Higher order embeddings of projective curves

Anna Stasica

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Abstract

In this note we show that every projective curve admits a k-very ample linear series of dimension at most $2k+1$. We also show how the order of the canonical embedding is related to the gonality of the given curve.

1 Introduction

Let X be a smooth projective variety of dimension n. Let L be a line bundle on X and let $V \subset H^0(X, L)$ be a linear system of divisors. We say that V is *k-very ample* if the restriction map

$$V \longrightarrow H^0(X, L \otimes \mathcal{O}_Z)$$

is surjective for every 0-dimensional subscheme $Z \subset X$ of length $k+1$. In particular L is k-very ample if $V = H^0(X, L)$. Note that 0-very ampleness is equivalent to base point freeness and 1-very ampleness to the usual very

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ampleness. If V is k-very ample for $k \geq 0$ then the associated mapping $\varphi_V \longrightarrow \mathbb{P}(V^*)$ is called an embedding of order k. Higher order embeddings play an important role in algebraic geometry, in particular in the investigation of local positivity of line bundles.

Our main goal is to generalise two basic results of the theory of projective curves and canonical embeddings. We prove the following theorems:

Theorem 3.1. *For any smooth projective curve there exists a k-very ample linear series V such that* $\dim V \leq 2k + 1$.

Theorem 3.2. *The canonical divisor K_X is k-very ample if and only if the curve X is not r-gonal for $r = 1, \dots, k + 1$.*

The first statement generalizes the well known fact that every projective curve can be embedded into $\mathbb{P}^3$ (see e.g. [H]). The second one generalizes the result which says that the canonical divisor on a curve X of genus $g \geq 2$ is very ample if and only if X is not hyperelliptic (Proposition 5.2, IV [H]).

1.1 Notation

In this paper we work over the field $\mathbb{C}$ of complex numbers. We denote by K_X the canonical divisor of a smooth projective curve X. If L is a line bundle on X then by $\mid L \mid$ we denote the complete linear series associated to L and by $h^0(L)$ the dimension of the vector space $H^0(X, L)$ of global sections of the bundle L. Similarly for an effective divisor D on X we denote the linear series $\mid \mathcal{O}_X(D) \mid$ by $\mid D \mid$. Linear equivalence of divisors D_1, D_2 is denoted by $D_1 \sim D_2$

2 Auxiliary results

In this part we recall some useful properties of k-very ample divisor on curves. First we note that if X is a curve, V is k-very ample if an only if

$$V \longrightarrow H^0(X, L \otimes (\mathcal{O}_X/\mathfrak{m}_{x_1}^{k_1} \otimes \cdots \otimes \mathfrak{m}_{x_r}^{k_r}))$$

is surjective, where $1 \leq r \leq k+1$, $k_1 + \cdots + k_r = k+1$ and $\mathfrak{m}_{x_i}$ denotes the maximal ideal of the point $x_i \in X$ for $i = 1, \dots, r$. In fact, a 0-dimensional subscheme of length k on a curve X consists of k points counted with multiplicity. In particular we have a nice description of k-very ampleness: the linear series $\mid L \mid$ on curve X is k-very ample if and only if

$$\dim \mid L - (x_1 + \cdots + x_{k+1}) \mid = \dim \mid L \mid - (k+1) \tag{1}$$

for any set of $k+1$ points on X (possibly some of these points are equal). We say that $k+1$ points impose independent conditions on divisors in the linear system $\mid L \mid$.

The following lemma gives a sufficient condition for a divisor on a smooth curve to be k-very ample.

Lemma 2.1 *Let X be a smooth curve of genus g and let D be a divisor on X.*
If $\deg D \geq 2g+k$, then the linear series $\mid D \mid$ is k-very ample.

Proof. Let P be a divisor of the form $P = P_1 + \cdots + P_{k+1}$, where $P_1, \ldots P_{k+1}$ are some points on X. By (1) it is enough to show that $\dim \mid D \mid = \dim \mid D - P \mid + k + 1$. Applying the Riemann-Roch theorem we get:

1. $\dim \mid D \mid - \dim \mid K_X - D \mid = 1 - g + \deg D$
2. $\dim \mid D - P \mid - \dim \mid K_X - D + P \mid = 1 - g + \deg D - k - 1$

but $\deg K_X - D = 2g - 2 - \deg D < 0$ so $\dim \mid K_X - D \mid = 0$ and

$$\dim \mid D \mid = 1 - g + \deg D.$$

In the same way we can show that $\dim \mid K_X - D + P \mid = 0$ hence $\mid D \mid$ is k-very ample.

The next proposition shows that the inequality in previous lemma might not be sharp.

Proposition 2.2 *Let L be a line bundle of degree $2g+k-1$ on a curve X of genus g. Then L is k-very ample if an only if $h^0(L - K_X) = 0$.*

Proof. By the Riemann-Roch theorem we have an equality

$$h^0(L) = 1 - g + 2g + k - 1 = g + k. \tag{2}$$

Let $P = P_1 + \ldots + P_{k+1}$, where every $P_i \in X$ is a point. Note that $\deg(K_X - L + P) = 0$ and hence

$$h^0(L - P) = h^0(K_X - L + P) + g - 1. \tag{3}$$

If L is a k-very ample line bundle we get, due to (1) and (2), the equality

$$h^0(L - P) = h^0(L) - k - 1 = g - 1$$

and consequently $h^0(K_X - L + P) = 0$. It means that the linear system $\mid L - K_X \mid$ is empty, otherwise we would have an effective divisor P' such that $L - K_X \sim P'$, hence $h^0(K_X - L + P') \geq 1$, but this fact would contradict the equality $h^0(K_X - L + P) = 0$. In consequence we get $h^0(L - K_X) = 0$.
On the other hand if $h^0(L - K_X) = 0$ then the linear system $\mid L - K_X \mid$ is empty and hence for every divisor $P = P_1 + \ldots + P_{k+1}$ we have $h^0(K_X - L + P) = 0$. Combining the Riemann-Roch theorem with (2) and (1), we get the assertion.

Proposition 2.3 *If L is a k-very ample line bundle on a curve X then:*

1. $h^0(L) \geq k+1$
2. *the equality holds if and only if X is isomorphic to $\mathbb{P}^1$ and $L \simeq \mathcal{O}_{\mathbb{P}^1}(k)$.*

Proof. If L is k-very ample then

$$h^0(L) = \dim \mid L \mid + 1 = \dim \mid L - (k+1)P \mid + k + 2 \geq k+1.$$

Assuming that L is a line bundle on X with $h^0(L) = k+1$, we obtain that the image by the associated map $\varphi_L(X)$ is a curve in $\mathbb{P}^k$. To show that X is rational it is enough to notice that $\deg \varphi_L(X) = k$ (Corollary 7.7, [M]). Let $H \simeq \mathbb{P}^{k+1} \subset \mathbb{P}^k$ be a generic hyperplane. If $H.X > k$ then $H \cap X$ is a 0 - dimensional subscheme of lenght $H.X$, which contains a subscheme Z' of lenght $k+1$. But then Z' does not impose independent conditions on $H^0(X, L)$. This implies that the image of φ_L is of degree k.

At the end we would like to recall a useful lemma, due to Beltrametti and Sommese (Lemma 2.2, [BS]).

Lemma 2.4 *If L_1 and L_2 are k_1-, respectively k_2-very ample line bundles on a curve then $\mid L_1 \otimes L_2 \mid$ is $(k_1 + k_2)$ - very ample.*

3 Main results

Now we are going to show that for any projective curve there exists an embedding of order k into $\mathbb{P}^{2k+1}$.

Let $X \subset \mathbb{P}^N$ be a nondegenerate curve. Then generic $k+1$ points on X span $\mathbb{P}^k \subset \mathbb{P}^N$ for $k \leq N$.

Consider $S^k_X = \{(P_1, \ldots, \mathbb{P}_{k+1}, u) \mid u \in < P_1, \ldots, P_{k+1} > \} \subset X^{k+1} \times \mathbb{P}^N$

We will denote by pr_2 projection of this set into space $\mathbb{P}^N$ and call the closed set $\mathrm{Sec}^k(X) = \overline{pr_2(S_X^k)}$ the k-secant variety of X. It is clear that dimension of this variety is not greater than $2k+1$.

Theorem 3.1 *For any smooth projective curve there exists a k-very ample linear series V such that* $\dim V \leq 2k+1$.

Proof. Let X be a curve of genus g. Let D be any divisor of degree $2g+k$ on X. Due to lemma 2.1 the divisor D is k - very ample and hence the associated map φ_D is k-th order embedding into $\mathbb{P}^N$ ($N = \dim \mid D \mid$). If $N \leq 2k+1$ then we are done.

In the other case we consider the projection p_{x_1} from some point $x_1 \in \mathbb{P}^N \setminus \mathrm{Sec}^k(X)$ (such a point exists because $\dim \mathrm{Sec}^k(X) \leq 2k+1$)

$$p_{x_1} : X \longrightarrow \mathbb{P}^{N-1}.$$

Note that any subseries $V_1 - x_1 \subset \mid D \mid$ of codimension 1, given by hyperplanes passing through the point x_1 is k-very ample. Indeed, the equality

$$\dim p_{x_1}(< P_1, \ldots, P_{k+1} >) = \dim < P_1, \ldots, P_{k+1} >$$

holds thanks to the fact that $x_1 \notin \mathrm{Sec}^k(X)$.

We continue this process untill we get k-very ample linear series $V = V_s$ with $s = N - (2k+1)$. This series gives an embedding of X of order k into the space $\mathbb{P}^{2k+1}$ as we claimed.

Note that if we have a nondegenerate curve $X \subset \mathbb{P}^{2k+1}$ then its secant variety is exactly $\mathbb{P}^{2k+1}$, therefore, there is no k-very ample linear series of dimension $< 2k+1$ on such a curve. Put

$$l_0 = \min\{l \mid \mathrm{Sec}^l(X) = \mathbb{P}^{2k+1}\}.$$

If $l_0 = k$, then $\mathrm{Sec}^k(X) = \mathbb{P}^{2k+1}$. If $l_0 > k$, we can apply Example 1.6 in [Z] which says that

$$\dim \mathrm{Sec}^l(X) = \min\{2l+1, 2k+1\}.$$

This gives $\mathrm{Sec}^l(X) = 2l+1$ for $l < l_0$ and, particularly $\mathrm{Sec}^k(X) = 2k+1$.

Finally we will show that the order of canonical embedding of a curve is determined by its gonality and vice versa.

Let X be a curve of general type, i.e. $g(X) \geq 2$. We have the following characterization.

Theorem 3.2 *The canonical divisor K_X is k-very ample if and only if the curve X is not r-gonal for $r = 1, \ldots, k+1$.*

Proof. Assume that K_X is k-very ample. Take $P = x_1 + \ldots + x_{k+1}$, where $x_1, \ldots, x_{k+1} \in X$ are any points. By Riemann-Roch theorem we have

$$\dim \mid P \mid = 2 - g + k + \dim \mid K_X - P \mid = 0$$

since $\dim \mid K_X - P \mid = g - 1 - (k+1)$ by assumption. It means that any linear system of degree $k+1$ has dimension 0 and hence X is not $(k+1)$ - gonal (and of course not r-gonal for $r \leq k$).
Now we will assume that the canonical divisor is not k-very ample. Then there exist points $x_1, \ldots, x_{k+1} \in X$ such that

$$\dim \mid K_X - P \mid \geq \dim \mid K_X \mid -k,$$

where $P = x_1 + \ldots + x_{k+1}$.
Again by Riemann-Roch theorem, we have

$$\dim \mid P \mid \geq 1 - g + k + 1 + g - 1 - k = 1.$$

Let $y_1, \ldots, y_t$ be all the base points of the linear system $\mid P \mid$. Than the system $\mid P - y_1 - \ldots - y_t \mid$ is globally generated and its dimension is equal to the dimension of $\mid P \mid$. Since any generic subseries $V \subset \mid P - y_1 - \ldots - y_t \mid$ of dimension 1 is base points free we have a morphism $\varphi_V : X \to \mathbb{P}^1$ of degree at most $k+1$ which implies that X is not $(k+1)$-gonal.

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Instytut Matematyki
Uniwersytet Jagielloński
ul. Reymonta 4, 30-059 Kraków
E-mail: astasica@im.uj.edu.pl