Geometric degree of generically finite extensions

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Abstract

Let V, W are irreducible algebraic subsets of $\mathbb{C}^n$. Every dominant, generically finite mapping $f : V \to W$ can be extented to a dominant, generically finite mapping $F : \mathbb{C}^n \to \mathbb{C}^n$. We show that if f is a projection then there exists a dominant, generically finite extension F of f with the same geometric degree. The same is shown for an arbitrary f with the assumption that V, W are smooth sets and $4 \cdot \dim V + 2 \leq n$.

1 Introduction

Let $f : V \to W$ be any polynomial mapping of irreducible algebraic subsets of $\mathbb{C}^n$. It is known (see [2] Lemma 5.4) that the mapping f can be extended to a dominant, generically finite mapping $F : \mathbb{C}^n \to \mathbb{C}^n$. If f is, also, dominant generically finite mapping, then there is a natural question about relation

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between the number $\mathrm{gdeg}\, F$ (defined as the number of points in the generic fiber of F and called geometric degree of the mapping F) and the number $\mathrm{gdeg}\, f$.

If f is finite mapping, then f can be extended to a finite mapping $F : \mathbb{C}^n \to \mathbb{C}^n$ (see [7]). For relations between $\mathrm{gdeg}\, F$ and $\mathrm{gdeg}\, f$ in this situation see [3], [4], [5] and [6].

In this paper we prove that any generically finite projection $\pi : V \to \pi(V)$ can be extended to a dominant, generically finite mapping with the same geometric degree. Applying this fact we will show that any dominant, generically finite mapping between smooth algebraic sets can be extended to a dominant, generically finite mapping with the same geometric degree.

2 Notations and basic facts

Let $f : V \to W$ be any polynomial mapping of irreducible algebraic sets. The mapping f is called generically finite if there exists an open and dense subset U of W such that $\# f^{-1}(y)$ is finite for all $y \in U$. The mapping f is called dominant if $\overline{f(V)} = W$.

It is known that if f is a dominant and generically finite mapping then $\mathbb{C}(V)$ is a finite field extension of the field $f^*(\mathbb{C}(W)) \simeq \mathbb{C}(W)$, where f^* denote the homomorphism of coordinate rings $f^* : \mathbb{C}[V] \ni \varphi \mapsto \varphi \circ f \in \mathbb{C}[W]$, and its extension to the homomorphism of the fields $f^* : \mathbb{C}(V) \to \mathbb{C}(W)$. In this situation $\mathrm{gdeg}\, f = [\mathbb{C}(V) : \mathbb{C}(W)] = \dim{}_{\mathbb{C}(W)}\mathbb{C}(V)$ (see [8]). If f is dominant and generically finite, then $\dim V = \dim W$. Conversely, if $\dim V = \dim W$, then $f : V \to W$ is dominant if and only if f is generically finite.

3 Projections

Now we will prove the following

Theorem 3.1 *Let $V \subset \mathbb{C}^k \times \mathbb{C}^n$ be an irreducible algebraic set, $\pi : V \to 0 \times \mathbb{C}^n$ natural projection. If $\pi : V \to \overline{\pi(V)}$ is generically finite, then there exists a generically finite mapping $\Pi : \mathbb{C}^k \times \mathbb{C}^n \to \mathbb{C}^k \times \mathbb{C}^n$ such that $\Pi|_V = \pi$ and*

$$\mathrm{gdeg}\, \Pi = \mathrm{gdeg}\, \pi.$$

Proof. Let $x=(x_1,\dots,x_k)$ be coordinates in $\mathbb{C}^k$ and let $y=(y_1,\dots,y_n)$ be coordinates in $\mathbb{C}^n$

For the beginning let us assume that $k=1$. We have that $\mathbb{C}(V)$ is a finite field extension of the field $\mathbb{C}(\overline{\pi(V)})$. Thus $x_1|_V$ is algebraic over $\mathbb{C}(\overline{\pi(V)})$. Let $\tilde{P}=T^l+\tilde{a}_1T^{l-1}+\ldots+\tilde{a}_l\in\mathbb{C}(\overline{\pi(V)})$ be the minimal polynomial for $x_1|_V$ over $\mathbb{C}(\overline{\pi(V)})$. Multiplying $\tilde{P}$ by a minimal common multiply of the denominators of $\tilde{a}_1,\dots,\tilde{a}_l$ we obtain polynomial $P=a_0T^l+a_1T^{l-1}+\ldots+a_l\in\mathbb{C}[\overline{\pi(V)}]$ such that $P(x_1|_V)=0$ and $a_0\neq 0$. By extension of coefficients $a_0,\dots,a_l$ from the set $\overline{\pi(V)}$ to the whole space $0\times\mathbb{C}^n$ we obtain polynomial $\bar{P}$ such that $\bar{P}|_V=0$ and $\deg\bar{P}=\deg\tilde{P}$. Now let $\Pi:\mathbb{C}\times\mathbb{C}^n\ni(x,y)\mapsto(\bar{P}(x,y),y)\in\mathbb{C}\times\mathbb{C}^n$. It is easy to see that Π is a generically finite mapping such that $\Pi|_V=\pi$, and $\operatorname{gdeg}\Pi=\deg\bar{P}=\deg\tilde{P}=\operatorname{gdeg}\pi$.

For $k>1$ we proceed by induction. Choose a system of coordinates for $\mathbb{C}^k$ in such a way that $\pi_1:V\to\overline{\pi_1(V)}$ is generically finite mapping, where $\pi_1:V\ni((x_1,\dots,x_k),y)\mapsto((x_2,\dots,x_k),y)\in\mathbb{C}^k\times\mathbb{C}^n$. It follows that $\pi_2:\overline{\pi_1(V)}\to\overline{\pi(V)}$ is a generically finite mapping, where $\pi_2:\overline{\pi_1(V)}\ni((x_2,\dots,x_k),y)\mapsto y\in 0\times\mathbb{C}^n$. Of course we have $\pi=\pi_2\circ\pi_1$ and $\operatorname{gdeg}\pi=\operatorname{gdeg}\pi_2\cdot\operatorname{gdeg}\pi_1$. By induction, there exists a generically finite mapping $\tilde{\Pi}_2:\mathbb{C}^{k-1}\times\mathbb{C}^n\to\mathbb{C}^{k-1}\times\mathbb{C}^n$ such that $\tilde{\Pi}_2|_{\pi_1(V)}=\pi_2$ and $\operatorname{gdeg}\tilde{\Pi}_2=\operatorname{gdeg}\pi_2$. By the first part of the proof, there exists a generically finite mapping $\Pi_1:\mathbb{C}^k\times\mathbb{C}^n\to\mathbb{C}^k\times\mathbb{C}^n$ such that $\Pi_1|_V=\pi_1$ and $\operatorname{gdeg}\Pi_1=\operatorname{gdeg}\pi_1$. Set $\Pi_2:\mathbb{C}^k\times\mathbb{C}^n\ni((x_1,\dots,x_k),y)\mapsto(x_1,\tilde{\Pi}_2((x_2,\dots,x_k),y))\in\mathbb{C}^k\times\mathbb{C}^n$, and

$$\Pi=\Pi_2\circ\Pi_1.$$

It is easy to check that $\Pi:\mathbb{C}^k\times\mathbb{C}^n\to\mathbb{C}^k\times\mathbb{C}^n$ is a generically finite mapping such that $\Pi|_V=\pi$ and $\operatorname{gdeg}\Pi=\operatorname{gdeg}\pi$. $\square$

Let us notice that there can exists a dominant, generically finite extension with a smaller geometric degree than the geometric degree of the extended mapping. We have the following

Example 1 *Let $F:\mathbb{C}^n\ni(x_1,\dots,x_n)\mapsto(x_1x_2,x_2,x_3,\dots,x_n)\in\mathbb{C}^n$, then F is generically finite and* $\deg F=1$. *Let $W\in\mathbb{C}[x_3,\dots,x_n][x_1]$ be any irreducible polynomial of degree $k\in\mathbb{N}$ (with respect to x_1) and set $V=\{(x_1,\dots,x_n)\in\mathbb{C}^n : x_2=0, W(x_1,x_3,\dots,x_n)=0\}$. Then $F|_V:V\to F(V)$ is a generically finite mapping and* $\operatorname{gdeg}(F|_V)=k$.

By the above example it is natural to state the following

Corollary 3.2 *Under the assumptions of Theorem 3.1 there exists a generically finite mapping* $\Pi : \mathbb{C}^k \times \mathbb{C}^n \to \mathbb{C}^k \times \mathbb{C}^n$ *such that* $\Pi|_V = \pi$ *and*

$$\operatorname{gdeg} \Pi \leq \operatorname{gdeg} \pi.$$

Since, for any projection π, we have $\pi = \pi \circ \pi$, it is natural to ask if there exists a generically finite mapping $\Pi : \mathbb{C}^k \times \mathbb{C}^n \to \mathbb{C}^k \times \mathbb{C}^n$ such that $\Pi|_V = \pi$, $\operatorname{gdeg} \Pi \leq \operatorname{gdeg} \pi$ and $\Pi = \Pi \circ \Pi$. But if $F : \mathbb{C}^n \to \mathbb{C}^n$ is dominant and $F = F \circ F$ then $F = \mathrm{id}_{\mathbb{C}^n}$. Thus in the general situation such Π does not exists.

4 Mappings from smooth variety

Let us recall some facts about embeddings. A polynomial mapping $f : V \to \mathbb{C}^n$ is called embedding if f is an isomorphism on its image $f(V) = \overline{f(V)}$. We have the following well-known lemma (see e.g.[1])

Lemma 4.1 *If* $X \subset \mathbb{C}^n$ *is a closed algebraic smooth set,* $\dim X = k$ *and* $n > 2k+1$, *then we can change coordinates in such a way that the projection*

$$\phi : X \ni (x, y) \mapsto (0, y) \in 0 \times \mathbb{C}^{2k+1},$$

is an embedding.

We also have the following

Theorem 4.2 *([1] Thm 1.2) Let* $X \subset \mathbb{C}^n$ *be a closed algebraic set which is smooth and not necessarily irreducible of dimension (not necessarily pure)* k. *Let* $\phi : X \to \mathbb{C}^n$ *be an embedding. If* $n \geq 4k+2$ *then there exists an isomorphism* $\Phi : \mathbb{C}^n \to \mathbb{C}^n$ *such that*

$$\Phi|_X = \phi.$$

Now we are in a position to prove the following

Theorem 4.3 *Let $V, W \subset \mathbb{C}^n$ are smooth, irreducible algebraic sets and let $f : V \to W$ be a dominant, generically finite mapping. If V, W are smooth and $4 \cdot \dim V + 2 \leq n$, then there exists a generically finite mapping $F : \mathbb{C}^n \to \mathbb{C}^n$ such that $F|_V = f$ and*

$$\operatorname{gdeg} F = \operatorname{gdeg} f.$$

Proof: By Lemma 4.1 we can assume that the projections:

$$\phi_1 : V \to 0 \times \mathbb{C}^{2k+1} \text{ and } \phi_2 : W \to 0 \times \mathbb{C}^{n-2k-1}$$

are embeddings. Put

$$\tilde{V} = \phi_1(V), \qquad \tilde{W} = \phi_2(W)$$

and

$$\tilde{f} = \phi_2 \circ f \circ \phi_1^{-1} : \tilde{V} \to \tilde{W}.$$

The mapping $\tilde{f}$ is generically finite with $\operatorname{gdeg} \tilde{f} = \operatorname{gdeg} f$.

Since $\tilde{V} \subset 0 \times \mathbb{C}^{2k+1}$, $\tilde{W} \subset 0 \times \mathbb{C}^{n-2k-1}$, we can consider the sets $\tilde{V}$ and $\tilde{W}$ as subsets of $\mathbb{C}^{2k+1}$ and $\mathbb{C}^{n-2k-1}$, respectively. Put the isomorphism:

$$\psi : \tilde{V} \ni x \mapsto (x, \tilde{f}(x)) \in \mathbb{C}^{2k+1} \times \mathbb{C}^{n-2k-1}$$

and the set $\hat{V} = \psi(\tilde{V})$. Now consider the projection:

$$\pi : \hat{V} \ni (x, y) \mapsto (0, y) \in 0 \times \mathbb{C}^{n-2k-1}.$$

We have $\tilde{f} = \pi \circ \psi$, and ψ is an isomorphism, so $\pi : \hat{V} \to \tilde{W} = \overline{\pi(\hat{V})}$ is a generically finite mapping and $\operatorname{gdeg} \pi = \operatorname{gdeg} \tilde{f} = \operatorname{gdeg} f$. By Theorem 3.1 there exists a generically finite mapping $\Pi : \mathbb{C}^{2k+1} \times \mathbb{C}^{n-2k-1} \to \mathbb{C}^{2k+1} \times \mathbb{C}^{n-2k-1}$ such that $\Pi|_{\hat{V}} = \pi$ and:

$$\operatorname{gdeg} \Pi = \operatorname{gdeg} \pi.$$

Applying Theorem 4.2 to $\psi : \tilde{V} \to \hat{V}$, $\phi_1 : V \to \tilde{V}$ and $\phi_2 : W \to \tilde{W}$ there exist isomorphisms Ψ, Φ_1, $\Phi_2 : \mathbb{C}^n \to \mathbb{C}^n$ such that:

$$\Psi|_{\tilde{V}} = \psi,\ \Phi_1|_V = \phi_1,\ \Phi_2|_W = \phi_2.$$

Putting

$$F = \Phi_2^{-1} \circ \Pi \circ \Psi \circ \Phi_1$$

we have a generically finite extension of f such that:

$$\operatorname{gdeg} F = \operatorname{gdeg} f. \square$$

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