r) + c.$$

Thus for those j

$$g^\varepsilon - c \leq g_j^\varepsilon \leq g^\varepsilon + c \quad \text{on} \quad \overline{\Omega}$$

and the theorem follows. □

In the proof of Theorem 1.1 we will also need to approximate g^ε. If $0 \leq \varepsilon < r$ and $0 \leq \delta \leq 1$ define

$$g^{\varepsilon,\delta} := \sup\{v \in PSH \cap L^\infty(\Omega) : v \leq g^\varepsilon, \ Mv \geq \delta \ \text{ in } \Omega^\varepsilon\}.$$

Note that $g^{\varepsilon,\delta}$ is increasing in ε and decreasing in δ. We also have

$$g^\varepsilon + \delta(|z-p^1|^2 - R^2) \leq g^{\varepsilon,\delta} \leq g^\varepsilon. \tag{2.3}$$

Proposition 2.2. *$g^{\varepsilon,\delta} \in PSH(\Omega)$, $Mg^{\varepsilon,\delta} = \delta$ in Ω^ε. If Ω is hyperconvex and $0 < \varepsilon < r_0$ then $g^{\varepsilon,\delta}$ is continuous on $\overline{\Omega}$. If Ω is C^∞ smooth and strictly pseudoconvex, $0 < \varepsilon < r_0$ and $0 < \delta \le 1$ then $g^{\varepsilon,\delta} \in C^\infty(\overline{\Omega^\varepsilon})$.*

Proof. We use standard procedures. Let

$$\mathcal{B} = \{v \in PSH(\Omega) : v \le g^\varepsilon,\ Mv \ge \delta \ \text{ in } \Omega^\varepsilon\}.$$

By the Choquet lemma there exists a sequence $v_j \in \mathcal{B}$ such that $(g^{\varepsilon,\delta})^* = (\sup_j v_j)^*$. ($u^*$ denotes the upper semicontinuous regularization of u.) If $w_j = \max\{v_1, \dots, v_j\}$ then $Mw_j \ge \delta$ in Ω^ε (see e.g. [Bł2]) and thus $w_j \in \mathcal{B}$. Therefore $w_j \uparrow (g^{\varepsilon,\delta})^*$ almost everywhere and by the approximation theorem from [BT2] $M(g^{\varepsilon,\delta})^* \ge \delta$ in Ω^ε. We conclude that $g^{\varepsilon,\delta} \in PSH(\Omega)$ and $Mg^{\varepsilon,\delta} \ge \delta$ in $\Omega\varepsilon$. The balayage procedure gives $Mg^{\varepsilon,\delta} = \delta$ in Ω^ε.

Now assume that Ω is hyperconvex and $0 < \varepsilon < r_0$. By [Bł1] there exists $\psi \in PSH(\Omega) \cap C(\overline{\Omega})$ with $\psi = 0$ on $\partial\Omega$ and $M\psi \ge 1$ in Ω. For A big enough

$$A\psi \le g^{\varepsilon,\delta} \le 0 \quad \text{in } \ \Omega. \tag{2.4}$$

Let v be given by Proposition 2.1 applied to a ball containing Ω. Then

$$v(z) \le g^{\varepsilon,\delta}(z) \le \mu_i \log \frac{|z - p^i|}{r} \quad \text{if } \ \varepsilon \le |z - p^i| \le r. \tag{2.5}$$

For small $h \in \mathbb{C}^n$ and $z \in \Omega^\varepsilon$ with $|h| < \operatorname{dist}(z, \partial\Omega^\varepsilon) < 2|h|$ we have

$$|g^{\varepsilon,\delta}(z+h) - g^{\varepsilon,\delta}(z)| \le C(|h|).$$

By the comparison principle (see [BT2]) applied to $g^{\varepsilon,\delta}$ and $g^{\varepsilon,\delta}(\cdot + h)$ the above inequality holds for all z with $\operatorname{dist}(z, \partial\Omega^\varepsilon) > |h|$. By (2.4) and (2.5)

$$\lim_{h \to 0} C(|h|) = 0$$

which means that $g^{\varepsilon,\delta}$ is continuous.

The last part of the proposition follows from Proposition 2.1 and Theorem 1.1 from [Gu]. Instead of [Gu] one can use Theorem 1 from [CKNS] (p. 211) and Theorems 4.1, 4.2 below. □

3. Gradient estimates

Theorem 1.3 will follow immediately from the next result applied to $\delta = 0$.

Theorem 3.1. *Fix $0 \le \delta \le 1$. Assume that*

$$\limsup_{z \to \partial\Omega} \frac{|g^{0,\delta}(z)|}{\operatorname{dist}(z, \partial\Omega)} \le B < \infty.$$

Then for ε satisfying Proposition 2.1 we have

$$|\nabla g^{\varepsilon,\delta}(z)| \le \frac{C}{\min_i |z - p^i|}, \quad z \in \Omega^\varepsilon,$$

where C is a constant depending only on n, k, R, r, m, M and B.

The assumption of Theorem 3.1 is satisfied uniformly for $\delta \le 1$ for example if Ω is smooth and strictly pseudoconvex.

Proof of Theorem 3.1. Let $\rho > 0$ be such that

$$-g^{\varepsilon,\delta}(z) \le -g^{0,\delta}(z) \le 2B \operatorname{dist}(z, \partial\Omega) \ \text{ if } \ \operatorname{dist}(z, \partial\Omega) \le \rho.$$

For h sufficiently small

$$g^{\varepsilon,\delta}(z+h) - g^{\varepsilon,\delta}(z) \le 2B|h| \quad \text{if} \quad \operatorname{dist}(z, \partial\Omega) = |h|,$$

and, since by Proposition 2.1

$$\mu_i \log(\varepsilon/r) \le g^{\varepsilon,\delta}(z) \le \mu_i \log \frac{|z-p^i|}{r} \quad \text{if } \ \varepsilon \le |z-p^i| \le r,$$

we have

$$g^{\varepsilon,\delta}(z+h) - g^{\varepsilon,\delta}(z) \le \mu_i \log \frac{|z-p^i+h|}{\varepsilon} \le 2\frac{\mu_i}{\varepsilon}|h|, \qquad \text{if } \ z \in \partial B(p^i, \varepsilon + |h|), \ i = 1, \dots, k.$$

From the comparison principle we get

$$g^{\varepsilon,\delta}(z+h) - g^{\varepsilon,\delta}(z) \le 2\max\{B, \frac{M}{\varepsilon}\}|h| \quad \text{if} \quad \||h| \le \min\{\rho, \operatorname{dist}(z, \partial\Omega^\varepsilon)\},$$

and thus

$$|\nabla g^{\varepsilon,\delta}| \le \frac{C_1}{\varepsilon} \quad \text{in } \Omega^\varepsilon. \tag{3.1}$$

We will need a lemma:

Lemma 3.2. *There exists a constant $\widetilde{C} = \widetilde{C}(k,n)$ such that for given $p^1, \dots, p^k \in \mathbb{C}^n$, $a \in \mathbb{C}^n \setminus \{p^1, \dots, p^k\}$ we can orthonormally change variables in $\mathbb{C}^n$ so that*

$$|a - p^i| \le \widetilde{C}|a_1 - p^i_1|, \quad i = 1, \dots, k.$$

Proof. By S denote the unit sphere in $\mathbb{C}^n$. We have to show that there exists $b \in S$ such that

$$|a - p^i| \le \widetilde{C}|\langle a - p^i, b\rangle|, \quad i = 1, \dots, k,$$

that is that

$$|\langle \frac{a-p^i}{|a-p^i|}, b\rangle| \ge \frac{1}{\widetilde{C}}.$$

Define

$$\widetilde{C} := \frac{1}{\min_{S^k} f},$$

where
$$f(\zeta^1,\dots,\zeta^k) := \max_{b\in S}\min_i |\langle \zeta^i, b\rangle|$$
is a continuous function on S^k. It remains to show that $f > 0$ on S^k. Fix $\zeta^1,\dots,\zeta^k \in S$ and define $K_i := \{b \in S : \langle b, \zeta^i\rangle = 0\}$, $i = 1,\dots,k$. Then $\bigcup_i K_i \neq S$ and thus for $b \in S\setminus\bigcup_i K_i$ we have
$$f(\zeta^1,\dots,\zeta^k) \geq \min_i |\langle \zeta^i, b\rangle| > 0. \qquad \square$$

End of proof of Theorem 3.1. Fix $a \in \Omega^\varepsilon$ and choose variables as in Lemma 3.2. Set
$$P(\lambda) := (\lambda - p_1^1)\dots(\lambda - p_1^k),$$
so that
$$\frac{|P(z_1)|}{|P(a_1)|} \leq C_2\frac{\min_i|z-p^i|}{\min_i|a-p^i|} \leq \frac{C_3}{\min_i|a-p^i|},\quad z\in\Omega.$$
For h sufficiently small let
$$\Omega'' := \{z\in\Omega : z + \frac{P(z_1)}{P(a_1)}h \in \Omega\}$$
and
$$\Omega' := \Omega''\setminus\bigcup_i \overline{B}(p^i,\varepsilon+\varepsilon'),$$
where
$$\varepsilon' = \min\{\varepsilon, r-\varepsilon, \operatorname{dist}(a,\partial\Omega^\varepsilon),\rho\}.$$
Set
$$v(z) := g^{\varepsilon,\delta}\left(z + \frac{P(z_1)}{P(a_1)}h\right) + \frac{C_4}{\min_i|a-p^i|}(|z-p^1|^2 - R^2)|h|,$$
so that if f C_4 is big enough then
$$Mv \geq \left|1 + \frac{P'(z_1)}{P(a_1)}h_1\right|^2\delta + \frac{C_4}{\min_i|a-p^i|}|h| \geq \delta.$$
For $z\in\partial\Omega''$ we have
$$v(z) - g^{\varepsilon,\delta}(z) \leq 2B\operatorname{dist}(z,\partial\Omega) \leq 2B\frac{C_3}{\min_i|a-p^i|}|h|,$$
whereas for $z\in\partial B(p^i,\varepsilon+\varepsilon')$
$$v(z) - g^\varepsilon(z) \leq \frac{C_1}{\varepsilon}\frac{|P(z_1)|}{|P(a_1)|}|h| \leq C_1C_2\frac{\varepsilon+\varepsilon'}{\varepsilon\min_i|a-p^i|} \leq \frac{C_5}{\min_i|a-p^i|}|h|.$$
Therefore, the comparison principle gives
$$g^\varepsilon(a+h) - g^\varepsilon(a) \leq \frac{C_6}{\min_i|a-p^i|}|h| \quad\text{if}\quad |h| \leq \varepsilon'\frac{\min_i|a-p^i|}{C_3}$$
and the theorem follows. $\square$

4. Estimates of the second derivative

Our goal will be to estimate $|\nabla^2 g^{\varepsilon,\delta}|$ for small ε, δ. First, we need such an estimate on $\partial\Omega^\varepsilon$. We will follow the method from [CKNS] (see also [Gu]). We shall prove two theorems.

Theorem 4.1. *Let Ω be a bounded strictly pseudoconvex domain in $\mathbb{C}^n$ and ψ a C^∞ psh defining function for Ω. Assume that $dd^c\psi \geq dd^c|z|^2$ and that there are positive constants A, a such that*

$$|\psi|, |\nabla\psi|, |\nabla^2\psi|, |\nabla^3\psi| \leq A \quad on\ \overline{\Omega},$$

$$|\nabla\psi| \geq a \quad on\ \partial\Omega.$$

For $\rho > 0$ denote $U = \{z \in \mathbb{C}^n : \operatorname{dist}(z, \partial\Omega) < \rho\}$. Let $u \in PSH(\Omega\cap U)\cap C^\infty(\overline{\Omega}\cap U)$ be such that $u = 0$ on $\partial\Omega$ and $u < 0$, $Mu = \delta$ in $\Omega \cap U$, where $0 < \delta \leq \delta_0$. Assume also that there are positive constants b, B such that

$$|\nabla u| \geq b \quad on\ \partial\Omega,$$

$$|\nabla u| \leq B \quad on\ \overline{\Omega} \cap U.$$

Then there is a constant $C = C(n, \rho, a, A, b, B, \delta_0)$ such that

$$|\nabla^2 u| \leq C \quad on\ \partial\Omega.$$

Theorem 4.2. *Fix $\alpha > 1$ and let $\Omega = \{z \in \mathbb{C}^n : 1 < |z| < \alpha\}$. Assume that $u \in PSH(\Omega) \cap C^\infty(\overline{\Omega})$ is such that $u = 0$ on ∂B_1 ($B_\alpha = B(0, \alpha)$), $u > 0$, $Mu = \delta > 0$ in Ω. Suppose moreover that there are positive constants β, b, B such that*

$$u \geq \beta \quad on\ \partial B_\alpha,$$

$$|\nabla u| \geq b \quad on\ \partial B_1,$$

$$|\nabla u| \leq B \quad on\ \overline{\Omega}.$$

Then there exist positive constants $\delta_0 = \delta_0(n, \alpha, \beta)$ and $C = C(n, \alpha, \beta, b, B)$ such that if $0 < \delta \leq \delta_0$, we have

$$|\nabla^2 u| \leq C \quad on\ \partial B_1.$$

Proof of Theorem 4.1. If $z_0 \in \partial\Omega$ then by N_{z_0} we will denote the outer normal of $\partial\Omega$ at z_0. For $z \in \overline{\Omega}$ we have

$$\partial_{N_{z_0}}\psi(z) = \langle \nabla\psi(z), \nabla\psi(z_0)/|\nabla\psi(z_0)|\rangle \geq |\nabla\psi(z_0)| - A|z - z_0|.$$

Hence, there are positive constants $\widetilde{a}, \widetilde{\rho}$ under control such that

$$\partial_{N_{z_0}}\psi(z) \geq \widetilde{a}, \quad z_0 \in \partial\Omega,\ z \in \overline{\Omega} \cap \overline{B}(z_0, \widetilde{\rho}).$$

Fix $z_0 \in \partial\Omega$. We may assume that $N_{z_0} = (0, \dots, 0, 1)$, so that $\partial_{N_{z_0}} = \partial/\partial x_n$. Since both ψ and u are C^∞ defining functions for Ω, there exists a C^∞ function v, defined in a neighborhood of $\partial\Omega$ such that $u = v\psi$ and $v > 0$ on $\overline{\Omega} \cap U$. Therefore, if $t, s \in \{x_1, y_1, \dots, x_{n-1}, y_{n-1}, y_n\}$, then

$$u_{ts}(z_0) = \frac{u_{x_n}(z_0)\psi_{ts}(z_0)}{\psi_{x_n}(z_0)} \tag{4.1}$$

and thus

$$|u_{ts}(z_0)| \le C_1. \tag{4.2}$$

Suppose now that we know that

$$|u_{tx_n}(z_0)| \le C_2 \tag{4.3}$$

and we want to estimate $|u_{x_n x_n}(z_0)|$. We have

$$u_{x_n x_n} = 4u_{n\overline{n}} - u_{y_n y_n}$$

and, by (4.1), (4.2), (4.3) and since $dd^c\psi \ge dd^c|z|^2$,

$$\delta_0 \ge \delta = \det(u_{i\overline{j}}(z_0)) \ge u_{n\overline{n}}(z_0)(a/A)^{n-1} - C_3.$$

It thus remains to show (4.3). On $\overline{\Omega} \cap \overline{B}(z_0, \widetilde{\rho})$ define

$$T := u_t - \frac{\psi_t}{\psi_{x_n}} u_{x_n},$$

so that

$$T = 0 \quad \text{on } \partial\Omega \cap \overline{B}(z_0, \widetilde{\rho}). \tag{4.4}$$

We have

$$T_{x_n}(z_0) = u_{tx_n}(z_0) - \frac{\psi_{tx_n}(z_0)}{\psi_{x_n}(z_0)} u_{x_n}(z_0),$$

and thus it is enough to prove that

$$|T_{x_n}(z_0)| \le C_4.$$

Set $f := \psi_t/\psi_{x_n}$, then

$$|\nabla f|, |\nabla^2 f| \le C_5 \quad \text{in } \overline{\Omega} \cap \overline{B}(z_0, \widetilde{\rho}). \tag{4.5}$$

Since $\det(u_{i\overline{j}})$ is constant, one can show that

$$u^{i\overline{j}} u_{i\overline{j}t} = u^{i\overline{j}} u_{i\overline{j}x_n} = 0.$$

(Here $(u^{i\bar{j}})$ denotes the inverse transposed matrix of $(u_{i\bar{j}})$.) Hence, we can compute

$$u^{i\bar{j}}T_{i\bar{j}} = -u_{x_n}u^{i\bar{j}}f_{i\bar{j}} - 2\mathrm{Re}\, u^{i\bar{j}}u_{ix_n}f_{\bar{j}} = -u_{x_n}u^{i\bar{j}}f_{i\bar{j}} - 2f_{x_n} - 2\mathrm{Im}\, u^{i\bar{j}}u_{iy_n}f_{\bar{j}}.$$

Since

$$u^{i\bar{j}}(u_{y_n}^2)_{i\bar{j}} = 2u^{i\bar{j}}u_{iy_n}u_{\bar{j}y_n},$$

the Schwartz inequality and (4.5) give

$$u^{i\bar{j}}(\pm T + \frac{1}{2}u_{y_n}^2)_{i\bar{j}} \geq \mp u_{x_n}u^{i\bar{j}}f_{i\bar{j}} \mp 2f_{x_n} - u^{i\bar{j}}f_i f_{\bar{j}} \geq -C_6(\sum_i u^{i\bar{i}} + 1).$$

On $\partial\Omega$ we have $u_{y_n} = u_{x_n}\psi_{y_n}/\psi_{x_n}$ and thus by (4.4)

$$|\pm T + \frac{1}{2}u_{y_n}^2| \leq C_7|z - z_0|^2, \quad z \in \partial\Omega \cap \overline{B}(z_0, \widetilde{\rho}).$$

Moreover,

$$|\pm T + \frac{1}{2}u_{y_n}^2| \leq C_8 \ \text{ in } \overline{\Omega} \cap \overline{B}(z_0, \widetilde{\rho}),$$

and we obtain that if $w = \pm T + \frac{1}{2}u_{y_n}^2 - C_9|z - z_0|^2$, where C_9 is big enough, then $w \leq 0$ on $\partial(\Omega \cap B(z_0, \widetilde{\rho}))$ and

$$u^{i\bar{j}}w_{i\bar{j}} \geq -C_{10}(\sum_i u^{i\bar{i}} + 1).$$

Therefore, if C_{11} and C_{12} are big enough then $w + C_{11}\psi + C_{12}u \leq 0$ on $\partial(\Omega \cap B(z_0, \widetilde{\rho}))$ and $u^{i\bar{j}}(w + C_{11}\psi + C_{12}u)_{i\bar{j}} \geq 0$ in $\Omega \cap B(z_0, \widetilde{\rho})$. By the maximum principle $w + C_{11}\psi + C_{12}u \leq 0$ in $\Omega \cap B(z_0, \widetilde{\rho})$ and thus

$$|T_{x_n}(z_0)| \leq C_{11}A + C_{12}B. \quad \square$$

Proof of Theorem 4.2. Set

$$\psi(z) = \lambda(|z|^2 - 1),$$

where $\lambda = \beta/(\alpha^2 - 1)$, so that $\psi \leq u$ in Ω for δ sufficiently small. We now follow the proof of Theorem 4.1. Fix $z_0 \in \partial B_1$, we may assume that $z_0 = (0, \dots, 0, 1)$. We may reduce the problem to the estimate

$$|u_{tx_n}(z_0)| \leq C_1.$$

Similarly as before we get that if $w = \pm T + \frac{1}{2}u_{y_n}^2 - C_2|z - z_0|^2$, where C_2 is big enough, then

$$u^{i\bar{j}}w_{i\bar{j}} \geq -C_3(\sum_i u^{i\bar{i}} + 1) \ \text{ in } \Omega \cap B(z_0, 1)$$

and $w \leq 0$ on $\partial(\Omega \cap B(z_0, 1))$.

Now by the inequality between arithmetic and geometric means we have

$$u^{i\bar{j}}(\psi - u)_{i\bar{j}} \geq \lambda \sum_i u^{i\bar{i}} - n \geq \frac{\lambda}{2} \sum_i u^{i\bar{i}} + n(\frac{\lambda}{2\delta^{1/n}} - 1) \geq \frac{\lambda}{2}(\sum_i u^{i\bar{i}} + 1)$$

for δ small enough. Thus

$$u^{i\bar{j}}(w + C_4(\psi - u))_{i\bar{j}} \geq 0 \text{ in } \Omega \cap B(z_0, 1)$$

if C_4 is sufficiently big and by the maximum principle we conclude that

$$|T_{x_n}(z_0)| \leq C_4 B. \quad \square$$

Proof of Theorem 1.1. Let ψ be a $C^{2,1}$ defining function for Ω with $dd^c\psi \geq dd^c|z|^2$ in Ω and

$$|\psi|, |\nabla\psi|, |\nabla^2\psi|, |\nabla^3\psi| \leq A \text{ on } \overline{\Omega},$$

$$|\nabla\psi| > a \text{ on } \partial\Omega$$

for some positive a and A. We can find $\widetilde{r} > 0$ such that for every $z_0 \in \partial\Omega$ there exists a ball $B(z_1, 2\widetilde{r})$, contained in Ω and tangent to $\partial\Omega$ at z_0. Then

$$g(z) \leq -\frac{\gamma}{\log 2} \log \frac{|z - z_1|}{2\widetilde{r}} \quad \text{if} \quad \widetilde{r} \leq |z - z_1| \leq 2\widetilde{r},$$

where

$$\gamma = \max_{\mathrm{dist}(z,\partial\Omega) \geq \widetilde{r}} g(z).$$

Therefore we can find b with

$$\liminf_{z \to \partial\Omega} \frac{|g(z)|}{\mathrm{dist}(z, \partial\Omega)} > b > 0.$$

Let $\psi_j = \psi * \rho_{1/j}$ be the standard regularization of ψ and let $\Omega_j = \{\psi_j < 0\}$. If j is big enough then the constants A, a and b are good also for ψ_j and Ω_j. Thus, we may assume that ψ (and thus Ω) is C^∞, provided that we prove that the constant in Theorem 1.1 depends only on n, k, r, R, m, M, A, a and b.

By Proposition 2.2 $g^{\varepsilon,\delta} \in C^\infty(\overline{\Omega^\varepsilon})$, if $0 < \varepsilon < r_0$, $0 < \delta \leq 1$. It is enough to show that for small positive ε and δ we have

$$|\nabla^2 g^{\varepsilon,\delta}(z)| \leq \frac{C_1}{\min_i |z - p^i|^2}, \quad z \in \Omega^\varepsilon.$$

Since $|\nabla g^{\varepsilon,\delta}| \geq b$ on $\partial\Omega$, by Theorems 3.1 and 4.1 we have

$$|\nabla^2 g^{\varepsilon,\delta}| \leq C_2 \quad \text{on} \quad \partial\Omega. \tag{4.6}$$

For $|w| \geq 1$ and fixed $i = 1, \dots, k$ set

$$u(w) := g^{\varepsilon,\delta}(p^i + \varepsilon w) - \mu_i \log(\varepsilon/r).$$

By (2.2) and (2.3)

$$u(w) \geq \mu_i \log |w| - C_3.$$

Thus, if α is so big that $\beta := m \log \alpha - C_3 > 0$ then for sufficiently small ε $u \geq \beta$ on ∂B_α. Moreover, $g^{\varepsilon,\delta} \geq -C_4$ on $\partial B(p^i, r)$. Thus by the comparison principle, for sufficiently small ε we have

$$\frac{\mu_i}{2} \log \frac{|z - p^i|}{r} + \frac{\mu_i}{2} \log(\varepsilon / r) + |z - p^i|^2 - \varepsilon^2 \leq g^{\varepsilon,\delta}(z) \ \text{ if } \ \varepsilon \leq |z - p^i| \leq r.$$

Therefore

$$|\nabla g^{\varepsilon,\delta}| \geq \frac{\mu_i}{2\varepsilon} \ \text{on} \ \ \partial B(p^i, \varepsilon)$$

and $|\nabla u| \geq \mu_i/2$ on ∂B_1. From Theorem 4.2 it follows that for δ small enough

$$|\nabla^2 u| \leq C_5 \quad \text{on} \ \ \partial B_1$$

which means that

$$|\nabla^2 g^{\varepsilon,\delta}| \leq \frac{C_5}{\varepsilon^2} \quad \text{on} \ \ \partial B(p^i, \varepsilon). \tag{4.7}$$

The rest of the proof will be a compilation of the methods from [Bł3] and from the proof of Theorem 3.1. Fix $a \in \Omega \setminus \{p^1, \dots, p^k\}$. From the fact that $g^{\varepsilon,\delta}$ is psh it follows that

$$|\nabla^2 g^{\varepsilon,\delta}(a)| = \limsup_{h \to 0} \frac{g^{\varepsilon,\delta}(a+h) + g^{\varepsilon,\delta}(a-h) - 2g^{\varepsilon,\delta}(a)}{|h|^2}. \tag{4.8}$$

Let P be as in the proof of Theorem 3.1 and let $\Omega'' \Subset \Omega' \Subset \Omega$, $\varepsilon' > 0$. For $z \in \overline{\Omega'} \setminus \bigcup_i B(p^i, \varepsilon + \varepsilon')$ and small h set

$$D(z, h) := g^{\varepsilon,\delta}\left(z + \frac{P(z_1)}{P(a_1)} h\right)$$

and

$$v(z, h) = D(z, h) + D(z, -h) + \frac{C_6}{|P(a_1)|^2} (|z - p^1|^2 - R^2) |h|^2,$$

so that

$$D(z, 0) = g^{\varepsilon,\delta}(z),$$

$$D(a, h) = g^{\varepsilon,\delta}(a + h),$$

v is psh in z and

$$v(a, h) \geq g^{\varepsilon,\delta}(a+h) + g^{\varepsilon,\delta}(a-h) - \frac{C_6 R^2}{|P(a_1)|^2} |h|^2. \tag{4.9}$$

If C_6 is sufficiently big and h sufficiently small then

$$\begin{aligned}&(Mv(\cdot,h))^{1/n}\\ &\quad\geq\left(\left|1+\frac{P'(z_1)}{P(a_1)}h_1\right|^{2/n}+\left|1-\frac{P'(z_1)}{P(a_1)}h_1\right|^{2/n}\right)\delta^{1/n}+\frac{C_6}{|P(a_1)|^2}|h|^2\\ &\quad\geq(2\delta)^{1/n}.\end{aligned}$$

The Taylor expansion of $D(z,\cdot)$ about the origin gives

$$v(z,h)\leq D(z,h)+D(z,-h)\leq 2g^{\varepsilon,\delta}(z)||\nabla^2(D(z,\cdot))||_{\overline{B}(0,|h|)}|h|^2.$$

Since

$$|\nabla^2(D(z,\cdot))(\widetilde{h})|=\frac{|P(z_1)|^2}{|P(a_1)|^2}\left|\nabla^2 g^{\varepsilon,\delta}\left(z+\frac{P(z_1)}{P(a_1)}\widetilde{h}\right)\right|,$$

we get

$$\begin{aligned}v(z,h)&\leq 2g^{\varepsilon,\delta}(z)+C'|h|^2, \quad z\in\partial\Omega',\\ v(z,h)&\leq 2g^{\varepsilon,\delta}(z)+C_i'|h|^2, \quad z\in\partial B(p^i,\varepsilon+\varepsilon'),\end{aligned}$$

where

$$\begin{aligned}C'&=C_7\frac{||\nabla^2 g^{\varepsilon,\delta}||_{\Omega\setminus\overline{\Omega''}}}{|P(a_1)|^2},\\ C_i'&=C_8\frac{(\varepsilon+\varepsilon')^2||\nabla^2 g^{\varepsilon,\delta}||_{B(p^i,\varepsilon+2\varepsilon')\cap\Omega^\varepsilon}}{|P(a_1)|^2}\end{aligned}$$

for h small enough. Now we can apply the comparison principle to v and $2g^{\varepsilon,\delta}$. We obtain

$$v(a,h)\leq 2g^{\varepsilon,\delta}(a)+\max\{C',C_1',\dots,C_k'\}|h|^2.$$

By (4.8) and (4.9)

$$|\nabla^2 g^{\varepsilon,\delta}(a)|\leq\max\{C',C_1',\dots,C_k'\}+\frac{C_6R^2}{|P(a_1)|^2}.$$

If we let $\Omega''\uparrow\Omega$, $\varepsilon'\downarrow 0$ and use (4.6), (4.7) then the desired estimate follows. $\square$

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