CONLEY INDEX CONTINUATION AND THIN DOMAIN PROBLEMS

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Abstract. Given $\epsilon > 0$ and a bounded Lipschitz domain Ω in $\mathbb{R}^M \times \mathbb{R}^N$ let $\Omega_\epsilon := \{\, (x, \epsilon y) \mid (x, y) \in \Omega \,\}$ be the *ϵ-squeezed domain*. Consider the reaction-diffusion equation

$$(\tilde{E}_\epsilon) \qquad u_t = \Delta u + f(u)$$

on Ω_ϵ with Neumann boundary condition. Here f is an appropriate nonlinearity such that $(\tilde{E}_\epsilon)$ generates a (local) semiflow $\tilde{\pi}_\epsilon$ on $H^1(\Omega_\epsilon)$. It was proved by Prizzi and Rybakowski (J. Diff. Equations, to appear), generalizing some previous results of Hale and Raugel, that there are a closed subspace $H^1_s(\Omega)$ of $H^1(\Omega)$, a closed subspace $L^2_s(\Omega)$ of $L^2(\Omega)$ and a sectorial operator A_0 on $L^2_s(\Omega)$ such that the semiflow π_0 defined on $H^1_s(\Omega)$ by the abstract equation

$$\dot{u} + A_0 u = \hat{f}(u)$$

is the limit of the semiflows $\tilde{\pi}_\epsilon$ as $\epsilon \to 0^+$.

In this paper we prove a singular Conley index continuation principle stating that every isolated invariant set K_0 of π_0 can be continued to a nearby family $\tilde{K}_\epsilon$ of isolated invariant sets of $\tilde{\pi}_\epsilon$ with the same Conley index. We present various applications of this result to problems like connection lifting or resonance bifurcation.

1. Introduction

Let M and N be positive integers and Ω be an arbitrary nonempty smooth bounded domain in $\mathbb{R}^M \times \mathbb{R}^N$. Write (x, y) for a generic point of $\mathbb{R}^M \times \mathbb{R}^N$. Given $\epsilon > 0$ squeeze Ω by the factor ϵ in the y-direction to obtain the squeezed domain Ω_ϵ. More precisely, let $T_\epsilon \colon \mathbb{R}^M \times \mathbb{R}^N \to \mathbb{R}^M \times \mathbb{R}^N$, $(x, y) \mapsto (x, \epsilon y)$, be the *squeezing operator* and define

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$\Omega_\epsilon := \{\, (x, \epsilon y) \mid (x, y) \in \Omega \,\}$. Consider the following reaction-diffusion equation on Ω_ϵ:

$$(\tilde{E}_\epsilon) \qquad \begin{aligned} &u_t = \Delta u + f(u), && t > 0,\ (x, y) \in \Omega_\epsilon, \\ &\partial_{\nu_\epsilon} u = 0, && t > 0,\ (x, y) \in \partial\Omega_\epsilon, \end{aligned}$$

where ν_ϵ is the exterior normal vector field on $\partial\Omega_\epsilon$ and $f\colon \mathbb{R} \to \mathbb{R}$ is a C^1-nonlinearity of appropriate polynomial growth which ensures that $(\tilde{E}_\epsilon)$ generates a (local) semiflow $\tilde{\pi}_\epsilon$ on $H^1(\Omega_\epsilon)$.

If f is *dissipative* in the sense that

$$\limsup_{|s|\to\infty} f(s)/s \leq -\delta_0 \quad \text{for some } \delta_0 > 0$$

then the semiflow $\tilde{\pi}_\epsilon$ possesses a global attractor $\tilde{\mathcal{A}}_\epsilon$.

As $\epsilon \to 0$ the thin domain Ω_ϵ degenerates to a one-dimensional interval. One may therefore ask what happens in the limit to the family $(\tilde{\pi}_\epsilon)_{\epsilon>0}$ of semiflows and to the family $(\tilde{\mathcal{A}}_\epsilon)_{\epsilon>0}$ of attractors. Is there a limit semiflow and a corresponding limit attractor?

This problem was first considered by Hale and Raugel in [21] for the case when $N = 1$, U is a bounded domain in $\mathbb{R}^M$ and the domain Ω is the *ordinate set* of a smooth positive function g defined on $\operatorname{cl} U$, i.e.

$$\Omega = \{\, (x, y) \mid x \in U \text{ and } 0 < y < g(x) \,\}.$$

The authors prove that, in this case, there exists a limit semiflow $\tilde{\pi}_0$, which is defined by the Neumann boundary value problem

$$(HR_0) \qquad \begin{aligned} &u_t = (1/g) \sum_{i=1}^{M} (g u_{x_i})_{x_i} + f(u), && t > 0,\ x \in U \\ &\partial_\nu u = 0, && t > 0,\ x \in \partial U. \end{aligned}$$

Moreover, if f is dissipative, then $\tilde{\pi}_0$ has a global attractor $\tilde{\mathcal{A}}_0$ and, in some sense, the family $(\tilde{\mathcal{A}}_\epsilon)_{\epsilon\geq 0}$ is upper-semicontinuous at $\epsilon = 0$.

Hale and Raugel also prove that one can modify the nonlinearity f in such a way that each modified semiflow $\tilde{\pi}'_\epsilon$ possesses an invariant C^1-manifold $\tilde{\mathcal{M}}_\epsilon$ of some fixed dimension ν which includes the attractor $\tilde{\mathcal{A}}_\epsilon$ of the original semiflow $\tilde{\pi}_\epsilon$. The semiflows $\tilde{\pi}_\epsilon$ and $\tilde{\pi}'_\epsilon$ coincide on the attractor $\tilde{\mathcal{A}}_\epsilon$ and, as $\epsilon \to 0$, the reduced flow on $\tilde{\mathcal{M}}_\epsilon$ converges in the C^1-sense to the reduced flow on $\tilde{\mathcal{M}}_0$.

If the domain Ω is not the ordinate set of some function (e.g. if Ω has holes or different horizontal branches) then an equation of the form (HR_0) cannot be a limiting equation for the family $(\tilde{E}_\epsilon)$, $\epsilon > 0$. Nevertheless, as it was proved by Prizzi and Rybakowski in [33] the family $\tilde{\pi}_\epsilon$, $\epsilon > 0$ still has a limit semiflow. Moreover, if f is dissipative then there exists a limit global attractor and the upper-semicontinuity result continues to hold.

In order to describe the main results of [33] we first transfer the family $(\tilde{E}_\epsilon)$ to boundary value problems on the fixed domain Ω. More explicitly, we use the linear isomorphism $\Phi_\epsilon\colon H^1(\Omega_\epsilon) \to H^1(\Omega)$, $u \mapsto u \circ T_\epsilon$, to transform problem $(\tilde{E}_\epsilon)$ to the equivalent problem

$$(E_\epsilon) \qquad \begin{aligned} &u_t = \Delta_x u + (1/\epsilon^2)\Delta_y u + f(u), && t > 0,\ (x, y) \in \Omega \\ &\nu_1 \cdot \nabla_x u + (1/\epsilon^2)\nu_2 \cdot \nabla_y u = 0, && t > 0,\ (x, y) \in \partial\Omega. \end{aligned}$$

Here, $\nu = (\nu_1, \nu_2)$ is the exterior normal vector field on $\partial\Omega$.

Note that equation (E_ϵ) can be written in the abstract form

$$\dot{u} + A_\epsilon u = \hat{f}(u)$$

where $\hat{f}\colon H^1(\Omega) \to L^2(\Omega)$ is the Nemitski operator generated by the function f and A_ϵ is the linear operator defined by

$$A_\epsilon u = -\Delta_x u - \frac{1}{\epsilon^2}\Delta_y u \in L^2(\Omega) \text{ for } u \in H^2(\Omega) \text{ with } \nu_1 \cdot \nabla_x u + \frac{1}{\epsilon^2}\nu_2 \cdot \nabla_y u = 0 \text{ on } \partial\Omega.$$

Equation (E_ϵ) defines a semiflow π_ϵ on $H^1(\Omega)$ which is conjugated to $\tilde{\pi}_\epsilon$ via the isomorphism Φ_ϵ. Therefore, whatever property we prove about the transformed semiflow π_ϵ translates into a corresponding property about the original semiflow $\tilde{\pi}_\epsilon$, and vice versa. In particular, if f is dissipative then π_ϵ has the global attractor $\mathcal{A}_\epsilon := \Phi_\epsilon(\tilde{\mathcal{A}}_\epsilon)$, consisting of the orbits of all full bounded solutions of (E_ϵ).

The operator A_ϵ is, in the usual way, induced by the following bilinear form

$$a_\epsilon(u,v) := \int_\Omega (\nabla_x u \cdot \nabla_x v + \frac{1}{\epsilon^2}\nabla_y u \cdot \nabla_y v)\,\mathrm{d}x\,\mathrm{d}y, \quad u, v \in H^1(\Omega).$$

Notice that, for every fixed $\epsilon > 0$ and $u \in H^1(\Omega)$, the formula

$$|u|_\epsilon = \Big(a_\epsilon(u,u) + |u|^2_{L^2(\Omega)}\Big)^{1/2}$$

defines a norm on $H^1(\Omega)$ which is equivalent to $|\cdot|_{H^1(\Omega)}$. However, $|u|_\epsilon \to \infty$ as $\epsilon \to 0^+$ whenever $\nabla_y u \neq 0$ in $L^2(\Omega)$.

In fact, we see that for $u \in H^1(\Omega)$

$$\lim_{\epsilon \to 0^+} a_\epsilon(u,u) = \begin{cases} \int_\Omega \nabla_x u \cdot \nabla_x u \,\mathrm{d}x\,\mathrm{d}y, & \text{if } \nabla_y u = 0 \\ +\infty, & \text{otherwise.} \end{cases}$$

Thus the family $a_\epsilon(u,u)$, $\epsilon > 0$, of real numbers has a finite limit (as $\epsilon \to 0$) if and only if $u \in H^1_s(\Omega)$, where we define

$$H^1_s(\Omega) := \{\, u \in H^1(\Omega) \mid \nabla_y u = 0 \,\}.$$

This is a closed linear subspace of $H^1(\Omega)$.

The corresponding limit bilinear form is given by the formula:

$$a_0(u,v) := \int_\Omega \nabla_x u \cdot \nabla_x v \mathrm{d}x\,\mathrm{d}y, \quad u, v \in H^1_s(\Omega).$$

The form a_0 uniquely determines a densely defined selfadjoint linear operator

$$A_0\colon D(A_0) \subset H^1_s(\Omega) \to L^2_s(\Omega)$$

by the usual formula

$$a_0(u,v) = \langle A_0 u, v\rangle_{L^2(\Omega)}, \qquad \text{for } u \in D(A_0) \text{ and } v \in H^1_s(\Omega).$$

Here, $L^2_s(\Omega)$ is the closure of $H^1_s(\Omega)$ in the L^2-norm, so $L^2_s(\Omega)$ is a closed linear subspace of $L^2(\Omega)$.

It follows that the Nemitski operator $\hat{f}$ maps the space $H^1_s(\Omega)$ into $L^2_s(\Omega)$. Consequently the abstract parabolic equation

$$(E_0) \qquad \dot{u} = -A_0 u + \hat{f}(u)$$

defines a semiflow π_0 on the space $H^1_s(\Omega)$. This is the limit semiflow of the family π_ϵ. In the ordinate set case considered by Hale and Raugel, equation (E_0) reduces to their equation (HR_0) stated above. For a large class of two-dimensional domains (called *nicely decomposed*) equation (E_0) is equivalent to a parabolic equation defined on a finite one-dimensional graph. For general higher dimensional domains and $M \geq 2$ it does not seem possible to give a more explicit expression for equation (E_0). Nevertheless, many qualitative properties of the limit semiflow π_0 may be deduced from the abstract equation (E_0).

For example, if f is dissipative then the semiflow π_0 has a global attractor $\mathcal{A}_0$ and the family of attractors $(\mathcal{A}_\epsilon)_{\epsilon\geq 0}$ is upper-semicontinuous at $\epsilon = 0$. More precisely, we have the following result:

Upper semicontinuity of attractors. *(see Theorem 5.10 in* [33]*) The family of attractors $(\mathcal{A}_\epsilon)_{\epsilon\in[0,1]}$ is upper-semicontinuous at $\epsilon = 0$ with respect to the family $|\cdot|_\epsilon$ of norms, i.e.*

$$\lim_{\epsilon\to 0^+} \sup_{u\in\mathcal{A}_\epsilon} \inf_{v\in\mathcal{A}_0} |u-v|_\epsilon = 0.$$

Notice that each attractor $\mathcal{A}_\epsilon$ for $\epsilon \geq 0$ is a compact isolated π_ϵ-invariant set with Conley index $h(\pi_\epsilon, \mathcal{A}_\epsilon) \equiv \Sigma^0$. Therefore the above theorem implies that the invariant set $\mathcal{A}_0$ can be continued to a family of invariant sets with the same Conley index.

We will prove in this paper that the latter statement holds true for an arbitrary isolated invariant set of the limit semiflow:

Conley index continuation theorem. *Let $K_0 \subset H^1_s(\Omega)$ be a compact isolated invariant set of the limit semiflow π_0. Then there is an $\epsilon_0 > 0$ and for each $\epsilon \in\,]0, \epsilon_0]$ there is a compact isolated invariant set $K_\epsilon \subset H^1(\Omega)$ of the semiflow π_ϵ such that $h(\pi_\epsilon, K_\epsilon) \equiv h(\pi_0, K_0)$ and the family $(K_\epsilon)_{\epsilon\in[0,\epsilon_0]}$ is upper-semicontinuous at $\epsilon = 0$ with respect to the family $|\cdot|_\epsilon$ of norms, i.e.*

$$\lim_{\epsilon\to 0^+} \sup_{u\in K_\epsilon} \inf_{v\in K_0} |u-v|_\epsilon = 0.$$

Note that in this theorem the function f need not be dissipative and it may depend not only on u but also on ϵ, x, y and ∇u. For a precise statement see Theorem 3.5 and its Corollary 3.7 in Section 3 below.

The reader should notice that our Conley index continuation Theorem 3.5 is not a direct consequence of the classical continuation principle in Conley index theory. In fact, we are dealing here with a singular perturbation problem and the limiting semiflow (unlike in the regular perturbation case) lives in the Banach space $H^1_s(\Omega)$ which is different from the Banach space $H^1(\Omega)$ on which the approximating semiflows are defined. Thus, although the proof of Theorem 3.5 follows the lines of the classical Conley continuation principle for admissible semiflows (as given in [37, 39]) several new ideas are required to cope with the present, more difficult situation (see Section 3 and the Appendix for details).

For the proof of Theorem 3.5 we need some convergence results, as $\epsilon \to 0$, of the semiflows π_ϵ and the linear semigroups generated by the operators A_ϵ. These results, presented in Section 2, strengthen and extend the convergence results previously obtained in [33].

In section 4 we present various applications of Theorem 3.5. First we show in Subsection 4.1 that, under appropriate assumptions, an orbit σ_0 connecting equilibria u_0' and u_0''

of the limit semiflow π_0 can be continued, for all $\epsilon > 0$ small enough, to a 'nearby' family σ_ϵ of orbits connecting 'nearby' equilibria u'_ϵ and u''_ϵ of π_ϵ.

This is then applied to a specific connection problem in parabolic equations on an interval, considered by Fiedler and Rocha [14].

We then prove that, generically in the domain, the family of the j-th eigenvalues $\lambda_{\epsilon,j}$ of the operators A_ϵ is strictly monotone decreasing in $\epsilon > 0$, for $j \geq 2$ (Subsection 4.2). Using this, together with Theorem 3.5 and the Conley index product formula we obtain, under suitable resonance hypotheses for the limit semiflow π_0, the existence of nontrivial equilibria of π_ϵ bifurcating from the trivial equilibrium 0 of π_0 (Subsection 4.3). Finally, we show that these resonance hypotheses are satisfied for very simple two-dimensional domains (Subsection 4.3).

After the completion of this work we became aware of the preprint [25] of Q. Huang, in which the author proves an analogue of our Theorem 3.5 for ordinate set domains with $g \equiv 1$, i.e. domains Ω of the type $\Omega = U \times]0,1[$, where U is a bounded domain in $\mathbb{R}^M$ with $M = 1$ or 2. The corresponding squeezed domain Ω_ϵ is then of the form $\Omega_\epsilon = U \times]0,\epsilon[$. (See Theorem 6.1 in [25].) In the case $M = 1$ such domains are just rectangles.

The author obtains his result as an application of an abstract Conley index continuation theorem (see Theorem A in [25]). One could use that theorem, together with the convergence results established in Section 2 of the present paper and some additional estimates to give an alternative proof of our Theorem 3.5.

On the other hand, our proof of Theorem 3.5 uses only some general properties of the family π_ϵ of semiflows and the family Q_ϵ of projectors (cf. Section 2 and the Appendix below). By abstracting these properties one can establish a Conley index continuation result which is more general than Theorem A of [25]. In fact, we can replace the compactness condition ($\mathcal{C}$) of [25] by a weaker admissibility assumption of the type introduced in [37]. Details are given in the paper [7]. Let us only remark here that this more general abstract result, unlike the results of [25], can be applied to singular perturbation problems for semiflows generated by damped wave equations (cf. [20, 22, 8]), neutral functional differential equations or infinite delay equations. The solution operators of such semiflows are not compact and yet the admissibility condition is satisfied.

Let us also remark that it is possible to formulate a general continuation principle for the Conley connection matrix in singular perturbation problems. See the forthcoming paper [9] for details.

For more results on the squeezed domain problems see the paper [34], which, among other things, contains an existence theorem for inertial manifolds containing the attractors $\mathcal{A}_\epsilon$ of π_ϵ, improving and generalizing to arbitrary domains the results of Hale and Raugel mentioned above. Some other papers on thin domain problems are contained in the References, although the list is far from being complete.

The Reference-section also lists a few papers on the theory and applications of various versions of the Conley index.

2. Squeezed domains

In this section we will prove various convergence results for linear semigroups and non-linear semiflows generated by reaction-diffusion equations on squeezed domains. These results, of interest in their own right, play an essential role in establishing the Conley index continuation principle of Section 3.

Let us first recall some definitions and basic results about squeezed domains proved in [33, 34], referring the reader to those papers and to the paper [21] of Hale and Raugel for more details on the subject. In this paper all vector spaces are over the real numbers.

Definition 2.1. *Let V be a vector space and $a\colon V\times V\to\mathbb{R}$ and $b\colon V\times V\to\mathbb{R}$ be symmetric bilinear forms on V. If $\lambda\in\mathbb{R}$, $u\in V\setminus\{0\}$ satisfy*

$$a(u,v)=\lambda b(u,v)\qquad \textit{for all } v\in V$$

then we say that λ is an eigenvalue of the pair (a,b) *and u is* an eigenvector of the pair (a,b), corresponding to λ. *The dimension of the span of all eigenvectors of (a,b) corresponding to λ is called* the multiplicity of λ. *If the set of eigenvalues of (a,b) is countably infinite, contains a smallest element and if each eigenvalue has finite multiplicity then* the repeated sequence of the eigenvalues of (a,b) *is the uniquely determined nondecreasing sequence $(\lambda_n)_{n\in\mathbb{N}}$ which contains exactly the eigenvalues of (a,b) and the number of occurrences of each eigenvalue in this sequence is equal to its multiplicity.*

The following proposition is well-known:

Proposition 2.2. *Let V, H be two infinite dimensional Hilbert spaces. Suppose $V\subset H$ with compact inclusion, and V is dense in H. Let $\|\cdot\|$ and $|\cdot|$ denote the norms of V and H respectively, and $\langle\cdot,\cdot\rangle$ be the inner product of H and b be the restriction of this inner product to $V\times V$. Let $a\colon V\times V\to\mathbb{R}$ be a symmetric bilinear form on V. Assume that there are constants d, C, $\alpha\in\mathbb{R}$, $\alpha>0$, such that, for all u, $v\in V$,*

$$|a(u,v)|\le C\|u\|\|v\|$$
$$a(u,u)\ge \alpha\|u\|^2-d|u|^2.$$

Then the set of eigenvalues of (a,b) is countably infinite, it has a smallest element and each eigenvalue has finite multiplicity. Let $(\lambda_n)_{n\in\mathbb{N}}$ be the repeated sequence of the eigenvalues of (a,b). Then the following properties are satisfied:

1. *There exists an H-orthogonal sequence $(u_n)_{n\in\mathbb{N}}$ such that for every $k\in\mathbb{N}$, u_k is an eigenvector of (a,b) corresponding to λ_k.*
2. *Whenever $(u_n)_{n\in\mathbb{N}}$ is an H-orthogonal sequence such that for every $k\in\mathbb{N}$, u_k is an eigenvector of (a,b) corresponding to λ_k, then $(u_n)_{n\in\mathbb{N}}$ is H-complete. Moreover, the space V is characterized in the following way:*

$$V=\{\,v\in H\mid \sum_{n=1}^{\infty}\lambda_n|\langle v,u_n\rangle|^2/|u_n|^2<\infty\,\}.$$

3. *Let D be the set of all $u\in V$ such that there is a $w=w_u\in H$ with the property that*

$$a(u,v)=\langle w,v\rangle$$

for all $v \in V$. Then w_u is uniquely determined by u, the set D is a dense linear subspace both of V and of H, and the map

$$A\colon D \to H, \quad u \mapsto w_u$$

is a linear selfadjoint operator on $(H, \langle\cdot,\cdot\rangle)$ with compact resolvent. We call A the linear operator generated by the pair $(a, \langle\cdot,\cdot\rangle)$. A pair (λ, u) is an eigenvalue-eigenvector pair of A if and only if (λ, u) is an eigenvalue-eigenvector pair of (a, b). □

Let Ω be an arbitrary nonempty bounded domain in $\mathbb{R}^M \times \mathbb{R}^N$ with Lipschitz boundary and let $\epsilon > 0$ be arbitrary. Write (x, y) for a generic point of $\mathbb{R}^M \times \mathbb{R}^N$. As in the Introduction, let $T_\epsilon\colon \mathbb{R}^M \times \mathbb{R}^N \to \mathbb{R}^M \times \mathbb{R}^N$, $(x, y) \mapsto (x, \epsilon y)$, be the squeezing operator and $\Omega_\epsilon := T_\epsilon(\Omega)$ be the squeezed domain. Define the symmetric bilinear forms

$$\tilde{a}_\epsilon\colon H^1(\Omega_\epsilon) \times H^1(\Omega_\epsilon) \to \mathbb{R}$$

by

$$\tilde{a}_\epsilon(u, v) := \int_{\Omega_\epsilon} \nabla u \cdot \nabla v \, \mathrm{d}x \, \mathrm{d}y$$

and

$$a_\epsilon\colon H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$$

by

$$a_\epsilon(u, v) := \int_\Omega \left(\nabla_x u \cdot \nabla_x v + \frac{1}{\epsilon^2} \nabla_y u \cdot \nabla_y v \right) \mathrm{d}x \, \mathrm{d}y.$$

Note that the assignment

$$\Phi_\epsilon\colon u \mapsto u \circ T_\epsilon$$

restricts to linear isomorphisms $L^2(\Omega_\epsilon) \to L^2(\Omega)$ and $H^1(\Omega_\epsilon) \to H^1(\Omega)$.

Let $\tilde{b}_\epsilon$ be the restriction of $\langle\cdot,\cdot\rangle_{L^2(\Omega_\epsilon)}$ to $H^1(\Omega_\epsilon) \times H^1(\Omega_\epsilon)$ and let b be the restriction of $\langle\cdot,\cdot\rangle_{L^2(\Omega)}$ to $H^1(\Omega) \times H^1(\Omega)$.

It follows that a pair (λ, u) is an eigenvalue-eigenvector pair of $(\tilde{a}_\epsilon, \tilde{b}_\epsilon)$ if and only if the pair $(\lambda, \Phi_\epsilon u)$ is an eigenvalue-eigenvector pair of (a_ϵ, b). The linear operators $\tilde{A}_\epsilon$ (resp. A_ϵ) defined by $(\tilde{a}_\epsilon, \tilde{b}_\epsilon)$ (resp. (a_ϵ, b)) satisfy the following properties:

(1) $D(A_\epsilon) = \Phi_\epsilon(D(\tilde{A}_\epsilon))$;
(2) $A_\epsilon(\Phi_\epsilon u) = \Phi_\epsilon(\tilde{A}_\epsilon u)$ for $u \in D(\tilde{A}_\epsilon)$.

Proposition 2.2 implies that there exists a nondecreasing sequence $(\lambda_{\epsilon,j}, w_{\epsilon,j})_{j\in\mathbb{N}}$ of eigenvalue-eigenvector pairs of (a_ϵ, b) such that $(w_{\epsilon,j})_{j\in\mathbb{N}}$ is a complete orthonormal system on $L^2(\Omega)$. Moreover, $\lambda_{\epsilon,1} \equiv 0$ and $\lambda_{\epsilon,j} > 0$ for all $\epsilon > 0$ and $j \geq 2$. For convenience we set $\lambda_{\epsilon,0} := -\infty$.

Now define the "limit" space $H^1_s(\Omega)$ by

$$H^1_s(\Omega) = \{\, u \in H^1(\Omega) \mid \nabla_y u = 0 \,\}.$$

Note that $H^1_s(\Omega)$ is a closed linear subspace of $H^1(\Omega)$. It can be proved that $H^1_s(\Omega)$ has infinite dimension.

Let us also define the space $L^2_s(\Omega)$ to be the closure of the set $H^1_s(\Omega)$ in $L^2(\Omega)$. It follows that $L^2_s(\Omega)$ is a Hilbert space under the scalar product of $L^2(\Omega)$.

Now let $a_0\colon H^1_s(\Omega)\times H^1_s(\Omega)\to\mathbb{R}$ be the "limit" bilinear form defined by

$$a_0(u,v):=\int_\Omega \nabla u\cdot\nabla v\,\mathrm{d}x\,\mathrm{d}y=\int_\Omega \nabla_x u\cdot\nabla_x v\,\mathrm{d}x\,\mathrm{d}y.$$

The bilinear form a_0 on $H^1_s(\Omega)$ together with the scalar product

$$\langle u,v\rangle_{L^2_s(\Omega)}:=\int_\Omega uv\,\mathrm{d}x\,\mathrm{d}y\quad\text{on } L^2_s(\Omega)$$

satisfy the hypotheses of Proposition 2.2. Let b_0 be the restriction of $\langle\cdot,\cdot\rangle_{L^2(\Omega)}$ to $H^1_s(\Omega)\times H^1_s(\Omega)$.

Proposition 2.2 implies that there exists a nondecreasing sequence $(\lambda_{0,j},w_{0,j})_{j\in\mathbb{N}}$ of eigenvalue-eigenvector pairs of (a_0,b_0) such that $(w_{0,j})_{j\in\mathbb{N}}$ is a complete orthonormal system on $L^2_s(\Omega)$. Moreover, $\lambda_{0,1}\equiv 0$ and $\lambda_{0,j}>0$ for all $j\geq 2$. For convenience we set $\lambda_{0,0}:=-\infty$.

Denote by A_0 the operator defined by the pair (a_0,b_0).

Let us now recall the concept of a semiflow:

Let X be a topological space, let D be an open subset of $[0,\infty[\times X$ and $\pi\colon D\to X$ be a continuous map. We write $x\pi t:=\pi(t,x)$ for $(t,x)\in D$. The map π is called a *local semiflow on* X if the following properties are satisfied:

(1) For every $x\in X$ there is a number $\omega_x=\omega_x^\pi\in\,]0,\infty]$ such that $(t,x)\in D$ if and only if $0\leq t<\omega_x$.
(2) $x\pi 0=x$ for all $x\in X$.
(3) If $(t,x)\in D$ and $(s,x\pi t)\in D$ then $(t+s,x)\in D$ and

$$x\pi(t+s)=(x\pi t)\pi s.$$

If $\omega_x=\infty$ for every $x\in X$, then π is called a *global semiflow on* X.

Let J be an arbitrary interval in $\mathbb{R}$. A map $\sigma\colon J\to X$ is called a *solution of* π if for all $t\in J$ and $s\in[0,\infty[$ for which $t+s\in J$, it follows that $\sigma(t)\pi s$ is defined and $\sigma(t)\pi s=\sigma(t+s)$. If $0\in J$ and $\sigma(0)=x$, we say that σ is a *solution through* x. If $J=\mathbb{R}$ (respectively $J=\,]-\infty,0]$), then σ is called a *full solution* (respectively *full left solution*) relative to π.

Let π and π' be local semiflows on the topological space X and Y be an arbitrary subset of X. We say that π and π' *coincide on* Y if whenever J is an interval in $\mathbb{R}$ and $\sigma\colon J\to Y$ is a map then σ is a solution of π if and only if σ is a solution of π'.

Now let X and X' be topological spaces and let π (resp. π') be a local semiflow on X (resp. on X'). We say that the map $\Phi\colon X\to X'$ *conjugates* π *with* π' if Φ is a homeomorphism and $\sigma\colon J\to X$ is a solution of π if and only if $\Phi\circ\sigma\colon J\to X'$ is a solution of π'.

Example 2.3. *Let X be a Banach space and A be a sectorial operator in X generating the family X^β, $\beta\geq 0$, of fractional power spaces. Fix an $\alpha\in[0,1[$ and suppose U is an open set in X^α and $f\colon U\to X$ is a locally Lipschitzian map. The equation*

$$\dot u=-Au+f(u)$$

defines, in the usual way, a local semiflow $\pi_{A,f}$ on X^α. (see [24] *or* [39]*). If f is globally Lipschitzian on U, then $\pi_{A,f}$ is a global semiflow.*

Example 2.4. *Let A and U be as in the Example 2.3. Moreover, let $f\colon U \to X$ and $f'\colon U \to X$ be locally Lipschitzian maps. If Y is a subset of U and $f(u) = f'(u)$ on Y, then $\pi_{A,f}$ coincides with $\pi_{A,f'}$ on Y. This easily follows from the definition of the solution of parabolic equations (see* [24] *or* [39]*).*

For every $\epsilon > 0$ the operator A_ϵ (resp. $\tilde{A}_\epsilon$) is sectorial on $X = L^2(\Omega)$ (resp. on $X = L^2(\Omega_\epsilon)$) and the corresponding fractional power space X^α with $\alpha = 1/2$ satisfies $X^\alpha = H^1(\Omega)$ (resp. $X^\alpha = H^1(\Omega_\epsilon)$). If $f_\epsilon\colon H^1(\Omega) \to L^2(\Omega)$ (resp. $\tilde{f}_\epsilon\colon H^1(\Omega_\epsilon) \to L^2(\Omega_\epsilon)$) is a locally Lipschitzian map we thus obtain the corresponding local semiflow $\pi_{\epsilon,f_\epsilon} := \pi_{A_\epsilon,f_\epsilon}$ (resp. $\tilde{\pi}_{\epsilon,\tilde{f}_\epsilon} := \pi_{\tilde{A}_\epsilon,\tilde{f}_\epsilon}$) on $H^1(\Omega)$ (resp. on $H^1(\Omega_\epsilon)$). Note that, given a locally Lipschitzian map $\tilde{f}_\epsilon\colon H^1(\Omega_\epsilon) \to L^2(\Omega_\epsilon)$, the map $\Phi_\epsilon(u) = u \circ T_\epsilon$ introduced above conjugates the semiflow $\tilde{\pi}_{\epsilon,\tilde{f}_\epsilon}$ with the semiflow $\pi_{\epsilon,f_\epsilon}$ where $f_\epsilon := \Phi_\epsilon \circ \tilde{f}_\epsilon \circ {\Phi_\epsilon}^{-1}$.

Now let $g_\epsilon\colon \Omega_\epsilon \times \mathbb{R} \times \mathbb{R}^M \times \mathbb{R}^N \to \mathbb{R}$ be a given function and $\tilde{f}_\epsilon := \hat{g}_\epsilon$ be the Nemitski operator generated by g_ϵ, i.e. for $u\colon \Omega_\epsilon \to \mathbb{R}$ set

$$\tilde{f}_\epsilon(u)(x,y) := g_\epsilon((x,y), u(x,y), \nabla_x u(x,y), \nabla_y u(x,y)), \text{ for } (x,y) \in \Omega_\epsilon.$$

Suppose $\tilde{f}_\epsilon$ restricts to a locally Lipschitzian map from $H^1(\Omega_\epsilon)$ to $L^2(\Omega_\epsilon)$. Then the map $f_\epsilon := \Phi_\epsilon \circ \tilde{f}_\epsilon \circ {\Phi_\epsilon}^{-1}$ is clearly given by

(1)

$$f_\epsilon(u)(x,y) := g_\epsilon((x,\epsilon y), u(x,y), \nabla_x u(x,y), \frac{1}{\epsilon}\nabla_y u(x,y)), \text{ for } u \in H^1(\Omega) \text{ and } (x,y) \in \Omega.$$

Now note that the "limit" operator A_0 is sectorial on $X = L^2_s(\Omega)$ and the corresponding fractional power space X^α with $\alpha = 1/2$ satisfies $X^\alpha = H^1_s(\Omega)$. If $f_0\colon H^1_s(\Omega) \to L^2_s(\Omega)$ is a locally Lipschitzian map we thus obtain the corresponding local semiflow $\pi_{0,f_0} := \pi_{A_0,f_0}$ on $H^1_s(\Omega)$. Again, if f_0 is globally Lipschitzian, then π_{0,f_0} is a global semiflow.

From now on, unless otherwise specified, we will work in a fixed bounded domain Ω and will often write $|\cdot|_{H^1}$ for $|\cdot|_{H^1(\Omega)}$ and $|\cdot|_{L^2}$ for $|\cdot|_{L^2(\Omega)}$. Moreover, we will write $\langle u, v\rangle$ to denote the scalar product in $L^2(\Omega)$.

For every $\epsilon > 0$ one can define on $H^1(\Omega)$ the norm

$$|u|_\epsilon := \left(a_\epsilon(u,u) + |u|^2_{L^2(\Omega)}\right)^{1/2}.$$

This norm is equivalent to $|\cdot|_{H^1(\Omega)}$ for every fixed $\epsilon > 0$, but $|u|_\epsilon \to \infty$ as $\epsilon \to 0^+$ whenever $\nabla_y u \neq 0$ in $L^2(\Omega)$. Note that $|u|_\epsilon = |u|_{H^1(\Omega)}$, whenever $\epsilon > 0$ and $u \in H^1_s(\Omega)$. It is easily seen that for every $u \in H^1(\Omega)$

$$|u|^2_\epsilon = \sum_{j=1}^{\infty} (\lambda_{\epsilon,j} + 1)|\langle u, w_{\epsilon,j}\rangle_{L^2(\Omega)}|^2.$$

Moreover, for every $u \in H^1_s(\Omega)$

$$|u|_{H^1(\Omega)} = \sum_{j=1}^{\infty} (\lambda_{0,j} + 1)|\langle u, w_{0,j}\rangle_{L^2(\Omega)}|^2.$$

The following concept will play a crucial in the convergence results established below:

Definition 2.5. *Given ϵ_0 with $0 < \epsilon_0 \leq 1$ we say that the family $(f_\epsilon)_{\epsilon\in[0,\epsilon_0]}$ of maps* satisfies hypothesis $(A1)$ *if the following properties are satisfied:*

(1) $f_\epsilon\colon H^1(\Omega) \to L^2(\Omega)$ *for every* $\epsilon \in]0, \epsilon_0]$ *and* $f_0\colon H^1_s(\Omega) \to L^2_s(\Omega)$.
(2) $\lim_{\epsilon\to 0^+} |f_\epsilon(u) - f_0(u)|_{L^2} = 0$ *for every* $u \in H^1_s(\Omega)$.
(3) *For every* $M \in [0, \infty[$ *there is an* $L \in [0, \infty[$ *such that*

$$|f_\epsilon(u) - f_\epsilon(v)|_{L^2} \le L|u - v|_\epsilon$$

for $\epsilon \in]0, \epsilon_0]$ *and* u, $v \in H^1(\Omega)$ *satisfying* $|u|_\epsilon$, $|v|_\epsilon \le M$. *Moreover,*

$$|f_0(u) - f_0(v)|_{L^2} \le L|u - v|_{H^1}$$

for u, $v \in H^1_s(\Omega)$ *satisfying* $|u|_{H^1}$, $|v|_{H^1} \le M$.

We say that $(f_\epsilon)_{\epsilon\in[0,\epsilon_0]}$ satisfies hypothesis $(A2)$ *if the following properties are satisfied:*

(1) $f_\epsilon\colon H^1(\Omega) \to L^2(\Omega)$ *for every* $\epsilon \in]0, \epsilon_0]$ *and* $f_0\colon H^1_s(\Omega) \to L^2_s(\Omega)$.
(2) $\lim_{\epsilon\to 0^+} |f_\epsilon(u) - f_0(u)|_{L^2} = 0$ *for every* $u \in H^1_s(\Omega)$.
(3) *There is an* $L \in [0, \infty[$ *such that*

$$|f_\epsilon(u) - f_\epsilon(v)|_{L^2} \le L|u - v|_\epsilon$$

for $\epsilon \in]0, \epsilon_0]$ *and all* u, $v \in H^1(\Omega)$. *Moreover,*

$$|f_0(u) - f_0(v)|_{L^2} \le L|u - v|_{H^1}$$

for all u, $v \in H^1_s(\Omega)$.

The following proposition shows how we can obtain, in applications, families of maps satisfying hypothesis $(A1)$:

Proposition 2.6. *Suppose the domain* Ω *satisfies the following condition:*

$$L^2_s(\Omega) = \{\, u \in L^2(\Omega) \mid \nabla_y u = 0 \,\}. \tag{2}$$

Let $G\colon \mathbb{R} \times \mathbb{R}^M \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^M \times \mathbb{R}^N \to \mathbb{R}$, $(\epsilon, x, y, \xi, \eta, \zeta) \mapsto G(\epsilon, x, y, \xi, \eta, \zeta)$, *be a* C^1-*function for which there are constants* β, γ *and* $C \in [0, \infty[$ *such that for all* $(\epsilon, x, y, \xi, \eta, \zeta) \in \mathbb{R} \times \mathbb{R}^M \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^M \times \mathbb{R}^N$ *the following estimates are satisfied:*

(1) $|\partial_\epsilon G(\epsilon, x, y, \xi, \eta, \zeta)| \le C(1 + |\xi|^\beta)$;
(2) $|\nabla_y G(\epsilon, x, y, \xi, \eta, \zeta)| \le C(1 + |\xi|^\beta)$;
(3) $|\partial_\xi G(\epsilon, x, y, \xi, \eta, \zeta)| \le C(1 + |\xi|^\gamma)$;
(4) $|\nabla_\eta G(\epsilon, x, y, \xi, \eta, \zeta)| + |\nabla_\zeta G(\epsilon, x, y, \xi, \eta, \zeta)| \le C$.

If $n := M + N > 2$ *then we also assume that* $\beta \le 2^*/2$ *and* $\gamma \le (2^*/2) - 1$, *where* $2^* := 2n/(n-2)$.

For $\epsilon > 0$ *define the function* $g_\epsilon\colon \Omega_\epsilon \times \mathbb{R} \times \mathbb{R}^M \times \mathbb{R}^N \to \mathbb{R}$ *by*

$$g_\epsilon((x,y), \xi, \eta, \zeta) = G(\epsilon, x, y, \xi, \eta, \zeta) \text{ for } ((x,y), \xi, \eta, \zeta) \in \Omega_\epsilon \times \mathbb{R} \times \mathbb{R}^M \times \mathbb{R}^N.$$

Then the Nemitski operator $\tilde f_\epsilon := \hat g_\epsilon$ *defined by the function* g_ϵ *is a well-defined map from* $H^1(\Omega_\epsilon)$ *to* $L^2(\Omega_\epsilon)$. *Define* $f_\epsilon\colon H^1(\Omega) \to L^2(\Omega)$ *by* $f_\epsilon = \Phi_\epsilon \circ \tilde f_\epsilon \circ {\Phi_\epsilon}^{-1}$. *More specifically,* f_ϵ *is given by* (1).

Furthermore, for $u \in H^1_s(\Omega)$ *define* $f_0(u)\colon \Omega \to \mathbb{R}$ *by*

$$f_0(u)(x,y) := G(0, x, 0, u(x,y), \nabla_x u(x,y), 0) \text{ for } (x,y) \in \Omega.$$

Then $f_0(u) \in L^2_s(\Omega)$ *and the family* $(f_\epsilon)_{\epsilon\in[0,\epsilon_0]}$ *satisfies hypothesis* $(A1)$.

Proof. Since $H^1(\Omega)$ is continuously contained in $L^\sigma(\Omega)$ where $\sigma \in [1,\infty[$ for $n = 2$ and $\sigma \in [1, 2^*]$ for $n > 2$, it follows by an application of the mean-value theorem and Hölder inequality, using estimates 1 - 4 above, that $f_\epsilon(H^1(\Omega)) \subset L^2(\Omega)$ for $\epsilon > 0$ and conditions 2 and 3 in the definition of hypothesis $(A1)$ are satisfied. Let $u \in H^1_s(\Omega)$ be arbitrary and $v := f_0(u)$. Then an application of Theorem 2.5 in [33] implies that $\nabla_y v = 0$. Now condition (2) implies that $v \in L^2_s(\Omega)$. This shows that hypothesis $(A1)$ is satisfied. The proof is complete. □

Remark 2.7. *1. Condition* (2) *is satisfied for domains* Ω *with connected vertical sections, i.e. domains such that the set*

$$\Omega_x := \{\, y \mid (x, y) \in \Omega \,\}$$

is connected for all $x \in \mathbb{R}^M$. *In fact in such cases functions* $u \in L^2(\Omega)$ *with* $\nabla_y u = 0$ *are actually functions depending only on the variables* x. *By appropriately regularizing these functions with respect to these variables we can prove that* $u \in L^2_s(\Omega)$. *Obvious details are omitted. In particular, condition* (2) *is satisfied for domains* Ω *considered by Hale and Raugel. Condition* (2) *is also satisfied for nicely decomposed domains. This follows from results in* [34].

2. If the function G *is independent of the variables* η *and* ζ, *then condition* (2) *is superfluous. In fact, the proof that* $f_0(H^1_s(\Omega)) \subset L^2_s(\Omega)$ *is, in that case, accomplished by a simple modification of the proof of Theorem 5.3 in* [33].

3. Without changing the proof we may relax the growth conditions 1 - 3 in Proposition 2.6. In fact, we may allow factors involving certain positive powers of $|\eta|$ *and* $|\zeta|$. *We do not give a precise statement here since our main interest is in functions* G *which do not depend on the variables* η *and* ζ, *so that the resulting semiflows* $\pi_{\epsilon,f_\epsilon}$ *are gradient-like.*

Condition $(A1)$ is stronger than condition $(A2)$. However, in many cases, one can modify a given family of maps satisfying $(A1)$ in such a way as to obtain a family satisfying $(A2)$ and so that both families coincide on a given bounded set. This is made more precise in the following result, which is obtained by a simple calculation, using Example 2.4:

Proposition 2.8. *Let* $0 < \epsilon_0 \le 1$ *and* $(f_\epsilon)_{\epsilon\in[0,\epsilon_0]}$ *satisfy hypothesis* $(A1)$. *Let* Y *be the open ball in* $H^1(\Omega)$ *at zero with radius* $r > 0$. *Choose a smooth function* $h\colon \mathbb{R} \to \mathbb{R}$ *such that*

$$h(s) = \begin{cases} 1 & \text{if } |s| < r, \\ 0 & \text{if } |s| > 2r. \end{cases}$$

For every $\epsilon \in]0, \epsilon_0]$ *define the map* $f'_\epsilon\colon H^1(\Omega) \to L^2(\Omega)$ *by*

$$f'_\epsilon(u) = h(|u|_\epsilon) f_\epsilon(u).$$

Moreover, define $f'_0\colon H^1(\Omega) \to L^2(\Omega)$ *by*

$$f'_0(u) = h(|u|_{H^1}) f_0(u).$$

Then the family $(f'_\epsilon)_{\epsilon\in[0,\epsilon_0]}$ *satisfies hypothesis* $(A2)$. *Moreover,* $f_\epsilon(u) = f'_\epsilon(u)$ *for* $\epsilon \in]0, \epsilon_0]$ *and* $u \in Y$. *Besides,* $f_0(u) = f'_0(u)$ *for* $u \in Y \cap H^1_s(\Omega)$. *Consequently, the local semiflows* $\pi_{\epsilon,f_\epsilon}$ *and* $\pi_{\epsilon,f'_\epsilon}$ *coincide on* $Y_\epsilon := \{\, u \in H^1(\Omega) \mid |u|_\epsilon < r \,\}$ *for* $\epsilon \in]0, \epsilon_0]$ *and on* $Y \cap H^1_s(\Omega)$ *for* $\epsilon = 0$. □

We can now state our first convergence result:

Theorem 2.9. *Let $(\epsilon_n)_{n\in\mathbb{N}}$ and $(t_n)_{n\in\mathbb{N}}$ be arbitrary sequences of positive numbers convergent to zero. Let $u_0 \in H^1_s(\Omega)$ and $(u_n)_{n\in\mathbb{N}}$ be a sequence in $H^1(\Omega)$ such that*

$$|u_n - u_0|_{\epsilon_n} \to 0 \quad \text{as } n \to \infty.$$

Then

$$\left|e^{-t_n A_{\epsilon_n}} u_n - u_0\right|_{\epsilon_n} \to 0 \quad \text{as } n \to \infty.$$

Proof. Given $v \in H^1(\Omega)$, $t > 0$ and $\epsilon > 0$ we have

$$\left|e^{-tA_\epsilon} v\right|_\epsilon^2 = \sum_{j=1}^\infty (\lambda_{\epsilon,j}+1)\left|\langle e^{-tA_\epsilon} v, w_{\epsilon,j}\rangle\right|^2 = \sum_{j=1}^\infty (e^{-t\lambda_{\epsilon,j}})^2(\lambda_{\epsilon,j}+1)\left|\langle v, w_{\epsilon,j}\rangle\right|^2.$$

Since $\lambda_{\epsilon,j} \geq 0$, it follows that

$$\left|e^{-tA_\epsilon} v\right|_\epsilon \leq |v|_\epsilon.$$

Hence we obtain

$$\begin{aligned}\left|e^{-t_n A_{\epsilon_n}} u_n - u_0\right|_{\epsilon_n} &\leq \left|e^{-t_n A_{\epsilon_n}} (u_n - u_0)\right|_{\epsilon_n} + \left|e^{-t_n A_{\epsilon_n}} u_0 - u_0\right|_{\epsilon_n} \\ &\leq |u_n - u_0|_{\epsilon_n} + \left|e^{-t_n A_{\epsilon_n}} u_0 - u_0\right|_{\epsilon_n}.\end{aligned}$$

Thus we only have to prove that

$$\left|e^{-t_n A_{\epsilon_n}} u_0 - u_0\right|_{\epsilon_n} \to 0 \quad \text{as } n \to \infty.$$

Let $\delta > 0$ be arbitrary. It follows that

$$|u_0|_{H^1}^2 = \sum_{j=1}^\infty (\lambda_{0,j}+1)\left|\langle u_0, w_{0,j}\rangle\right|^2 < \infty.$$

Hence there is a $k \in \mathbb{N}$ such that

$$d_0 := \sum_{j=k}^\infty (\lambda_{0,j}+1)\left|\langle u_0, w_{0,j}\rangle\right|^2 < \delta.$$

Let $P_n \colon L^2(\Omega) \to L^2(\Omega)$ be the L^2-orthogonal projection onto the span of the vectors $\{w_{\epsilon_n,1}, \ldots, w_{\epsilon_n,k-1}\}$. Then

$$|u_0|_{H^1}^2 = |u_0|_\epsilon^2 = \sum_{j=1}^\infty (\lambda_{\epsilon_n,j}+1)\left|\langle u_0, w_{\epsilon_n,j}\rangle\right|^2 = c_n + d_n,$$

where

$$c_n := \sum_{j=1}^{k-1} (\lambda_{\epsilon_n,j}+1)\left|\langle u_0, w_{\epsilon_n,j}\rangle\right|^2$$

and

$$d_n := \sum_{j=k}^\infty (\lambda_{\epsilon_n,j}+1)\left|\langle u_0, w_{\epsilon_n,j}\rangle\right|^2.$$

Theorem 3.3 in [33] implies that for an appropriate choice of the eigenvectors $(w_{0,j})_{j\in\mathbb{N}}$

$$c_n \to \sum_{j=1}^{k-1} (\lambda_{0_n,j}+1)\left|\langle u_0, w_{0,j}\rangle\right|^2 =: c_0 \quad \text{as } n \to \infty.$$

Consequently, for $n \to \infty$,

$$d_n \to |u_0|_{H^1}^2 - c_0 = d_0 < \delta.$$

Thus there is an n_0 such that $0 \leq d_n < \delta$ for $n \geq n_0$. Now

$$\left|e^{-t_n A_{\epsilon_n}} u_0 - u_0\right|_{\epsilon_n}^2 = \sum_{j=1}^{\infty} (e^{-t_n \lambda_{\epsilon_n,j}} - 1)^2 (\lambda_{\epsilon_n,j} + 1) \left|\langle u_0, w_{\epsilon_n,j}\rangle\right|^2 = c_n' + d_n',$$

where

$$c_n' := \sum_{j=1}^{k-1} (e^{-t_n \lambda_{\epsilon_n,j}} - 1)^2 (\lambda_{\epsilon_n,j} + 1) \left|\langle u_0, w_{\epsilon_n,j}\rangle\right|^2$$

and

$$d_n' := \sum_{j=k}^{\infty} (e^{-t_n \lambda_{\epsilon_n,j}} - 1)^2 (\lambda_{\epsilon_n,j} + 1) \left|\langle u_0, w_{\epsilon_n,j}\rangle\right|^2.$$

Now Theorem 3.3 in [33] implies that $t_n \cdot \lambda_{\epsilon_n,j} \to 0 \cdot \lambda_{0,j}$ as $n \to \infty$, so, again by Theorem 3.3 in [33],

$$c_n' \to 0 \quad \text{as } n \to \infty,$$

so there is an $n_1 \geq n_0$ such that

$$c_n' < \delta \quad \text{for } n \geq n_1.$$

Furthermore, $(e^{-t_n \lambda_{\epsilon_n,j}} - 1)^2 \leq 1$ so

$$d_n' \leq \sum_{j=k}^{\infty} (\lambda_{\epsilon_n,j} + 1) \left|\langle u_0, w_{\epsilon_n,j}\rangle\right|^2 = d_n < \delta.$$

Therefore

$$\left|e^{-t_n A_{\epsilon_n}} u_0 - u_0\right|_{\epsilon_n}^2 < 2\delta \quad \text{for } n \geq n_1.$$

Since δ is arbitrary, the theorem is proved. □

Remark 2.10. *Let $(\epsilon_n)_{n\in\mathbb{N}}$ and $(t_n)_{n\in\mathbb{N}}$ be arbitrary sequences of positive numbers convergent to zero. Let $u_0 \in L^2_s(\Omega)$ and $(u_n)_{n\in\mathbb{N}}$ be a sequence in $L^2(\Omega)$ converging to u_0 in $L^2(\Omega)$. Then*

$$\left|e^{-t_n A_{\epsilon_n}} u_n - u_0\right|_{L^2} \to 0 \quad \textit{as } n \to \infty.$$

The proof of this assertion is completely analogous to (and easier than) the proof of Theorem 2.9.

For the rest of this section let $\epsilon_0 > 0$ and $(f_\epsilon)_{\epsilon\in[0,\epsilon_0]}$ satisfy hypothesis $(A2)$. Write π_ϵ for $\pi_{\epsilon,f_\epsilon}$.

Before stating our next convergence result let us note that there is a positive real constant C_1 such that for all $\epsilon > 0$, $r > 0$ and $u \in L^2(\Omega)$

$$|e^{-A_\epsilon r} u|_\epsilon \leq \left(C_1 r^{-1/2} + 1\right) |u|_{L^2}. \tag{3}$$

Moreover, for all $u \in L^2_s(\Omega)$

$$|e^{-A_0 r} u|_\epsilon \leq \left(C_1 r^{-1/2} + 1\right) |u|_{L^2}. \tag{4}$$

We will need these estimates in the proofs to follow.

We can now state our second convergence theorem:

Theorem 2.11. *Let $(\epsilon_n)_{n\in\mathbb{N}}$ and $(t_n)_{n\in\mathbb{N}}$ be arbitrary sequences of positive numbers convergent to zero. Moreover, let $u_0 \in H^1_s(\Omega)$ and $(u_n)_{n\in\mathbb{N}}$ be a sequence in $H^1(\Omega)$ such that*

$$|u_n - u_0|_{\epsilon_n} \to 0 \quad \text{as } n \to \infty.$$

Then

$$\left|u_n \pi_{\epsilon_n} t_n - u_0\right|_{\epsilon_n} \to 0 \quad \text{as } n \to \infty.$$

Proof. We may assume that $t_n \le 1$ for all $n \in \mathbb{N}$. For every $t \in]0,1]$ we have

$$u_n\pi_{\epsilon_n} t - u_0 = e^{-tA_{\epsilon_n}} u_n + \int_0^t e^{-(t-s)A_{\epsilon_n}} (f_{\epsilon_n}(u_n\pi_{\epsilon_n} s) - f_{\epsilon_n}(u_0))\,\mathrm{d}s$$
$$+ \int_0^t e^{-(t-s)A_{\epsilon_n}} f_{\epsilon_n}(u_0)\,\mathrm{d}s - u_0.$$

Therefore the estimate (3) implies that

$$\left|u_n\pi_{\epsilon_n} t - u_0\right|_{\epsilon_n} \le \left|e^{-tA_{\epsilon_n}} u_n\right|_{\epsilon_n} + \int_0^t C_1 L((t-s)^{-1/2}+1)\left|u_n\pi_{\epsilon_n} s - u_0\right|_{\epsilon_n} \mathrm{d}s$$
$$+ \int_0^t C_1((t-s)^{-1/2}+1)|f_{\epsilon_n}(u_0)|_{L^2} + \left|u_0\right|_{\epsilon_n}.$$

By hypothesis $(A2)2$ we have that the sequence $f_{\epsilon_n}(u_0)$, $n \in \mathbb{N}$, is bounded in $L^2(\Omega)$. Since the sequence $(|u_n|_{\epsilon_n})_{n\in\mathbb{N}}$ is bounded there is a positive real constant C_2 such that for all $n \in \mathbb{N}$ and every $t \in]0,1]$

$$\left|u_n\pi_{\epsilon_n} t - u_0\right|_{\epsilon_n} \le C_2 + C_2\int_0^t (t-s)^{-1/2}\left|u_n\pi_{\epsilon_n} s - u_0\right|_{\epsilon_n} \mathrm{d}s.$$

Thus an application of Henry's Inequality ([24], Lemma 7.1.1) implies that

$$\left|u_n\pi_{\epsilon_n} t - u_0\right|_{\epsilon_n} \le C_2 + C_2\int_0^t \rho(t-s)\,\mathrm{d}s,$$

where

$$\rho(x) := \sum_{n=1}^{\infty} \frac{(C_2\Gamma(\beta))^n}{\Gamma(n\beta)} x^{n\beta-1}$$

with $\beta := 1/2$. The function $\rho\colon\ ]0,\infty[\to]0,\infty[$ is well defined and continuous on $]0,\infty[$ and it satisfies the estimate

$$\rho(x) \le C_2 x^{-1/2} + C_3 \quad \text{for } x \in]0,1],$$

where C_3 is a constant. It follows that there is a constant M such that for all $n \in \mathbb{N}$ and every $t \in]0,1]$

$$\left|u_n\pi_{\epsilon_n} t - u_0\right|_{\epsilon_n} \le M.$$

It follows that for all $n \in \mathbb{N}$ and every s with $0 < s \le 1$

$$|f_{\epsilon_n}(u_n\pi_{\epsilon_n} s)|_{L^2} \le |f_{\epsilon_n}(u_n\pi_{\epsilon_n} s) - f_{\epsilon_n}(u_0)|_{L^2} + |f_{\epsilon_n}(u_0)|_{L^2} \le LM + |f_{\epsilon_n}(u_0)|_{L^2} \le C_4$$

for some constant C_4. Since

$$u_n\pi_{\epsilon_n} t_n - u_0 = e^{-t_nA_{\epsilon_n}} u_n - u_0 + \int_0^{t_n} e^{-(t_n-s)A_{\epsilon_n}} f_{\epsilon_n}(u_n\pi_{\epsilon_n} s)\,\mathrm{d}s$$

we conclude that

$$\left|u_n\pi_{\epsilon_n}t_n - u_0\right|_{\epsilon_n} \leq \left|e^{-t_nA_{\epsilon_n}}u_n - u_0\right|_{\epsilon_n} + C_5\int_0^{t_n}(t_n-s)^{-1/2}\,\mathrm{d}s$$

for some constant C_5. Now an application of Theorem 2.9 completes the proof of the theorem. □

Remark 2.12. *Let $(\epsilon_n)_{n\in\mathbb{N}}$ and $(t_n)_{n\in\mathbb{N}}$ be arbitrary sequences of positive numbers convergent to zero. Let $u_0 \in L^2_s(\Omega)$ and $(u_n)_{n\in\mathbb{N}}$ be a sequence in $L^2(\Omega)$ converging to u_0 in $L^2(\Omega)$. Then*

$$\left|u_n\pi_{\epsilon_n}t_n - u_0\right|_{L^2(\Omega)} \to 0 \quad \text{as } n\to\infty.$$

The proof of this assertion is analogous to (and easier than) the proof of Theorem 2.11.

We can also state the following generalization of Theorem 5.1 in [33]:

Theorem 2.13. *Let $(\epsilon_n)_{n\in\mathbb{N}}$ be an arbitrary sequence of positive numbers convergent to zero. Moreover let $t\in\,]0,\infty[$ and $(t_n)_{n\in\mathbb{N}}$ be a sequence in $]0,\infty[$ converging to t. Finally, let $u_0\in H^1_s(\Omega)$ and $(u_n)_{n\in\mathbb{N}}$ be a sequence in $H^1(\Omega)$ such that*

$$|u_n - u_0|_{L^2} \to 0 \qquad \text{as } n\to\infty.$$

Then

$$\left|u_n\pi_{\epsilon_n}t_n - u_0\pi_0 t\right|_{\epsilon_n} \to 0 \quad \text{as } n\to\infty.$$

Proof. We modify the initial part of proof of Theorem 5.1 in [33] and therefore use the notation of that proof: write $|\cdot|_n := |\cdot|_{\epsilon_n}$, $A_n := A_{\epsilon_n}$, $A := A_0$, $\pi_n := \pi_{\epsilon_n}$, $\pi := \pi_0$ and $u := u_0$. Let $b\in[0,\infty[$ be arbitrary with $t_n\in\,]0,b]$, $n\in\mathbb{N}$, and $t\in\,]0,b]$. For every $t\in[0,b]$ we have, by the variation-of-constants formula,

$$u_n\pi_n t - u\pi t = e^{-A_nt}u_n - e^{-At}u +$$
$$\int_0^t e^{-A_n(t-s)}\big(f_{\epsilon_n}(u_n\pi_n s) - f_{\epsilon_n}(u\pi s)\big)\,\mathrm{d}s + \int_0^t\big(e^{-A_n(t-s)}f_{\epsilon_n}(u\pi s) - e^{-A(t-s)}f_0(u\pi s)\big)\,\mathrm{d}s.$$

Define the function $g_n\colon[0,b]\times[0,b]\to\mathbb{R}$ as follows: If $0<s<t$ then set

$$g_n(t,s) = |e^{-A_n(t-s)}f_{\epsilon_n}(u\pi s) - e^{-A(t-s)}f_0(u\pi s)|_n$$

and set $g_n(t,s)=0$ otherwise. The function g_n restricted to the set of (s,t) with $0<s<t$ is continuous. Thus g_n is measurable on $[0,b]\times[0,b]$. By Fubini's theorem the function

$$c_n(t) := \int_0^b g_n(t,s)\,\mathrm{d}s = \int_0^t g_n(t,s)\,\mathrm{d}s$$

is a.e. defined and measurable on $[0,b]$. Set

$$a_n(t) := |e^{-A_nt}u_n - e^{-At}u|_n + c_n(t) \quad \text{for } t\in\,]0,b]$$

and $a_n(0):=0$. It follows that a_n is measurable on $[0,b]$. Using the estimates (3) and (4) we obtain

$$|g_n(t,s)| \leq 2C_2\big(C_1(t-s)^{-1/2}+1\big) \quad \text{for } 0<s<t$$

where

$$C_2 := \max\{\sup_{s\in[0,b]}\sup_{n\in\mathbb{N}}|f_{\epsilon_n}(u\pi s)|_{L^2},\ \sup_{s\in[0,b]}|f_0(u\pi s)|_{L^2}\}.$$

Note that, by hypothesis $(A2)$

$$|f_{\epsilon_n}(u\pi s)|_{L^2} \leq |f_{\epsilon_n}(u\pi s) - f_{\epsilon_n}(u)|_{L^2} + |f_{\epsilon_n}(u)|_{L^2} \leq L|u\pi s - u|_n + |f_{\epsilon_n}(u)|_{L^2} \leq M$$

for some constant $M < \infty$, independent of $n \in \mathbb{N}$ and $s \in [0, b]$. Similarly, we may assume that

$$|f_0(u\pi s)|_{L^2} \le M, \ s \in [0, b].$$

This shows that $C_2 < \infty$.

The remaining part of the proof is now almost identical to the corresponding part of the proof of Theorem 5.1 in [33] and so we omit it here. $\square$

We thus obtain the following corollary:

Corollary 2.14. *Let $(\epsilon_n)_{n\in\mathbb{N}}$ be an arbitrary sequence of positive numbers convergent to zero. Moreover let $t \in [0, \infty[$ and $(t_n)_{n\in\mathbb{N}}$ be a sequence in $[0, \infty[$ converging to t. Finally, let $u_0 \in H^1_s(\Omega)$ and $(u_n)_{n\in\mathbb{N}}$ be a sequence in $H^1(\Omega)$ such that*

$$|u_n - u_0|_{\epsilon_n} \to 0 \quad \text{as } n \to \infty.$$

Then

$$\left|u_n\pi_{\epsilon_n}t_n - u_0\pi_0 t\right|_{\epsilon_n} \to 0 \quad \text{as } n \to \infty.$$

Proof. We may assume, of course, that all numbers t_n are positive. If $t = 0$, then the assertion follows from Theorem 2.11. If $t > 0$, the assertion follows from Theorem 2.13. $\square$

We also need the following

Proposition 2.15. *Let $(\epsilon_n)_{n\in\mathbb{N}}$ be an arbitrary sequence of positive numbers convergent to zero. Moreover, let $(t_n)_{n\in\mathbb{N}}$ be a sequence in $[0, \infty[$ converging to ∞. For every $n \in \mathbb{N}$ let $\sigma_n\colon [-t_n, 0] \to H^1(\Omega)$ be a solution of π_{ϵ_n}. Assume that there is a $C \in [0, \infty[$ such that*

$$|\sigma_n(t)|_{\epsilon_n} \le C \quad \text{for all } n \in \mathbb{N} \text{ and } t \in [-t_n, 0].$$

Then there exists a subsequence of $(\sigma_n)_{n\in\mathbb{N}}$, denoted by the same symbol $(\sigma_n)_{n\in\mathbb{N}}$, and there is a full left solution $\sigma\colon\ ]-\infty, 0] \to H^1_s(\Omega)$ of π_0, such that for every $t \in]-\infty, 0]$,

$$|\sigma_n(t) - \sigma(t)|_{\epsilon_n} \to 0 \quad \text{as } n \to \infty.$$

Remark 2.16. *Note that for every $t \in]-\infty, 0]$, there is some n_0 with $-t_n \le t$ for all $n \ge n_0$, so the sequence $(\sigma_n(t))$ is well defined for $n \ge n_0$.*

Proof. The proof of this result is completely analogous to that of Corollary 5.2 in [33]. We therefore omit the details. $\square$

Note that, for every fixed $\epsilon > 0$, the space $H^1(\Omega)$ is a Hilbert space under the scalar product

$$(u, v)_\epsilon := a_\epsilon(u, v) + \langle u, v\rangle, \quad u, v \in H^1(\Omega).$$

Since $H^1_s(\Omega)$ is a closed subspace of $H^1(\Omega)$, there is a map

$$Q_\epsilon\colon H^1(\Omega) \to H^1(\Omega)$$

which is the orthogonal projector onto $H^1_s(\Omega)$ with respect to the scalar product $(\cdot, \cdot)_\epsilon$. This projector will play a crucial role in the sequel. Note that, for every $u \in H^1(\Omega)$,

$$|u|^2_\epsilon = |Q_\epsilon u|^2_\epsilon + |(I - Q_\epsilon)u|^2_\epsilon,$$

where, as usual, I is the identity map on $H^1(\Omega)$.

Remark 2.17. *If Ω is a product domain, i.e. if $\Omega = U \times V$ with $U \subset \mathbb{R}^M$ and $V \subset \mathbb{R}^N$, and if κ denotes the measure of V then, as it is easily proved, the projector Q_ϵ has the explicit form*

$$(Q_\epsilon u)(x) = \frac{1}{\kappa}\int_V u(x,y)\,\mathrm{d}y, \text{ for } u \in H^1(\Omega) \text{ and } x \in U$$

i.e. Q_ϵ is the mean-value operator with respect to the variable $y \in V$. No such explicit form of Q_ϵ is known for general domains Ω.

Definition 2.18. *Given a subset V of $H^1_s(\Omega)$, $\eta > 0$ and $\epsilon > 0$, define the 'inflated' subsets $]V[_{\epsilon,\eta}$ and $[V]_{\epsilon,\eta}$ of $H^1(\Omega)$ as follows:*

$$]V[_{\epsilon,\eta} := \{\, u \in H^1(\Omega) \mid Q_\epsilon u \in V \text{ and } \left|(I - Q_\epsilon)u\right|_\epsilon < \eta \,\},$$
$$[V]_{\epsilon,\eta} := \{\, u \in H^1(\Omega) \mid Q_\epsilon u \in V \text{ and } \left|(I - Q_\epsilon)u\right|_\epsilon \le \eta \,\}.$$

Lemma 2.19. *Let V be a subset of $H^1_s(\Omega)$. Then for every $\epsilon > 0$ and $\eta > 0$*

$$\mathrm{cl}_\epsilon\,]V[_{\epsilon,\eta} = [\mathrm{cl}\, V]_{\epsilon,\eta}\,.$$

Proof. Since $[\mathrm{cl}\, V]_{\epsilon,\eta}$ is closed in $H^1(\Omega)$, we obtain that $\mathrm{cl}_\epsilon\,]V[_{\epsilon,\eta} \subset [\mathrm{cl}\, V]_{\epsilon,\eta}$. Conversely, let $u \in [\mathrm{cl}\, V]_{\epsilon,\eta}$. This implies that $Q_\epsilon u \in \mathrm{cl}\, V$ and $\left|(I - Q_\epsilon)u\right|_\epsilon \le \eta$. Therefore there exists a sequence $(w_n)_{n\in\mathbb{N}}$, $w_n \in V$, such that

$$\left|w_n - Q_\epsilon u\right|_{H^1} \to 0 \text{ as } n \to \infty.$$

Define

$$z_n := (I - Q_\epsilon)\frac{n}{n+1}u \text{ and } v_n := w_n + z_n.$$

It is clear that $v_n \in\,]V[_{\epsilon,\eta}$ and $\left|v_n - u\right|_\epsilon \to 0$ for $n \to \infty$. The lemma is proved. □

The following result is an immediate consequence of Proposition 2.15 and the proof of Corollary 5.2 in [33]:

Lemma 2.20. *Let S be a closed bounded set in $H^1_s(\Omega)$, $\eta > 0$ and $(\epsilon_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers converging to zero. For every $n \in \mathbb{N}$ let $\sigma_n \colon J_n \to [S]_{\epsilon_n,\eta}$ be a solution of π_{ϵ_n}. If $J_n \equiv \mathbb{R}$ for every $n \in \mathbb{N}$ (resp. if $J_n = [-t_n, 0]$, $n \in \mathbb{N}$, where $t_n \to \infty$), then there is a subsequence of $(\sigma_n)_{n\in\mathbb{N}}$, again denoted by $(\sigma_n)_{n\in\mathbb{N}}$ and there is a solution $\sigma \colon \mathbb{R} \to S$ (resp. $\sigma \colon\,]-\infty, 0] \to S$) of π_0 such that for every $t \in \mathbb{R}$ (resp. every $t \in\,]-\infty, 0]$)*

$$|\sigma_n(t) - \sigma(t)|_{\epsilon_n} \to 0 \text{ as } n \to \infty$$

so

$$|Q_{\epsilon_n}\sigma_n(t) - \sigma(t)|_{H^1} \to 0 \text{ and } |(I - Q_{\epsilon_n})\sigma_n(t)|_{\epsilon_n} \to 0 \text{ as } n \to \infty.$$

□

3. A Conley index continuation theorem

In this section, we will state our basic Conley index continuation theorem for thin domain problems. The proof of this theorem will be given in the Appendix.

We will first review some basic definitions and results concerning the Conley index theory for semiflows defined on a metric space. The reader is referred to [37] or [39] for the proofs of several of the results stated below and for further details on the subject.

Given a topological space Z and an arbitrary set Y, we define the quotient space Z/Y as follows: fix an arbitrary $p \notin Z$. Define the set Z/Y as

$$Z/Y := (Z \setminus Y) \cup \{p\}$$

and the map $q\colon Z \to Z/Y$ as

$$q(z) = \begin{cases} z & \text{if } z \in Z \setminus Y, \\ p & \text{otherwise.} \end{cases}$$

We commonly write $[z]$ instead of $q(z)$. We also write $[Y]$ instead of p.

Call a subset V of Z/Y *open in* Z/Y if and only if $q^{-1}(V)$ is open in Z. This defines a (quotient) topology on Z/Y. Note that if $Y \cap Z \neq \emptyset$, then q is a (surjective) identification map. If $Y \cap Z = \emptyset$, then $Z/Y = Z \cup \{p\}$ and $V \subset Z/Y$ is open in Z/Y if and only if $V \cap Z$ is open in Z.

For the rest of this section, unless specified otherwise, let X be a metric space and let π be a local semiflow on X.

Suppose that Y is a subset of X. We define the following subsets of X:

$$\begin{aligned} A_\pi^+(Y) :=& \{\, u \in X \mid u\pi\,[0, \omega_u[\,\subset Y \,\}, \\ A_\pi^-(Y) :=& \{\, u \in X \mid \text{there exists a solution } \sigma\colon\ ]-\infty, 0] \to X \text{ through } u \text{ with} \\ & \quad \sigma(]-\infty, 0]) \subset Y \,\}, \\ A_\pi(Y) :=& A_\pi^+(Y) \cap A_\pi^-(Y). \end{aligned}$$

A subset Y of X is called *invariant* (respectively *positively invariant*, respectively *negatively invariant*) relative to π if $Y = A_\pi(Y)$ (respectively $Y = A_\pi^+(Y)$, respectively $Y = A_\pi^-(Y)$).

Let N be a closed subset of X such that $K := A_\pi(N)$ is closed and $K \subset \operatorname{Int} N$. Then N is called an *isolating neighborhood of K relative to π* and K is called an *isolated invariant set relative to π*.

Let $B \subset X$ be a closed set and $u \in \partial B$. The point u is called a *strict egress* (respectively *strict ingress*, respectively *bounce-off*) of B, if for every solution $\sigma\colon\ [-\delta_1, \delta_2] \to X$ through u, with $\delta_1 \geq 0$ and $\delta_2 > 0$, the following properties hold:

(1) There exists an $\epsilon_2 \in\ ]0, \delta_2[$ such that $\sigma(t) \notin B$ (respectively $\sigma(t) \in \operatorname{Int} B$, respectively $\sigma(t) \notin B$), for $t \in\ ]0, \epsilon_2]$.
(2) If $\delta_1 > 0$, then there exists an $\epsilon_1 \in\ ]0, \delta_1[$ such that $\sigma(t) \in \operatorname{Int} B$ (respectively $\sigma(t) \notin B$, respectively $\sigma(t) \notin B$), for $t \in [-\epsilon_1, 0[$.

The set of all strict egress points (respectively strict ingress, respectively bounce-off) of the closed set B will be denoted by B^e (respectively B^i, respectively B^b). A closed set $B \subset X$ is called an *isolating block*, if $\partial B = B^e \cup B^i \cup B^b$ and $B^- := B^e \cup B^b$ is closed.

Let N be a closed subset of X. Then N is called *strongly π-admissible* if the following properties are satisfied:

(1) Whenever $u\pi t \in N$ for all $t \in [0, \omega_u[$ then $\omega_u = \infty$.

(2) Whenever $(u_n)_{n\in\mathbb{N}}$ is a sequence in X and $(t_n)_{n\in\mathbb{N}}$ is a sequence in $[0,\infty[$ such that $t_n \to \infty$ as $n \to \infty$ and $u_n\pi\,[0,t_n] \subset N$ for all $n \in \mathbb{N}$, then the sequence $(u_n\pi t_n)_{n\in\mathbb{N}}$ of endpoints has a convergent subsequence.

Theorem 3.1. *Let K be an isolated π-invariant set and N be a strongly π-admissible isolating neighborhood of K. Then there exists an open set V such that $B := \operatorname{cl} V$ is an isolating block such that $K \subset V \subset B \subset N$ and $\partial V = \partial B$.*

Proof. By Theorem 5.1 in Chapter 1 of [39] there exists an isolating block B' such that $K \subset \operatorname{Int} B' \subset B' \subset N$. Set $V := \operatorname{Int} B'$ and $B := \operatorname{cl} V$. It is easily verified that $B := \operatorname{cl} V$ is an isolating block such that $K \subset V \subset B \subset N$ and $\partial V = \partial B$. □

Let N, Y be subsets of X such that $Y \subset N$. The set Y is called *N-positively invariant* relative to π, if whenever $u \in X$, $t \geq 0$ are such that $u\pi\,[0,t] \subset N$ and $u \in Y$, then $u\pi\,[0,t] \subset Y$.

Definition 3.2. *Let N be a closed set in X and N_1 and N_2 be closed subsets of N. The pair (N_1, N_2) is called a* pseudo-index pair in N *if:*

1. *N_1 and N_2 are N-positively invariant;*
2. *whenever $u \in N_1$ and $u\pi t_0 \notin N$ for some $t_0 \in [0,\infty[$, then there exists a $t' \in [0,t_0]$ such that $u\pi\,[0,t'] \subset N$ and $u\pi t' \in N_2$.*

A pseudo index pair (N_1, N_2) in N is called an index pair in N *if $A_\pi(N)$ is closed and $A_\pi(N) \subset \operatorname{Int}(N_1 \setminus N_2)$.*

An isolating block B determines a special index pair in B, namely (B, B^-).

Let K be an isolated invariant set, N be a strongly π-admissible isolating neighborhood of K and (N_1, N_2) be an index pair in N. Then the homotopy type of the pointed space $(N_1/N_2, [N_2])$ depends only on the semiflow π and the isolated invariant set K (see Theorem 10.1 in [39]). The *Conley index*, $h(\pi, K)$, of the isolated invariant set K with respect to π is defined to be the homotopy type of $(N_1/N_2, [N_2])$.

Remark 3.3. *If N is a strongly π-admissible isolating neighborhood relative to π, we will sometimes write $h(\pi, N)$ to denote $h(\pi, A_\pi(N))$. This will not lead to confusion.*

The following result is obvious:

Proposition 3.4. *Let π and π' be local semiflows on the metric space X. Let Y be a subset of X on which π and π' coincide. Suppose N is closed in X and $N \subset Y$. Then $A_\pi(N) = A_{\pi'}(N)$. Moreover, N is a strongly admissible isolating neighborhood relative to π if and only if N is a strongly admissible isolating neighborhood relative to π' and in this case*

$$h(\pi, N) = h(\pi', N).$$

We can now state the main result of this section:

Theorem 3.5. *Let $\epsilon_0 > 0$ and $(f_\epsilon)_{\epsilon\in[0,\epsilon_0]}$ be a family of maps satisfying hypothesis (A1). Let N be a bounded isolating neighborhood for π_{0,f_0}. Then for every $\eta > 0$ there exists an $\epsilon^{\mathrm{c}} = \epsilon^{\mathrm{c}}(\eta) > 0$ such that for every $\epsilon \in\,]0, \epsilon^{\mathrm{c}}]$ the set $[N]_{\epsilon,\eta}$ is a strongly admissible isolating neighborhood relative to $\pi_{\epsilon,f_\epsilon}$ and*

$$h(\pi_{\epsilon,f_\epsilon}, [N]_{\epsilon,\eta}) = h(\pi_{0,f_0}, N). \tag{5}$$

Theorem 3.5 can also be reworded in terms of the semiflows $\tilde{\pi}_{\epsilon,\tilde{f}_\epsilon}$ generated by the original reaction-diffusion equations on the squeezed domains Ω_ϵ:

Theorem 3.6. *Let $\epsilon_0 > 0$ and $(f_\epsilon)_{\epsilon\in[0,\epsilon_0]}$ be a family of maps satisfying hypothesis $(A1)$. For $\epsilon > 0$ define $\tilde{f}_\epsilon\colon H^1(\Omega_\epsilon) \to L^2(\Omega_\epsilon)$ by $\tilde{f}_\epsilon := {\Phi_\epsilon}^{-1} \circ f_\epsilon \circ \Phi_\epsilon$. Let N be a bounded isolating neighborhood for π_{0,f_0}. Then for every $\eta > 0$ there exists an $\epsilon^{\mathrm{c}} = \epsilon^{\mathrm{c}}(\eta) > 0$ such that for every $\epsilon \in]0, \epsilon^{\mathrm{c}}]$ the set ${\Phi_\epsilon}^{-1}([N]_{\epsilon,\eta})$ is a strongly admissible isolating neighborhood relative to $\tilde{\pi}_{\epsilon,\tilde{f}_\epsilon}$ and*

$$h(\tilde{\pi}_{\epsilon,\tilde{f}_\epsilon}, {\Phi_\epsilon}^{-1}([N]_{\epsilon,\eta})) = h(\pi_{0,f_0}, N).$$

Proof. The isomorphism Φ_ϵ conjugates the local semiflow $\tilde{\pi}_{\epsilon,\tilde{f}_\epsilon}$ to the local semiflow $\pi_{\epsilon,f_\epsilon}$. Thus whenever S is a strongly admissible isolating neighborhood with respect to $\pi_{\epsilon,f_\epsilon}$, then ${\Phi_\epsilon}^{-1}(S)$ is a strongly admissible isolating neighborhood with respect to $\tilde{\pi}_{\epsilon,\tilde{f}_\epsilon}$ and

$$h(\pi_{\epsilon,f_\epsilon}, S) = h(\tilde{\pi}_{\epsilon,\tilde{f}_\epsilon}, {\Phi_\epsilon}^{-1}(S)).$$

Now Theorem 3.5 completes the proof. □

Theorem 3.5 yields the following corollary, which was stated, somewhat less precisely, in the Introduction:

Corollary 3.7. *For every $\eta > 0$ the family $(K_{\epsilon,\eta})_{\epsilon\in[0,\epsilon^{\mathrm{c}}(\eta)]}$ of invariant sets, where $K_{\epsilon,\eta} := A_{\pi_\epsilon}([N]_{\epsilon,\eta})$, $\epsilon \in]0, \epsilon^{\mathrm{c}}(\eta)]$, and $K_{0,\eta} = K_0 := A_{\pi_0}(N)$, is upper semicontinuous at $\epsilon = 0$ with respect to the family $|\cdot|_\epsilon$ of norms. In other words,*

$$\lim_{\epsilon\to 0^+} \sup_{u\in K_{\epsilon,\eta}} \inf_{v\in K_0} |u - v|_\epsilon = 0.$$

Proof. If the corollary is not true, then there are numbers η and $\beta > 0$, a sequence $\epsilon_n \to 0^+$ and a sequence $u_n \in K_{\epsilon_n,\eta}$, $n \in \mathbb{N}$, such that

$$\inf_{v\in K_0} |u_n - v|_\epsilon > \beta.$$

For every $n \in \mathbb{N}$ let $\sigma_n\colon \mathbb{R} \to [N]_{\epsilon_n,\eta}$ be a solution of π_{ϵ_n} with $\sigma_n(0) = u_n$. Using Lemma 2.20 we may assume that whenever $t \in \mathbb{R}$, then $|\sigma_n(t) - \sigma(t)|_{\epsilon_n} \to 0$ as $n \to \infty$, where $\sigma\colon \mathbb{R} \to N$ is a solution of π_0. In particular, $|u_n - v|_{\epsilon_n} \to 0$ as $n \to \infty$ where $v := \sigma(0) \in K_0$. This is a contradiction, which proves the corollary. □

Theorem 3.5 will be proved in the Appendix. In the next section we discuss a few applications of this result. More applications will be given in a forthcoming publication.

4. Applications

4.1. **Connection lifting.** Until further notice assume that $\epsilon_0 > 0$ and that $(f_\epsilon)_{\epsilon\in[0,\epsilon_0]}$ is a family of maps satisfying hypothesis $(A1)$. Write $\pi_\epsilon := \pi_{\epsilon,f_\epsilon}$ for $\epsilon \in [0, \epsilon_0]$. Using Theorem 3.5 we will show in this subsection that, under certain hypotheses, heteroclinic orbits of the limit semiflow π_0 can be 'lifted' to the semiflows π_ϵ and thus, by conjugacy, to the corresponding semiflows $\tilde{\pi}_\epsilon$ on the squeezed domains Ω_ϵ, for $\epsilon > 0$ small. We first treat an abstract problem and then apply the results obtained to an equation considered by Fiedler and Rocha.

Proposition 4.1. *Suppose that the local semiflow π_0 is gradient-like with respect to a Liapunov function $V_0 \colon H^1_s(\Omega) \to \mathbb{R}$. Assume also that N, N' and N'' are bounded isolating neighborhoods with respect to π_0 with $N' \cap N'' = \emptyset$, $N' \cup N'' \subset N$ and there are points u' and u'' such that $V_0(u') > V_0(u'')$, $\{u'\} = A_{\pi_0}(N')$ and $\{u''\} = A_{\pi_0}(N'')$. Finally, assume that*

$$h(\pi_0, N) \neq h(\pi_0, N') \vee h(\pi_0, N'') \tag{6}$$

and there are no equilibria of π_0 lying in $N \setminus (N' \cup N'')$. Then there is a solution $\sigma_0 \colon \mathbb{R} \to N$ of π_0 with $\{u'\} = \alpha(\sigma_0)$ and $\{u''\} = \omega(\sigma_0)$.

Remark 4.2. *Recall that, given a metric space X and an arbitrary map $\sigma \colon \mathbb{R} \to X$, the α-limit set $\alpha(\sigma)$ (resp. the ω-limit set $\omega(\sigma)$) of σ is defined as the set of all points $x \in X$ for which there is a sequence $(t_n)_{n \in \mathbb{N}}$ in $\mathbb{R}$ with $t_n \to -\infty$ (resp. $t_n \to \infty$) as $n \to \infty$ such that $\sigma(t_n) \to x$ as $n \to \infty$.*

Proof. From (6) we see that there is a solution $\sigma_0 \colon \mathbb{R} \to N$ of π_0 with $\sigma_0(\mathbb{R}) \not\subset N' \cup N''$. Since π_0 is gradient-like, it follows that $\alpha(\sigma_0)$ and $\omega(\sigma_0)$ are nonempty sets of equilibria of π_0. Our assumptions imply that the only equilibria of π_0 lying in N are u' and u''. In particular, σ_0 is a nonconstant solution, so V_0 is strictly decreasing along σ_0. This concludes the proof. □

Using Theorem 3.5 we can, in some sense, 'lift' the connection σ_0 to the semiflows π_ϵ, for $\epsilon > 0$ small:

Theorem 4.3. *Assume the hypotheses of Proposition 4.1. In addition, suppose that for some $\epsilon_0 > 0$ and every $\epsilon \in]0, \epsilon_0]$ the local semiflow π_ϵ is gradient-like with respect to a Liapunov function $V_\epsilon \colon H^1(\Omega) \to \mathbb{R}$. Assume that whenever $\epsilon_n \to 0^+$, $(u_n)_{n \in \mathbb{N}}$ is a sequence in $H^1(\Omega)$ and $|u_n - u_0|_{\epsilon_n} \to$ as $n \to \infty$, where $u_0 \in H^1_s(\Omega)$, then $V_{\epsilon_n}(u_n) \to V_0(u_0)$ as $n \to \infty$. Then for every $\eta > 0$ there is an $\epsilon^\eta > 0$ such that for every $\epsilon \in]0, \epsilon^\eta]$ there exists a solution $\sigma_\epsilon \colon \mathbb{R} \to [N]_{\epsilon,\eta}$ of π_ϵ such that $\alpha(\sigma_\epsilon) \subset [N']_{\epsilon,\eta}$ and $\omega(\sigma_\epsilon) \subset [N'']_{\epsilon,\eta}$.*

Proof. Choose numbers γ_1 and γ_2 such that

$$V_0(u'') < \gamma_1 < \gamma_2 < V_0(u').$$

Given $\eta > 0$, Theorem 3.5 implies that there is an $\epsilon^\eta > 0$ such that for $\epsilon \in]0, \epsilon^\eta]$

$$\begin{aligned} h(\pi_\epsilon, [N]_{\epsilon,\eta}) &= h(\pi_0, N) \\ h(\pi_\epsilon, [N']_{\epsilon,\eta}) &= h(\pi_0, N') \\ h(\pi_\epsilon, [N'']_{\epsilon,\eta}) &= h(\pi_0, N'') \end{aligned} \tag{7}$$

and so there exists a solution $\sigma_\epsilon \colon \mathbb{R} \to [N]_{\epsilon,\eta}$ of π_ϵ such that $\sigma_\epsilon(0) \in [N \setminus (N' \cup N'')]_{\epsilon,\eta}$.

An application of Proposition 2.15 shows that we may also assume that for $\epsilon \in]0, \epsilon^\eta]$ there are no equilibria of π_ϵ lying in $[N \setminus (N' \cup N'')]_{\epsilon,\eta}$. This implies that

$$\alpha(\sigma_\epsilon) \cup \omega(\sigma_\epsilon) \subset [N']_{\epsilon,\eta} \cup [N'']_{\epsilon,\eta}$$

and so

$$\alpha(\sigma_\epsilon) \cup \omega(\sigma_\epsilon) \subset A_{\pi_\epsilon}([N']_{\epsilon,\eta} \cup [N'']_{\epsilon,\eta}) = A_{\pi_\epsilon}([N']_{\epsilon,\eta}) \cup A_{\pi_\epsilon}([N'']_{\epsilon,\eta})$$

for $\epsilon \in]0, \epsilon^\eta]$. The latter equality follows since N' and N'' are disjoint.

Using Proposition 2.15 again we may assume that

$$V_\epsilon(u''_\epsilon) < \gamma_1 < \gamma_2 < V_\epsilon(u'_\epsilon)$$

whenever $\epsilon \in]0, \epsilon^\eta]$, $u'_\epsilon \in A_{\pi_\epsilon}([N']_{\epsilon,\eta})$ and $u''_\epsilon \in A_{\pi_\epsilon}([N'']_{\epsilon,\eta})$. By the same token, we may assume that for every $\epsilon \in]0, \epsilon^\eta]$ there is a t_ϵ such that $\gamma_1 < V_\epsilon(\sigma_\epsilon(t_\epsilon)) < \gamma_2$. We also have

$$V_\epsilon(u''_\epsilon) < V_\epsilon(\sigma_\epsilon(t_\epsilon)) < V_\epsilon(u'_\epsilon)$$

whenever $\epsilon \in]0, \epsilon^\eta]$, $u'_\epsilon \in \alpha(\sigma_\epsilon)$ and $u''_\epsilon \in \omega(\sigma_\epsilon)$. Thus whenever $\epsilon \in]0, \epsilon^\eta]$ and $u'_\epsilon \in \alpha(\sigma_\epsilon)$ then for every $u''_\epsilon \in A_{\pi_\epsilon}([N'']_{\epsilon,\eta})$ we obtain that

$$V_\epsilon(u''_\epsilon) < \gamma_1 < V_\epsilon(\sigma_\epsilon(t_\epsilon)) < V_\epsilon(u'_\epsilon)$$

so $u'_\epsilon \notin A_{\pi_\epsilon}([N'']_{\epsilon,\eta})$. Hence $\alpha(\sigma_\epsilon) \subset A_{\pi_\epsilon}([N']_{\epsilon,\eta})$. Similarly we prove that $\omega(\sigma_\epsilon) \subset A_{\pi_\epsilon}([N'']_{\epsilon,\eta})$. □

Assuming hyperbolicity, we can refine the results of Theorem 4.3. To this end, we need the following result:

Proposition 4.4. *Assume that for every $\epsilon \in [0, \epsilon_0]$ the map f_ϵ is Fréchet differentiable. Moreover, suppose that, whenever u_0, $v \in H^1_s(\Omega)$, $\epsilon_n \to 0^+$ as $n \to \infty$ and $(u_n)_{n\in\mathbb{N}}$ is a sequence in $H^1(\Omega)$ with $|u_n - u_0|_{\epsilon_n} \to 0$ as $n \to \infty$, then*

$$|Df_{\epsilon_n}(u_n)v - Df_0(u_0)v|_{L^2} \to 0 \text{ as } n \to \infty.$$

Let u_0 be a hyperbolic equilibrium of π_0 and N be a bounded isolating neighborhood of $\{u_0\}$. Then for every $\eta > 0$ there is an $\epsilon_1 = \epsilon_1(\eta)$, $0 < \epsilon_1 \le \epsilon_0$, such that for every $\epsilon \in]0, \epsilon_1]$ there is only a finite number of equilibria of π_ϵ in the set $[N]_{\epsilon,\eta}$ and all of them are hyperbolic.

Proof. In view of Proposition 2.8 it is not restrictive to assume that the family $(f_\epsilon)_{\epsilon\in[0,\epsilon_0]}$ satisfies hypothesis $(A2)$. Let $\eta > 0$ be arbitrary. We claim that for $\epsilon > 0$ small, every equilibrium of π_ϵ in the set $[N]_{\epsilon,\eta}$ is hyperbolic. If the claim is not true, then we obtain a strictly decreasing sequence $\epsilon_n \to 0^+$ and a sequence $u_n \in [N]_{\epsilon_n,\eta}$, $n \in \mathbb{N}$, such that u_n is a nonhyperbolic equilibrium of π_{ϵ_n}. It follows that for every $n \in \mathbb{N}$ there is a $v_n \in D(A_{\epsilon_n})$ with $|v_n|_{\epsilon_n} = 1$ and

$$A_{\epsilon_n} v_n = Df_{\epsilon_n}(u_n)v_n.$$

For $\epsilon \in [0, \epsilon_0]$ define $F_\epsilon := Df_\epsilon(u_n)$ if there is an $n \in \mathbb{N}$ with $\epsilon = \epsilon_n$ and set $F_\epsilon := Df_\epsilon(u_0)$ otherwise. Since $|u_n - u_0|_{\epsilon_n} \to 0$ as $n \to \infty$ it follows that the family $(F_\epsilon)_{\epsilon\in[0,\epsilon_0]}$ satisfies hypothesis $(A2)$. Since $\sigma_n(t) \equiv v_n$ is a solution of $\pi_{\epsilon_n,F_{\epsilon_n}}$ for every $n \in \mathbb{N}$, we may use Proposition 2.15 and assume, taking a subsequence if necessary, that $|v_n - v_0|_{\epsilon_n} \to 0$ as $n \to \infty$, where v_0 is an equilibrium of π_{0,F_0}. It follows that $|v_0|_{H^1} = 1$ and

$$A_0 v_0 = Df_0(u_0)v_0.$$

Thus u_0 is not hyperbolic, a contradiction, proving our claim. Since for $\epsilon > 0$ small, every equilibrium of π_ϵ in the set $[N]_{\epsilon,\eta}$ is hyperbolic and since the largest π_ϵ-invariant set in $[N]_{\epsilon,\eta}$ is compact, it follows that there can be only a finite number of equilibria in $[N]_{\epsilon,\eta}$. This proves the proposition. □

Combining Theorem 4.3 and Proposition 4.4 we arrive at the following

Corollary 4.5. *Assume the hypotheses of Theorem 4.3. In addition, suppose that for every $\epsilon \in [0, \epsilon_0]$ the map f_ϵ is Fréchet differentiable and that*

$$|Df_{\epsilon_n}(u_n)v - Df_0(u_0)v|_{L^2} \to 0 \text{ as } n \to \infty$$

whenever u_0, $v \in H^1_s(\Omega)$, $\epsilon_n \to 0^+$ as $n \to \infty$ and $(u_n)_{n\in\mathbb{N}}$ is a sequence in $H^1(\Omega)$ with $|u_n - u_0|_{\epsilon_n} \to 0$ as $n \to \infty$. Finally, assume that the equilibria u' and u'' are hyperbolic.

Under these assumptions, for every $\eta > 0$ there is an $\epsilon^\eta > 0$ such that for every $\epsilon \in]0, \epsilon^\eta]$ there exists a solution $\sigma_\epsilon \colon \mathbb{R} \to [N]_{\epsilon,\eta}$ and hyperbolic equilibria $u'_\epsilon \in [N']_{\epsilon,\eta}$ and $u''_\epsilon \in [N'']_{\epsilon,\eta}$ of π_ϵ such that $\{u'_\epsilon\} = \alpha(\sigma_\epsilon)$ and $\{u''_\epsilon\} = \omega(\sigma_\epsilon)$. □

For the rest of this subsection suppose that $M = 1$ so $\Omega \subset \mathbb{R}\times\mathbb{R}^N$ and let $P \colon \mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ be the projection onto the first coordinate. Define $J := P(\Omega)$. Suppose also that Ω has *connected vertical sections,* i.e. that for every $x \in J$ the x-section

$$\Omega_x := \{\, y \in \mathbb{R}^N \mid (x, y) \in \Omega \,\}$$

is connected.

For every $x \in J$ let $p(x) > 0$ be the N-dimensional Lebesgue measure of the vertical section Ω_x. Define the space

$$H_p := \{\, v \mid v \colon J \to \mathbb{R} \text{ and } p^{1/2}v \in L^2(J)\}.$$

This is a Hilbert space under the scalar product $(u, v)_{H_p} = \int_J puv \,\mathrm{d}x$.

Furthermore, define the space

$$V_p := \{\, v \in H_p \mid \ v' \in L^1_{\mathrm{loc}}(J) \text{ and } p^{1/2}v' \in L^2(J)\,\}.$$

This is a Hilbert space under the scalar product $(u, v)_{V_p} = \int_J puv \,\mathrm{d}x + \int_J pu'v' \,\mathrm{d}x$. If p and $1/p$ are bounded on J, then H_p is isomorphic to $L^2(J)$ and V_p is isomorphic to $H^1(J)$. An obvious modification of the proof of Proposition 3.6 in [34] shows that a function $u \colon \Omega \to \mathbb{R}$ lies in $L^2_s(\Omega)$ if and only if there is a function $\tilde{u} \colon J \to \mathbb{R}$ such that $u(x, y) = \tilde{u}(x)$ a.e. in Ω and $p^{1/2}\tilde{u} \in L^2(J)$. The assignment $u \mapsto \tilde{u}$ defines an isomorphism $\Phi \colon L^2_s(\Omega) \to H_p$. Moreover, $\Phi(H^1_s(\Omega)) = V_p$ and $\Phi|_{H^1_s(\Omega)}$ is an isomorphism of $H^1_s(\Omega)$ onto V_p.

A modification of the proof of Theorem 6.6 in [33] yields a characterization of the operator A_0. In fact, $u \in D(A_0) \subset H^1_s(\Omega)$ and $A_0u = w \in L^2_s(\Omega)$ if and only if, defining $\tilde{u} := \Phi(u) \in V_p$ and $\tilde{w} := \Phi(w) \in H_p$, we have that

(1) The distributional derivative $(p\tilde{u}')'$ of $p\tilde{u}'$ is an $L^1_{\mathrm{loc}}(J)$-function with $(p\tilde{u}')' = -p\tilde{w}$;
(2) $(p\tilde{u}')(x) = 0$ for $x \in \partial J$.

Assume that $p(x) \geq c > 0$ on J and $p \in C^3(\mathrm{cl}\, J)$. It follows that $u \in D(A_0)$ and $A_0u = w$ if and only if the functions $\tilde{u} := \Phi(u)$ and $\tilde{w} := \Phi(w)$ are such that $u \in H^2(\Omega)$, $\tilde{u}'(x) = 0$ for $x \in \partial J$ and $\tilde{w} = -\tilde{u}'' - p'\tilde{u}'/p$ a.e. in J.

Let $f \colon \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$, $(\epsilon, x, y, u) \mapsto f(\epsilon, x, y, u)$ be a function of class C^2 for which there are constants β, γ and $C \in [0, \infty[$ such that for all $(\epsilon, x, y, u) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}$ the following estimates are satisfied:

(1) $|\partial_\epsilon f(\epsilon, x, y, u)| \leq C(1 + |u|^\beta)$,
(2) $|\nabla_y f(\epsilon, x, y, u)| \leq C(1 + |u|^\beta)$,
(3) $|\partial_u f(\epsilon, x, y, u)| \leq C(1 + |u|^\gamma)$,
(4) $|\partial_\epsilon\partial_u f(\epsilon, x, y, u)| \leq C(1 + |u|^\beta)$,
(5) $|\nabla_y\partial_u f(\epsilon, x, y, u)| \leq C(1 + |u|^\beta)$,
(6) $|\partial_u\partial_u f(\epsilon, x, y, u)| \leq C(1 + |u|^\gamma)$.

If $n := M+1 > 2$, we also assume that $\beta \leq 2^*/2$ and $\gamma \leq (2^*/2)-1$, where $2^* := 2n/(n-2)$.

Given $\epsilon \geq 0$ and a function $u \colon \Omega \to \mathbb{R}$ define the function $\hat{f}_\epsilon(u) \colon \Omega \to \mathbb{R}$ by

$$\hat{f}_\epsilon(u)(x, y) := f(\epsilon, x, \epsilon y, u(x, y)) \text{ for } (x, y) \in \Omega.$$

Lemma 4.6. *The following conditions are satisfied:*

1. *If $u \in H^1(\Omega)$, then $\hat{f}_\epsilon(u) \in L^2(\Omega)$.*
2. *The family $(\hat{f}_\epsilon)_{\epsilon\in[0,1]}$ of maps satisfies hypothesis (A1).*
3. *For every $\epsilon \in [0,1]$ the map $\hat{f}_\epsilon$ is Fréchet differentiable and*
$$|D\hat{f}_{\epsilon_n}(u_n)v - D\hat{f}_0(u_0)v|_{L^2} \to 0 \text{ as } n \to \infty$$
whenever $u_0,\ v \in H^1_s(\Omega)$, $\epsilon_n \to 0^+$ as $n \to \infty$ and $(u_n)_{n\in\mathbb{N}}$ is a sequence in $H^1(\Omega)$ with $|u_n - u_0|_{\epsilon_n} \to 0$ as $n \to \infty$.

Proof. Assertions 1 and 2 follows from Proposition 2.6. The proof of that proposition, using the estimates on $\partial_\epsilon\partial_u f$, $\nabla_y\partial_u f$ and $\partial_u\partial_u f$ yields assertion 3. ☐

Define the function $F\colon \mathbb{R}\times\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}\to\mathbb{R}$ by
$$F(\epsilon,x,y,s) := \int_0^s f(\epsilon,x,y,r)\,\mathrm{d}r \text{ for } (\epsilon,x,y,s)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}.$$
An easy calculation shows that for every $\epsilon > 0$ the semiflow $\pi_\epsilon := \pi_{\epsilon,\hat{f}_\epsilon}$ is gradient-like with respect to the Liapunov function $V_\epsilon\colon H^1(\Omega)\to\mathbb{R}$ defined by
$$V_\epsilon(u) := (1/2)a_\epsilon(u,u) - \int_\Omega F(\epsilon,x,\epsilon y,u(x,y))\,\mathrm{d}x\,\mathrm{d}y, \text{ for } u\in H^1(\Omega).$$
Moreover, the semiflow $\pi_0 := \pi_{0,\hat{f}_0}$ is gradient-like with respect to the Liapunov function $V_0\colon H^1_s(\Omega)\to\mathbb{R}$ defined by
$$V_0(u) := (1/2)a_0(u,u) - \int_\Omega F(0,x,0,u(x,y))\,\mathrm{d}x\,\mathrm{d}y, \text{ for } u\in H^1_s(\Omega).$$
It is easily seen that whenever $\epsilon_n\to 0^+$, $(u_n)_{n\in\mathbb{N}}$ is a sequence in $H^1(\Omega)$ and $|u_n - u_0|_{\epsilon_n}\to$ as $n\to\infty$, where $u_0\in H^1_s(\Omega)$, then $V_{\epsilon_n}(u_n)\to V_0(u_0)$ as $n\to\infty$.

Furthermore, it is clear that the isomorphism $\Phi|_{H^1_s(\Omega)}$ conjugates the semiflow π_0 to the semiflow $\tilde{\pi}_0$ generated by the following scalar semilinear parabolic equation with Neumann boundary conditions:
$$\begin{aligned} &\tilde{u}_t = \tilde{u}_{xx} + p'\tilde{u}_x/p + f(0,x,0,\tilde{u}), && t>0, x\in J,\\ &\tilde{u}_x = 0, && t>0, x\in\partial J. \end{aligned} \tag{8}$$
Therefore Proposition 3.2 in Chapter 2 of [39] implies that a set N is a strongly admissible isolating neighborhood with respect to π_0 if and only if $\tilde{N} := \Phi(N)$ is a strongly admissible isolating neighborhood with respect to $\tilde{\pi}_0$, and, in this case, $A_{\tilde{\pi}_0}(\tilde{N}) = \Phi(A_{\pi_0}(N))$ and $h(\pi_0, A_{\pi_0}(N)) = h(\tilde{\pi}_0, A_{\tilde{\pi}_0}(\tilde{N}))$.

Notice that the equation (8) satisfies the hypotheses of [14].

Therefore, we obtain the following result:

Theorem 4.7. *Let p and f satisfy the above hypotheses. Assume $\tilde{u}'$ and $\tilde{u}''$ are hyperbolic equilibria for the equation (8) such that $h(\tilde{\pi}_0,\{\tilde{u}'\}) = \Sigma^{i(\tilde{u}')}$ and $h(\tilde{\pi}_0,\{\tilde{u}''\}) = \Sigma^{i(\tilde{u}'')}$ with $i(\tilde{u}') = i(\tilde{u}'')+1$. Moreover assume that the connections from $\tilde{u}'$ to $\tilde{u}''$ are not blocked in the sense of* [14].

Under these assumptions there exist bounded isolating neighborhoods N, N' and N'' with respect to π_0 with $N'\cap N'' = \emptyset$, $N'\cup N''\subset N$ such that $\{\Phi^{-1}(\tilde{u}')\} = A_{\pi_0}(N')$ and

$\{\Phi^{-1}(\tilde{u}'')\} = A_{\pi_0}(N'')$. *Furthermore for every* $\eta > 0$ *there is an* $\epsilon^\eta > 0$ *such that for every* $\epsilon \in]0, \epsilon^\eta]$ *there exists a solution* $\sigma_\epsilon \colon \mathbb{R} \to [N]_{\epsilon,\eta}$ *and hyperbolic equilibria* $u'_\epsilon \in [N']_{\epsilon,\eta}$ *and* $u''_\epsilon \in [N'']_{\epsilon,\eta}$ *of* π_ϵ *such that* $\{u'_\epsilon\} = \alpha(\sigma_\epsilon)$ *and* $\{u''_\epsilon\} = \omega(\sigma_\epsilon)$.

Proof. Lemma 1.7 in [14] and the remarks preceding the statement of this theorem imply that the hypotheses of Proposition 4.1 hold. Note that the conditions of Theorem 4.3 are also satisfied. Therefore the result follows from Lemma 4.6 and Corollary 4.5. □

4.2. **Monotonicity.** Now recall that, given $\epsilon \geq 0$, we denote by $(\lambda_{\epsilon,j})_{j\in\mathbb{N}}$ the repeated sequence of eigenvalues of the operator A_ϵ.

As it was proved in [33], for every $j \in \mathbb{N}$ the family $(\lambda_{\epsilon,j})_{\epsilon>0}$ is monotone decreasing and converges to $\lambda_{0,j}$. Note that, $\lambda_{\epsilon,1} \equiv 0$. On the other hand, if $j \geq 2$, then, as we will prove now, the family $(\lambda_{\epsilon,j})_{\epsilon>0}$ is strictly monotone decreasing in many cases. We begin with the following result, which gives a necessary and sufficient condition for strict monotonicity of a given family $(\lambda_{\epsilon,j})_{\epsilon>0}$ of eigenvalues:

Theorem 4.8.
1. *Suppose that there is a* $j \in \mathbb{N}$ *and there are numbers* $0 < \epsilon_2 < \epsilon_1$, *such that* $\lambda_{\epsilon_2,j} = \lambda_{\epsilon_1,j}$. *Then there is a* $w \in H^1_s(\Omega)$, $w \neq 0$, *such that*

$$\langle \nabla w, \nabla v\rangle = \lambda \langle w, v\rangle \quad \textit{for all } v \in H^1(\Omega), \tag{9}$$

   where $\lambda = \lambda_{\epsilon_2,j}$.
2. *Conversely, if there exist a* $\lambda \in \mathbb{R}$ *and a* $w \in H^1_s(\Omega)$, $w \neq 0$ *such that* (9) *is satisfied, then there is a unique* $r \in \mathbb{N}$ *with* $\lambda = \lambda_{0,r} < \lambda_{0,r+1}$. *Furthermore, there is an* $\epsilon_0 > 0$ *with the property that*

$$\lambda_{\epsilon,r} \equiv \lambda \quad \textit{for all } \epsilon \in [0, \epsilon_0].$$

Proof. Assume the hypothesis of the first part of the theorem. Let E be the (j-dimensional) subspace of $H^1(\Omega)$ spanned by the vectors $w_{\epsilon_2,k}$, $k = 1, \ldots, j$. Then, by the min-max principle,

$$\lambda_{\epsilon_1,j} \leq \max_{u\in E\setminus\{0\}} \frac{a_{\epsilon_1}(u,u)}{\langle u,u\rangle}.$$

Hence there is a $w \in E$, $\langle w,w\rangle = 1$, such that

$$\lambda_{\epsilon_1,j} \leq a_{\epsilon_1}(w,w) = a_{\epsilon_2}(w,w) - \left(\frac{1}{\epsilon_2^2} - \frac{1}{\epsilon_1^2}\right)\langle \nabla_y w, \nabla_y w\rangle.$$

Now

$$a_{\epsilon_2}(w,w) = \sum_{k=1}^{j} \lambda_{\epsilon_2,k} |\langle w, w_{\epsilon_2,k}\rangle|^2.$$

We also have

$$\lambda_{\epsilon_2,j} = \lambda_{\epsilon_2,j}\langle w,w\rangle = \lambda_{\epsilon_2,j}\sum_{k=1}^{j} |\langle w, w_{\epsilon_2,k}\rangle|^2.$$

Since $\lambda_{\epsilon_2,j} = \lambda_{\epsilon_1,j}$ we obtain

$$\lambda_{\epsilon_2,j}\sum_{k=1}^{j} |\langle w, w_{\epsilon_2,k}\rangle|^2 \leq \sum_{k=1}^{j} \lambda_{\epsilon_2,k}|\langle w, w_{\epsilon_2,k}\rangle|^2 - \left(\frac{1}{\epsilon_2^2} - \frac{1}{\epsilon_1^2}\right)\langle \nabla_y w, \nabla_y w\rangle,$$

i.e.

$$\sum_{k=1}^{j}(\lambda_{\epsilon_2,j}-\lambda_{\epsilon_2,k})|\langle w,w_{\epsilon_2,k}\rangle|^2\leq-\left(\frac{1}{\epsilon_2^2}-\frac{1}{\epsilon_1^2}\right)\langle\nabla_y w,\nabla_y w\rangle.$$

Since the left-hand side of this inequality is nonnegative while the right-hand side is nonpositive, both sides must be equal to zero. This implies, on the one hand, that $\langle\nabla_y w,\nabla_y w\rangle = 0$ so $w\in H^1_s(\Omega)$ and, on the other hand, that $\langle w,w_{\epsilon_2,k}\rangle = 0$, whenever $\lambda_{\epsilon_2,j}-\lambda_{\epsilon_2,k}>0$. Thus w lies in the eigenspace of the eigenvalue λ (with respect to the pair (a_{ϵ_2},b)). In other words, for all $v\in H^1(\Omega)$,

$$\langle\nabla w,\nabla v\rangle=\langle\nabla_x w,\nabla_x v\rangle=\langle\nabla_x w,\nabla_x v\rangle+\frac{1}{\epsilon_2^2}\langle\nabla_y w,\nabla_y v\rangle=\lambda\langle w,v\rangle.$$

This proves the first part of the theorem. Now assume the hypothesis of the second part of the theorem. Since $w\in H^1_s(\Omega)$ it follows that for every $\epsilon>0$

$$a_\epsilon(w,v)=\langle\nabla w,\nabla v\rangle=\lambda\langle w,v\rangle\quad\text{ for all } v\in H^1(\Omega).$$

Thus for every $\epsilon\geq 0$ the number λ is an eigenvalue of the operator A_ϵ. In particular, let $r\in\mathbb{N}$ be the largest number with $\lambda=\lambda_{0,r}$. Then $\lambda_{0,r}<\lambda_{0,r+1}$ so there exists an $\epsilon_0>0$ such that $\lambda=\lambda_{0,r}<\lambda_{\epsilon,r+1}$ for $\epsilon\in\,]0,\epsilon_0]$. This implies that, for every $\epsilon\in\,]0,\epsilon_0]$, we have that $\lambda_{\epsilon,r}=\lambda$, since otherwise

$$\lambda_{\epsilon,r}<\lambda<\lambda_{\epsilon,r+1}$$

so λ is not an eigenvalue of A_ϵ, a contradiction. The theorem is proved. □

The following result implies that strict monotonicity of all the families $(\lambda_{\epsilon,j})_{\epsilon>0}$, $j\geq 2$, is an open dense (and so generic) property with respect to C^1-perturbations of the boundary of Ω:

Theorem 4.9. *Suppose that there is a nonempty open set J in $\mathbb{R}^M$ and there are disjoint open sets U and U' in $\mathbb{R}^M\times\mathbb{R}^N$ such that $\Omega\cap(J\times\mathbb{R}^N)=U\cup U'$, and for every $x\in J$, the vertical section*

$$U_x:=\{\,y\mid(x,y)\in U\,\}$$

is nonempty and connected. For $x\in J$ let $p(x)>0$ be the N-dimensional Lebesgue measure of the vertical section U_x. Assume that $p\in C^1(J)$ and $\nabla p\not\equiv 0$. Then for every $j\in\mathbb{N}$ with $j\geq 2$ the family $(\lambda_{\epsilon,j})_{\epsilon>0}$ is strictly monotone decreasing.

Proof. If the theorem is not true, then, by Theorem 4.8, there is a $\lambda\in\mathbb{R}$, $\lambda\neq 0$ and a $w\in H^1_s(\Omega)$, $w\neq 0$, such that

$$\langle\nabla w,\nabla v\rangle=\lambda\langle w,v\rangle\quad\text{ for all } v\in H^1(\Omega).$$

By the usual regularity theory it follows that w is real analytic on Ω and $\Delta w(x,y) = -\lambda w(x,y)$ for all $(x,y)\in\Omega$. Since $w\in H^1_s(\Omega)$ it follows that $\nabla_y w(x,y)\equiv 0$ for all $(x,y)\in\Omega$. Since U_x is connected for all $x\in J$ it follows that there is a function $\tilde{w}\colon J\to\mathbb{R}$ with $\tilde{w}(x)\equiv w(x,y)$ for all $(x,y)\in U$. This immediately implies that $\tilde{w}$ is real analytic on J and

$$\Delta_x\tilde{w}(x)=-\lambda\tilde{w}(x)\quad\text{ for } x\in J. \tag{10}$$

Now given $\tilde{v}\in C_0^\infty(J)$ define the function $v\colon\Omega\to\mathbb{R}$ by

$$v(x,y)=\begin{cases}\tilde{v}(x) & \text{if } (x,y)\in U,\\ 0 & \text{otherwise.}\end{cases}$$

Using our assumptions it is easy to prove that $v \in C^\infty(\Omega)$ with $\nabla_y v \equiv 0$ and

$$\nabla_x v(x,y) = \begin{cases} \nabla \tilde{v}(x) & \text{if } (x,y) \in U, \\ 0 & \text{otherwise.} \end{cases}$$

Thus v and ∇v are bounded on Ω and so $v \in H^1(\Omega)$.

We therefore obtain

$$\begin{aligned}\langle \nabla w, \nabla v \rangle = \int_U \nabla w \cdot \nabla v \,\mathrm{d}x\,\mathrm{d}y &= \int_J p(x) \nabla \tilde{w}(x) \cdot \nabla \tilde{v}(x)\,\mathrm{d}x \\ &= -\int_J p(x)\tilde{v}(x)\Delta\tilde{w}(x)\,\mathrm{d}x - \int_J \tilde{v}(x)\nabla p \cdot \nabla \tilde{w}\,\mathrm{d}x \\ &= \lambda \int_J p(x)\tilde{v}(x)\tilde{w}(x)\,\mathrm{d}x - \int_J \tilde{v}(x)\nabla p \cdot \nabla \tilde{w}\,\mathrm{d}x.\end{aligned}$$

On the other hand,

$$\langle \nabla w, \nabla v \rangle = \lambda \langle w, v \rangle = \lambda \int_U wv\,\mathrm{d}x\,\mathrm{d}y = \lambda \int_J p(x)\tilde{v}(x)\tilde{w}(x)\,\mathrm{d}x.$$

This shows that

$$\int_J \tilde{v}(x)\nabla p \cdot \nabla \tilde{w}\,\mathrm{d}x \equiv 0 \quad \text{for all } \tilde{v} \in C_0^\infty(J)$$

so

$$\nabla p \cdot \nabla \tilde{w} \equiv 0 \quad \text{on } J.$$

Since $\nabla p \not\equiv 0$ it follows that for some nonempty open connected subset J' of J,

$$\nabla \tilde{w}(x) \equiv 0 \quad \text{on } J'.$$

Hence $\tilde{w}$ is constant on J'. Since $\lambda \neq 0$, formula (10) implies that $\tilde{w}(x) \equiv 0$ on J' so

$$w(x,y) \equiv 0 \quad \text{for } (x,y) \in \Omega' := U \cap (J' \times \mathbb{R}^N).$$

Since J' is nonempty and U_x is nonempty for every $x \in J$, it follows that Ω' is nonempty. Since w is analytic, it follows that $w \equiv 0$, which is a contradiction. The theorem is proved. □

4.3. **Bifurcation at resonance.** For the rest of this subsection, suppose that $M = N = 1$ so $\Omega \subset \mathbb{R} \times \mathbb{R}$ and let $r \in \mathbb{N}$, $r \geq 2$, be such that $\lambda_{0,r} < \lambda_{0,r+1}$. Consider the following hypothesis:

(HR) $f \colon \mathbb{R} \to \mathbb{R}$ is a function of class C^2 such that $f(0) = 0$, $f'(0) = \lambda_{0,r}$ and there are constants p and $C \in]0,\infty[$ such that

$$|f''(s)| \leq C(|s|^p + 1)$$

for all $s \in \mathbb{R}$.

The following result is well-known, cf. Theorem 5.3 and Proposition 5.5 in [33]:

Lemma 4.10. *Let f satisfy hypothesis (HR). Then*

(1) *$f \circ v \in L^2(\Omega)$ whenever $v \in H^1(\Omega)$.*

(2) *The Nemitski operator* $\hat{f}\colon H^1(\Omega) \to L^2(\Omega)$ *given by*

$$\hat{f}(v) = f \circ v$$

is well-defined, Lipschitzian on bounded subsets of $H^1(\Omega)$ *and it maps bounded subsets of* $H^1(\Omega)$ *into bounded subsets of* $L^2(\Omega)$. *Moreover,* $\hat{f}(H^1_s(\Omega)) \subset L^2_s(\Omega)$.

(3) $\hat{f}$ *is Fréchet differentiable and*

$$D\hat{f}(v)(u) = f'(v) \cdot u,$$

for all v *and for all* u *in* $H^1(\Omega)$.

(4) $D\hat{f}$ *is continuous and Lipschitzian on bounded subsets of* $H^1(\Omega)$.

□

For the rest of this subsection assume hypothesis (HR) and suppose that

$$H^1_s(\Omega) \subset L^\infty(\Omega) \text{ with continuous inclusion.} \tag{11}$$

This latter property is satisfied, e.g., if Ω is a nicely decomposed domain in the sense defined in [33].

We write $\pi_\epsilon := \pi_{\epsilon,\hat{f}}$ for $\epsilon \geq 0$. Since $\hat{f}(0) = 0$, the set $K_0 = \{0\}$ is an invariant set for π_ϵ, for $\epsilon \geq 0$. We will prove in this subsection that, under appropriate conditions, there is a family u_ϵ, of nontrivial equilibria of π_ϵ, $\epsilon > 0$ small, bifurcating from the trivial equilibrium 0 of π_0. The precise statement is given in Theorem 4.15. Our main tool will be Theorem 3.5 and the Conley index product formula introduced in [38] or Chapter 2 of [39]. To set up the framework, define

$$L := A_0 - \lambda_{0,r} I.$$

Recall that $(w_{0,j})_{j\in\mathbb{N}}$ is a complete orthonormal system on $L^2_s(\Omega)$.

Define the following linear subspaces of $L^2_s(\Omega)$:

$$\begin{aligned}
X_1 &:= \operatorname{span}\{\, w_{0,j} \mid \lambda_{0,j} = \lambda_{0,r} \,\},\\
X_2 &:= \operatorname{span}\{\, w_{0,j} \mid \lambda_{0,j} < \lambda_{0,r} \,\},\\
X_3 &:= \operatorname{cl}_{L^2} \operatorname{span}\{\, w_{0,j} \mid \lambda_{0,j} > \lambda_{0,r} \,\}.
\end{aligned}$$

Notice that $m := \dim X_1 < \infty$, $m \geq 1$ and $\dim X_2 = r - m$. Furthermore, X_1, X_2 and X_3 are mutually orthogonal and L-invariant subspaces of $L^2_s(\Omega)$. Moreover

$$L^2_s(\Omega) = X_1 \oplus X_2 \oplus X_3. \tag{12}$$

For each $i = 1, 2, 3$ the restriction of the operator L to X_i will be denoted by L_i. Since $\sigma(L) = \{\lambda - \lambda_{0,r} \mid \lambda \in \sigma(A_0)\,\}$, we have

$$\begin{aligned}
&\lambda \in \sigma(L_1) \text{ implies that } \lambda = 0,\\
&\lambda \in \sigma(L_2) \text{ implies that } \lambda < 0,\\
&\lambda \in \sigma(L_3) \text{ implies that } \lambda > 0.
\end{aligned}$$

In what follows we will write $e_j := w_{0,r-m+j}$, $j = 1, \ldots, m$, so

$$X_1 = \operatorname{span}\{e_1, \ldots, e_m\}.$$

We will also write $g := \hat{f} - \lambda_{0,r} I$. Notice that we are in the setting of Theorem 2.1 in Chapter 2 of [39]. Therefore for each $i = 1, 2, 3$ there exists a neighborhood V_i of zero in

$X_i^{1/2} := H_s^1(\Omega) \cap X_i$ such that an m-dimensional local center manifold close to zero can be described by a mapping

$$\xi \colon V_1 \to V_2 \oplus V_3.$$

Furthermore, the coordinate representation π_ξ of the semiflow π_0 on the center manifold is the flow generated by the ordinary differential equation

$$\dot{u} + L_1 u = E_1 g(u + \xi(u)), \tag{13}$$

where $u \in V_1$ and E_1 denotes the projection of L_s^2 onto X_1 induced by direct sum described in (12). We have the following result:

Proposition 4.11. *Assume hypotheses (HR) and* (11)*. Moreover, suppose that*

$$f(s) = \lambda_{0,r} s + a s^k + \beta(s), \tag{14}$$

where $k \geq 2$ is a natural number, $a \in \mathbb{R}$ and $\beta(s) = O(s^{k+1})$ as $s \to 0$. Then the following statements hold:

(1) *If $a < 0$ and k is odd, then the set $\{0\}$ is an isolated invariant set relative to π_ξ and*
$$h(\pi_\xi, \{0\}) = \Sigma^0.$$
(2) *If k is even, $m = 1$ and $a\langle w_{0,r}^k, w_{0,r}\rangle \neq 0$, then the set $\{0\}$ is an isolated invariant set relative to π_ξ and*
$$h(\pi_\xi, \{0\}) = \bar{0}.$$

Proof. Notice that all norms are equivalent on the finite-dimensional space X_1. Write $I_m^k := \{1, \dots, m\}^k$ and let $\alpha = (\alpha_1, \dots, \alpha_k)$ denote a generic element of I_m^k. Recall that

$$X_1 = \operatorname{span}\{e_1, e_2, \dots, e_m\}.$$

Define the map $\phi \colon X_1 \to X_2^{1/2} \oplus X_3^{1/2}$ by

$$\phi(u) := \sum_{\alpha \in I_m^k} \langle u, e_{\alpha_1}\rangle \langle u, e_{\alpha_2}\rangle \dots \langle u, e_{\alpha_k}\rangle v(\alpha_1, \dots, \alpha_k),$$

with $v(\alpha_1, \dots, \alpha_k) \in D(A_0)$ and orthogonal to e_i for all $i = 1, \dots, m$ to be determined later. Obviously ϕ is of class C^∞. By (11) there are a constant C and a $\delta_0 > 0$ such that $|\phi(u)(x,y)| \leq C|u|_{L^2}^k$ for all $u \in X_1$ and all $(x,y) \in \Omega$. Moreover, $|\beta(u(x,y) + \phi(u)(x,y))| \leq M|u|_{L^2}^{k+1}$ for all $u \in X_1$ with $|u|_{L^2} \leq \delta_0$ and all $(x,y) \in \Omega$. It follows that

$$g(u + \phi(u)) = a u^k + \beta_1(u),$$

with $|\beta_1(u)|_{L^2} = O(|u|_{L^2}^{k+1})$ as $|u|_{L^2} \to 0$. Consequently,

$$E_1 g(u + \phi(u)) = a\langle u^k, e_1\rangle e_1 + \dots + a\langle u^k, e_m\rangle e_m + \beta_2(u),$$

with $\beta_2(u) = O(|u|_{L^2}^{k+1})$ as $|u|_{L^2} \to 0$. Define the map $\Delta \colon X_1 \to L^2(\Omega)$ by

$$\Delta(u) := D\phi(u)\,[L_1 u - E_1 g(u + \phi(u))] - L\phi(u) + g(u + \phi(u)) - E_1 g(u + \phi(u)). \tag{15}$$

Notice that $L_1 u = 0$. It is a simple task to prove that $|D\phi(u)\,[-E_1 g(u + \phi(u))]\,|_{L^2} = O(|u|_{L^2}^{k+1})$ as $|u|_{L^2} \to 0$. Therefore

$$\Delta(u) = \sum_{\alpha \in I_m^k} \langle u, e_{\alpha_1}\rangle \langle u, e_{\alpha_2}\rangle \dots \langle u, e_{\alpha_k}\rangle h(\alpha_1, \dots, \alpha_k) + \beta_3(u),$$

where

$$h(\alpha_1,\dots,\alpha_k) := -Lv(\alpha_1,\dots,\alpha_k) + ae_{\alpha_1}\dots e_{\alpha_k} - a\sum_{i=1}^{m}\langle e_{\alpha_1}\dots e_{\alpha_k}, e_i\rangle e_i$$

and $|\beta_3(u)|_{L^2} = O(|u|_{L^2}^{k+1})$ as $|u|_{L^2} \to 0$. Now choose $v(\alpha_1,\dots,\alpha_k) \in D(A_0)$, orthogonal to e_i, for $i = 1,\dots,m$ in such a way that $h(\alpha_1,\dots,\alpha_k) = 0$. Since $L = A_0 - \lambda_{0,r}$, the vector $v(\alpha_1,\dots,\alpha_k)$ with the above property exists and is unique. With this choice, we have

$$|\Delta(u)|_{L^2} = O(|u|_{L^2}^{k+1}).$$

Therefore, we are in the conditions of Theorem 2.3 in Chapter 2 of [39]. In other words,

$$|\xi(u) - \phi(u)|_{H^1_s} = O(|u|_{L^2}^{k+1}).$$

Reasoning as above, we see that the reduced equation (13) on the center manifold becomes:

$$\dot{u} = E_1 g(u+\xi(u)) = E_1 g(u+\phi(u)) + \beta_4(u) = aE_1(u^k) + \beta_5(u), \tag{16}$$

with $|\beta_4(u)|_{L^2} = O(|u|_{L^2}^{k+1})$ and $|\beta_5(u)|_{L^2} = O(|u|_{L^2}^{k+1})$ as $|u|_{L^2} \to 0$.

Suppose first that $a < 0$ and k is odd. Consider the positive definite functional $V\colon X_1 \to \mathbb{R}$ given by $V(u) = \langle u,u\rangle/2$. Let $t \mapsto u\pi_\xi t$ be a solution of π_ξ. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}V(u\pi_\xi t)\mid_{t=0} = a\langle E_1(u^k), u\rangle + \langle\beta_5(u), u\rangle = a\langle u^k, u\rangle + \beta_6(u),$$

where $\beta_6(u) = O(|u|_{L^2}^{k+2})$ as $|u|_{L^2} \to 0$. There is a constant C such that

$$|u|_{L^2} \le C|u|_{L^{k+1}} \text{ for all } u \in X_1.$$

Therefore there exist positive constants M and δ such that whenever $|u|_{L^2} \le \delta$ then

$$|\beta_6(u)| \le M|u|_{L^2}|u|_{L^2}^{k+1} \le MC^{k+1}|u|_{L^2}\langle u^k, u\rangle.$$

Choosing δ such that $MC^{k+1}\delta = -a/2$ we see that

$$\frac{\mathrm{d}}{\mathrm{d}t}V(u\pi_\xi t)\mid_{t=0} \le (a/2)\langle u^k, u\rangle < 0$$

if $|u|_{L^2} \le \delta$ and $u \ne 0$.

It follows that the closed ball

$$B := \{\, u \in X_1 \mid \langle u,u\rangle \le \delta \,\}$$

is an isolating block for π_ξ with $B^- = \emptyset$ and the largest invariant set in B is $\{0\}$. Thus $h(\pi_\xi,\{0\}) = h(\pi_\xi, B) = \Sigma^0$. This completes the proof of the first part of the proposition.

Now assume that k is even, $m = 1$ and $a\langle w_{0,r}^k, w_{0,r}\rangle \ne 0$. Note that $e_m = w_{0,r}$. Writing $u = ye_m$ we see that (16) is equivalent to the one-dimensional equation

$$\dot{y} = a\langle e_m^k, e_m\rangle y^k + \gamma(y),$$

where $\gamma(y) = O(|y|^{k+1})$ as $y \to 0$. Since k is even and $a\langle e_m^k, e_m\rangle \ne 0$, it is clear that $\{0\}$ is an isolated invariant set of the latter equation with Conley index $\bar{0}$. This proves the second part of the proposition. $\square$

Applying the index-product formula of [38] (or [39]) we thus arrive at the following

Proposition 4.12. *Under each of the alternative hypotheses of Proposition 4.11 the set* $K_0 = \{0\}$ *is an isolated invariant set for* π_0 *and*

$$h(\pi_0, K_0) = \begin{cases} \Sigma^{r-m} & \text{if } k \text{ is odd and } a < 0, \\ \bar{0} & k \text{ is even, } m = 1 \text{ and } a\langle w_{0,r}^k, w_{0,r}\rangle \neq 0. \end{cases} \qquad \square$$

The case for which $m = 1$ and $a\langle w_{0,r}^k, w_{0,r}\rangle = 0$ can be treated as in Section 2.4 of Chapter 2 in [39]. The index depends on the higher order terms in the expansion of the map f. We only treat a special case:

Proposition 4.13. *Assume hypotheses (HR) and* (11). *Moreover, suppose that* $m = 1$, $a\langle w_{0,r}^k, w_{0,r}\rangle = 0$ *and*

$$f(s) = \lambda_{0,r} s + as^k + bs^{2k-1} + \beta(s), \tag{17}$$

where k *is an even positive integer,* a *and* $b \in \mathbb{R}$ *are arbitrary and* $\beta(s) = O(s^{2k})$ *as* $s \to 0$. *Then for all* $b < 0$ *with* $|b|$ *large enough the set* $\{0\}$ *is an isolated invariant set relative to* π_ξ *and*

$$h(\pi_\xi, \{0\}) = \Sigma^0.$$

Consequently, the set $\{0\}$ *is an isolated invariant set relative to* π_0 *and*

$$h(\pi_0, \{0\}) = \Sigma^{r-1}.$$

Proof. Note again that $e_m = w_{0,r}$. Let $\phi\colon X_1 \to X_2^{1/2} \oplus X_3^{1/2}$ be a map defined by

$$\phi(u) = \langle u, e_m\rangle^k v + \langle u, e_m\rangle^{2k-1} w,$$

where $v, w \in D(A_0)$, v and w orthogonal to e_m, $Lv = ae_m^k + a\langle e_m^k, e_m\rangle e_m = ae_m^k$ and $Lw = ake_m^{k-1}v - ak\langle e_m^k, v\rangle e_m + be_m^{2k-1} - b\langle e_m^{2k-1}, e_m\rangle e_m$.

Proceeding exactly as in Section 2.4 of Chapter 2 in [39], we see that the reduced equation on the center manifold is

$$\dot{u} = E_1 g(u + \xi(u)) = \langle u, e_m\rangle^{2k-1} \Big(ak\langle e_m^k, v\rangle + b\langle e_m^{2k-1}, e_m\rangle\Big) e_m + \gamma(u),$$

where $|\gamma(u)|_{L^2} = O(|u|_{L^2}^{2k})$ as $|u|_{L^2} \to 0$. An equivalent ordinary differential equation on $\mathbb{R}$ reads as follows:

$$\dot{y} = \Big(ak\langle e_m^k, v\rangle + b\langle e_m^{2k-1}, e_m\rangle\Big) y^{2k-1} + \gamma_1(y),$$

where $|\gamma_1(y)| = O(|y|^{2k})$ as $|y| \to 0$. Since $\langle e_m^{2k-1}, e_m\rangle > 0$, it follows that $ak\langle e_m^k, v\rangle + b\langle e_m^{2k-1}, e_m\rangle < 0$ for $b < 0$ and $|b|$ large enough. It clearly follows that in this case the set $K_\xi := \{0\}$ is an isolated invariant set relative to π_ξ and its index is Σ^0. An application of the index product formula concludes the proof. $\square$

Proposition 4.14. *For every* $\epsilon > 0$ *let* π_ϵ *be as above. Suppose that the family* $(\lambda_{\epsilon,r})_{\epsilon>0}$ *is strictly monotone decreasing. Then there exists an* $\epsilon_1 > 0$ *such that for all* $\epsilon \in]0, \epsilon_1]$, *the set* $\{0\}$ *is an isolated invariant set relative to* π_ϵ *and*

$$h(\pi_\epsilon, \{0\}) = \Sigma^r.$$

Proof. Since $\lambda_{0,r} < \lambda_{0,r+1}$, our assumption implies that

$$\lambda_{\epsilon,r} < \lambda_{0,r} < \lambda_{\epsilon,r+1}$$

for all $\epsilon > 0$ small enough. This obviously implies the assertion. $\square$

We can now state the main result of this subsection:

Theorem 4.15. *Assume any of the alternative hypotheses of Proposition 4.11 or else assume the hypotheses of Proposition 4.13. Moreover, suppose that the family* $(\lambda_{\epsilon,r})_{\epsilon>0}$ *is strictly monotone decreasing. Suppose that* N_0 *is a bounded isolating neighborhood of* $\{0\}$ *for* π_0. *Then for every* $\eta > 0$ *there exists an* $0 < \epsilon_0 = \epsilon_0(\eta)$, *such that for every* $\epsilon \in]0, \epsilon_0]$,

$$K_{\epsilon,\eta} := A_{\pi_\epsilon}([N_0]_{\epsilon,\eta})$$

contains a nontrivial equilibrium u_ϵ *of* π_ϵ *and a nonconstant full solution* σ_ϵ *of* π_ϵ *whose* ω*-limit set or* α*-limit set is equal to* $\{0\}$. *In other words,* 0 *is connected to a set of nontrivial equilibria of* π_ϵ.

Proof. Fix $\eta > 0$. By Proposition 4.14, there exists an $\epsilon_1 > 0$ such that for all $\epsilon \in]0, \epsilon_1]$, the set $\{0\}$ is an isolated invariant set relative to π_ϵ and

$$h(\pi_\epsilon, \{0\}) = \Sigma^r.$$

Since $\Sigma^r \neq \Sigma^{r-m}$ and $\Sigma^r \neq \bar{0}$ Theorem 3.5 and Proposition 4.12 (or else Proposition 4.13) implies that

$$h(\pi_\epsilon, K_{\epsilon,\eta}) \neq \Sigma^r$$

so $K_{\epsilon,\eta} \neq \{0\}$ for all $\epsilon > 0$ small enough. Since π_ϵ is gradient-like, we obtain the existence of a nontrivial equilibrium u_ϵ. The existence of the connection σ_ϵ follows from the irreducibility of $K_{\epsilon,\eta}$ (see Lemma 11.4 and Theorem 11.6 in [39]). □

4.4. **C-shaped domains.** In this subsection we will define a class of very simple two-dimensional domains, which we call *C-shaped,* and show that all of the alternative assumptions of Theorem 4.15 are satisfied for an appropriate choice of the domain Ω belonging to that class.

Definition 4.16. *Let* a_i *and* b_i, $i = 1, 2, 3$, *be real numbers with* $a_1 < 0$, $a_2 > 0$, $a_3 > 0$ *and* $0 < b_1 < b_2 < b_3$. The C-shaped domain with parameters a_i and b_i, $i = 1, 2, 3$ *is the following set*

$$\Omega := \big(]a_1, a_2[\times]0, b_1[\big) \cup \big(]a_1, 0[\times]0, b_3[\big) \cup \big(]a_1, a_3[\times]b_2, b_3[\big).$$

(See Figure 1.) Setting $\Omega_1 :=]a_1, 0[\times]0, b_3[$, $\Omega_2 :=]0, a_2[\times]0, b_1[$, $\Omega_3 :=]0, a_3[\times]b_2, b_3[$ *and* $Z := \{0\} \times \mathbb{R}$ *we obtain a nice decomposition of* Ω *in the sense of* [33]. *In this case* $J_1 =]a_1, 0[$, $J_2 =]0, a_2[$ *and* $J_3 =]0, a_3[$, *while* $p_1(x) \equiv b_3$, $p_2(x) \equiv b_1$ *and* $p_3(x) \equiv b_3 - b_2$.

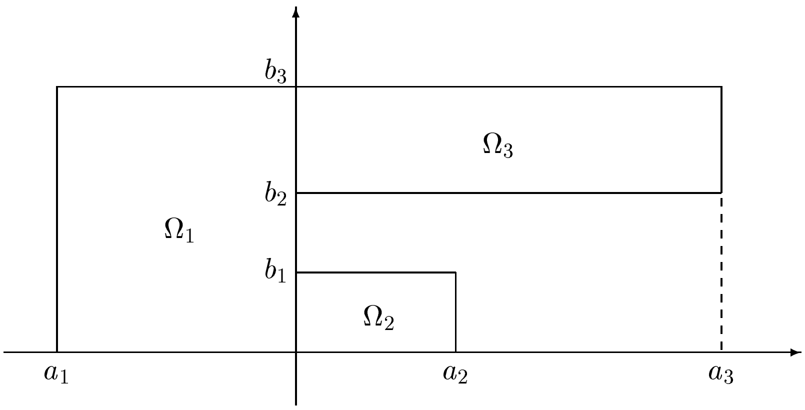

Figure 1.

For the rest of this section let Ω be a C-shaped domain with parameters a_i and b_i, $i = 1, 2, 3$. Theorem 6.6 in [33] provides conditions for a pair (λ, u) to be an eigenvalue-eigenvector pair of the operator A_0 on Ω. In fact, (λ, u) is an eigenvalue-eigenvector pair of A_0 if and only if $u \not\equiv 0$ and there exist absolutely continuous functions $u_l \colon J_l \to \mathbb{R}$, $l = 1, 2, 3$ and a null set S in $\mathbb{R} \times \mathbb{R}$ such that whenever $l = 1, 2, 3$, then $u(x, y) = u_l(x)$ for $(x, y) \in \Omega_l \setminus S$, $u_l, u_l' \in L^2(J_l)$ and the following properties hold:

(1) $u_l'' = -\lambda u_l$, $l = 1, 2, 3$, in the sense of distributions;
(2) The limits: $u_l(0) := \lim_{x\to 0} u_l(x)$, $l = 1, 2, 3$, exist and $u_1(0) = u_2(0) = u_3(0)$;
(3) The limits: $u_l'(a_l) := \lim_{x\to a_l} u_l'(x)$ and $u_l'(0) := \lim_{x\to 0} u_l'(x)$, $l = 1, 2, 3$, exist and $u_1'(a_1) = u_2'(a_2) = u_3'(a_3) = 0$ while $b_3 u_1'(0) = b_1 u_2'(0) + (b_3 - b_2) u_3'(0)$.

Notice that condition 1 is equivalent to

$$u_l(x) = \alpha_l \cos \sqrt{\lambda} x + \beta_l \sin \sqrt{\lambda} x,\ l = 1, 2, 3,\ x \in J_l,$$

where α_l and β_l, $l = 1, 2, 3$, are arbitrary real numbers. By a simple calculation we thus obtain the following

Proposition 4.17. *A pair (λ, u) is an eigenvalue-eigenvector pair of A_0 if and only if $u \not\equiv 0$ and there exist functions $u_l \colon J_l \to \mathbb{R}$, $l = 1, 2, 3$, real numbers α and β_l, $l = 1, 2, 3$, and a null set S in $\mathbb{R} \times \mathbb{R}$ such that whenever $l = 1, 2, 3$ then $u(x, y) = u_l(x)$ for all $(x, y) \in \Omega_l \setminus S$ and conditions*

$$u_l(x) = \alpha \cos \sqrt{\lambda} x + \beta_l \sin \sqrt{\lambda} x,\ l = 1, 2, 3,\ x \in J_l, \tag{18}$$

$$b_3 \sqrt{\lambda} \beta_1 = b_1 \sqrt{\lambda} \beta_2 + (b_3 - b_2) \sqrt{\lambda} \beta_3 \tag{19}$$

and

$$-\alpha \sqrt{\lambda} \sin a_l \sqrt{\lambda} + \beta_l \sqrt{\lambda} \cos a_l \sqrt{\lambda} = 0,\ l = 1, 2, 3 \tag{20}$$

hold.

Remark 4.18. *Proposition 4.17 immediately implies that $\lambda = 0$ is a simple eigenvalue of A_0. Now let $\lambda > 0$ be an arbitrary eigenvalue of A_0. If $\sin a_l \sqrt{\lambda} = 0$ for $l = 1, 2, 3$, then $\beta_l = 0$ for $l = 1, 2, 3$ by condition (20), so by condition (18) λ is a simple eigenvalue. If $\sin a_i \sqrt{\lambda} \neq 0$ for some i, then $\alpha = \beta_i \cot a_i \sqrt{\lambda}$ and so conditions (18) and (19) imply that the eigenspace of λ is at most two-dimensional. Thus there are no eigenvalues of A_0 of multiplicity higher than two.*

The following result is a strict monotonicity criterion for C-shaped domains:

Proposition 4.19. *Let $\lambda > 0$ be an eigenvalue of A_0 and let $r \in \mathbb{N}$, $r \geq 2$, be such that $\lambda = \lambda_{0,r} < \lambda_{0,r+1}$. Then the following properties are equivalent:*

(1) *$\sin(a_l \sqrt{\lambda}) = 0$ for $l = 1, 2, 3$;*
(2) *There is an $\epsilon_0 > 0$ such that $\lambda_{\epsilon,r} \equiv \lambda$ for $\epsilon \in [0, \epsilon_0]$.*

Proof. Suppose property 1 holds. Set $u(x, y) \equiv u(x) := \cos \sqrt{\lambda} x$, for $(x, y) \in \Omega$. Then $u \not\equiv 0$ and so Proposition 4.17 (with $\alpha = 1$ and $\beta_l \equiv 0$) implies that the pair (λ, u) is an eigenvalue-eigenvector pair of the operator A_0. In particular we have that $u \in H^1_s(\Omega)$.

The divergence theorem implies that for every $v \in H^1(\Omega)$

$$\begin{aligned}\langle \nabla u, \nabla v\rangle &= \sum_{l=1}^{3} \int_{\Omega_l} u'(x)\partial_x v(x,y)\,\mathrm{d}x\,\mathrm{d}y \\ &= \sum_{l=1}^{3} \int_{\partial\Omega_l} u'(x)v(x,y)\rho_l(x,y)\mathrm{d}\sigma(x,y) - \sum_{l=1}^{3} \int_{\Omega_l} u''(x)v(x,y)\,\mathrm{d}x\,\mathrm{d}y \\ &= \sum_{l=1}^{3} \int_{\partial\Omega_l} u'(x)v(x,y)\rho_l(x,y)\mathrm{d}\sigma(x,y) + \lambda\sum_{l=1}^{3} \int_{\Omega_l} u(x)v(x,y)\,\mathrm{d}x\,\mathrm{d}y \\ &= \sum_{l=1}^{3} \int_{\partial\Omega_l} u'(x)v(x,y)\rho_l(x,y)\mathrm{d}\sigma(x,y) + \lambda\langle u, v\rangle,\end{aligned}$$

where σ is the 1-dimensional Hausdorff measure on $\mathbb{R}^2$ and $\rho_l(x,y)$ is the x-component of the outer normal vector to $\partial\Omega_l$ at $(x,y) \in \partial\Omega_l$. Now $\rho_l(x,y) = 0$ on the horizontal part of $\partial\Omega_l$, while, by the definition of u, $u'(x) = 0$ for (x,y) lying on the vertical part of $\partial\Omega_l$. Thus

$$\sum_{l=1}^{3} \int_{\partial\Omega_l} u'(x)v(x,y)\rho_l(x,y)\mathrm{d}\sigma(x,y) = 0$$

and so

$$\langle \nabla u, \nabla v\rangle = \lambda\langle u, v\rangle \text{ for all } v \in H^1(\Omega).$$

Now Theorem 4.8 implies property 2.

Assume, conversely, that property 2 holds. Then Theorem 4.8 implies that there is a $u \in H^1_s(\Omega)$, $u \not\equiv 0$, such that

$$\langle \nabla u, \nabla v\rangle = \lambda\langle u, v\rangle \text{ for all } v \in H^1(\Omega). \tag{21}$$

In particular, (λ, u) is an eigenvalue-eigenvector pair of A_0. By Proposition 4.17 there exist functions $u_l\colon J_l \to \mathbb{R}$, real numbers α and β_l, $l = 1, 2, 3$, and a null set S in $\mathbb{R}\times\mathbb{R}$ such that whenever $l = 1, 2, 3$ then $u(x,y) = u_l(x)$ for all $(x,y) \in \Omega_l \setminus S$ and conditions (18), (19) and (20) hold. Formula (21) and the divergence theorem imply that for every $v \in H^1(\Omega)$

$$\begin{aligned}\lambda\langle u, v\rangle = \langle \nabla u, \nabla v\rangle &= \sum_{l=1}^{3} \int_{\Omega_l} u_l'(x)\partial_x v(x,y)\,\mathrm{d}x\,\mathrm{d}y \\ &= \sum_{l=1}^{3} \int_{\partial\Omega_l} u_l'(x)v(x,y)\rho_l(x,y)\mathrm{d}\sigma(x,y) - \sum_{l=1}^{3} \int_{\Omega_l} u_l''(x)v(x,y)\,\mathrm{d}x\,\mathrm{d}y \\ &= \sum_{l=1}^{3} \int_{\partial\Omega_l} u_l'(x)v(x,y)\rho_l(x,y)\mathrm{d}\sigma(x,y) + \lambda\sum_{l=1}^{3} \int_{\Omega_l} u_l(x)v(x,y)\,\mathrm{d}x\,\mathrm{d}y \\ &= \sum_{l=1}^{3} \int_{\partial\Omega_l} u_l'(x)v(x,y)\rho_l(x,y)\mathrm{d}\sigma(x,y) + \lambda\langle u, v\rangle.\end{aligned}$$

This implies that

$$\sum_{l=1}^{3} \int_{\partial\Omega_l} u_l'(x)v(x,y)\rho_l(x,y)\mathrm{d}\sigma(x,y) = 0 \text{ for all } v \in H^1(\Omega). \tag{22}$$

Letting v in (22) be the restriction to Ω of an arbitrary nonnegative test function on $\mathbb{R}^2$ whose support is included in the horizontal strip $\mathbb{R} \times]b_1, b_2[$ and which is nontrivial on $\{0\} \times]b_1, b_2[$ we see that $u_1'(0) = 0$, i.e. $\beta_1\sqrt{\lambda} = 0$, so $\beta_1 = 0$. Similarly, we prove that $\beta_2 = \beta_3 = 0$. Hence $\alpha \neq 0$ and so condition (20) implies that $\sin(a_l\sqrt{\lambda}) = 0$ for $l = 1, 2, 3$. The proposition is proved. □

The following result shows that, by choosing the parameters a_i and b_i, $i = 1, 2, 3$, appropriately, we can satisfy all the alternative hypotheses of Theorem 4.15:

Proposition 4.20. *Suppose that $b_l = l$ for $l = 1, 2, 3$ and let λ be an arbitrary real number.*

(1) *If $-a_1 = a_2 = a_3 =: a$ and $\cos a\sqrt{\lambda} = 0$, then λ is a double eigenvalue of A_0, whose eigenspace is formed by all functions u for which there are constants $\beta_2, \beta_3 \in \mathbb{R}$ such that:*
   1.1. *$u(x, y) = (1/3)(\beta_2 + \beta_3)\sin\sqrt{\lambda}x$ a.e. on Ω_1;*
   1.2. *$u(x, y) = \beta_l \sin\sqrt{\lambda}x$ a.e. on Ω_l, $l = 2, 3$.*

(2) *If $-a_1 = a_2 =: a$, $\cos a\sqrt{\lambda} = 0$ and $\cos a_3\sqrt{\lambda} \neq 0$, then λ is a simple eigenvalue of A_0, whose eigenspace is formed by all functions u for which there is a constant $\beta_2 \in \mathbb{R}$ such that:*
   2.1. *$u(x, y) = (1/3)\beta_2 \sin\sqrt{\lambda}x$ a.e. on Ω_1;*
   2.2. *$u(x, y) = \beta_2 \sin\sqrt{\lambda}x$ a.e. on Ω_2;*
   2.3. *$u(x, y) = 0$ a.e. on Ω_3;*
   If e is an arbitrary eigenfunction of λ and $k \in \mathbb{N}$ is even, then

$$\langle e^k, e\rangle = \int_\Omega e^{k+1}(x, y)\,\mathrm{d}x\,\mathrm{d}y \neq 0. \tag{23}$$

(3) *If $a_2 = a_3 =: a$, $\cos a\sqrt{\lambda} = 0$ and $\cos a_1\sqrt{\lambda} \neq 0$, then λ is a simple eigenvalue of A_0, whose eigenspace is formed by all functions u for which there is a constant $\beta_2 \in \mathbb{R}$ such that:*
   3.1. *$u(x, y) = 0$ a.e. on Ω_1;*
   3.2. *$u(x, y) = \beta_2 \sin\sqrt{\lambda}x$ a.e. on Ω_2;*
   3.3. *$u(x, y) = -\beta_2 \sin\sqrt{\lambda}x$ a.e. on Ω_3;*
   If e is an arbitrary eigenfunction of λ and $k \in \mathbb{N}$ is even, then

$$\langle e^k, e\rangle = \int_\Omega e^{k+1}(x, y)\,\mathrm{d}x\,\mathrm{d}y = 0. \tag{24}$$

In all the above cases, if r is such that $\lambda = \lambda_{0,r} < \lambda_{0,r+1}$ then the family $(\lambda_{\epsilon,r})_{\epsilon>0}$ is strictly monotone decreasing, so, in particular, $\lambda_{\epsilon,r} < \lambda$ for all $\epsilon > 0$.

Proof. Consider the first case. If u is any function satisfying properties 1.1 and 1.2 then conditions (18), (19) and (20) of Proposition 4.17 are satisfied with $\alpha = 0$ and $\beta_1 = (1/3)(\beta_2 + \beta_3)$, so, by that proposition, λ is an eigenvalue of A_0 and u lies in its eigenspace.

Now consider the second case. Suppose first that λ is an eigenvalue of A_0 and let u be a corresponding eigenvector. Then u is determined by constants α and β_l, $l = 1, 2, 3$, so that conditions (18), (19) and (20) of Proposition 4.17 hold. Since $\cos a\sqrt{\lambda} = 0$, we conclude that $\alpha = 0$ and since $\cos a_3\sqrt{\lambda} \neq 0$ we also conclude that $\beta_3 = 0$. Thus u must necessarily satisfy properties 2.1, 2.2 and 2.3, so λ is a simple eigenvalue. On the other hand, letting $u \not\equiv 0$ be an arbitrary function satisfying these properties, we conclude from

Proposition 4.17 that (λ, u) is an eigenvalue-eigenvector pair of A_0. Thus λ is, indeed, a simple eigenvalue of A_0. Formula (23) is obtained by a simple integration.

The third case is proved in the same way.

The last statement of the proposition is a consequence of Proposition 4.19. The proposition is proved. $\square$

5. Appendix

In this section we prove Theorem 3.5. In view of Proposition 2.8 it is not restrictive to assume that the family $(f_\epsilon)_{\epsilon\in[0,\epsilon_0]}$ satisfies hypothesis $(A2)$. Write $\pi_\epsilon := \pi_{\epsilon,f_\epsilon}$ for all $\epsilon \in [0, \epsilon_0]$.

We need a number of preliminary results.

Proposition 5.1. *For all sufficiently small $\epsilon > 0$ the set $[N]_{\epsilon,\eta}$ is a strongly π_ϵ-admissible isolating neighborhood for π_ϵ.*

Proof. Theorem 4.4 in [39] implies that $[N]_{\epsilon,\eta}$ is a strongly π_ϵ-admissible set for every $\eta > 0$ and $\epsilon > 0$. Suppose that there are an $\eta > 0$ and a sequence $(\epsilon_n)_{n\in\mathbb{N}}$ of positive numbers converging to zero such that for each $n \in \mathbb{N}$ the set $[N]_{\epsilon_n,\eta}$ is not an isolating neighborhood for π_{ϵ_n}. Then for each $n \in \mathbb{N}$ there exists a solution $\sigma_n : \mathbb{R} \to H^1(\Omega)$ of π_{ϵ_n} such that

$$\sigma_n(\mathbb{R}) \subset [N]_{\epsilon_n,\eta} \text{ and } \sigma_n(0) \in \partial [N]_{\epsilon_n,\eta}.$$

By Lemma 2.20 there exists a subsequence of $(\sigma_n)_{n\in\mathbb{N}}$ denoted again by $(\sigma_n)_{n\in\mathbb{N}}$ such that for every $t \in \mathbb{R}$

$$\left|Q_{\epsilon_n}\sigma_n(t) - \sigma(t)\right|_{H^1} \to 0 \text{ and } \left|(I - Q_{\epsilon_n})\sigma_n(t)\right|_{\epsilon_n} \to 0 \tag{25}$$

as $n \to \infty$, where $\sigma : \mathbb{R} \to N$ is a solution of π_0. Setting $t = 0$ in (25), we conclude that $\left|(I - Q_{\epsilon_n})\sigma_n(0)\right|_{\epsilon_n} < \eta$ and so $Q_{\epsilon_n}\sigma_n(0) \in \partial N$ for all n large enough. This implies that $\sigma(0) \in \partial N$, a contradiction to the fact that N is an isolating neighborhood for π_0. The proposition is proved. $\square$

Assume now that $K_0 := A_{\pi_0}(N) = \emptyset$. We claim that $K_{\epsilon,\eta} := A_{\pi_\epsilon}([N]_{\epsilon,\eta}) = \emptyset$ for every $\eta > 0$ and all sufficiently small $\epsilon > 0$. Otherwise, we find an $\eta > 0$ and a sequence $(\epsilon_n)_{n\in\mathbb{N}}$, of positive numbers converging to zero such that for each $n \in \mathbb{N}$,

$$K_{\epsilon_n,\eta} \neq \emptyset.$$

Let $\sigma_n : \mathbb{R} \to [N]_{\epsilon_n,\eta}$ be a full solution of π_{ϵ_n}. Lemma 2.20 implies that there exists a solution $\sigma : \mathbb{R} \to N$ of π_0. Since K_0 is the empty set, we obtain a contradiction. This proves our claim.

Thus Theorem 3.5 holds if $K_0 = \emptyset$. Therefore let us assume from now on that $K_0 \neq \emptyset$. We shall prove Theorem 3.5 by appropriately modifying the proof of the usual Conley index continuation theorem as given in [37] or [39].

Assume that $K_0 \neq \emptyset$ is an isolated invariant set of π_0. By Theorem 3.1, we may choose an open set U_0 in $H^1_s(\Omega)$ such that $N_0 := \operatorname{cl} U_0$ is an isolating block for π_0 and such that $K \subset U_0 \subset N_0 \subset N$ and $\partial U_0 = \partial N_0$.

Let $\alpha\colon [0,\infty[\to [1,2[$ be a monotone increasing C^∞-diffeomorphism. Let $s^+\colon N_0 \to \mathbb{R}\cup\{\,\infty\,\}$ be given by

$$s^+(u) := \sup\{\, t \geq 0 \mid u\pi_0\,[0,t] \subset N_0\,\},$$

$t^+\colon U_0 \to \mathbb{R}\cup\{\,\infty\,\}$ by

$$t^+(u) := \sup\{\, t \geq 0 \mid u\pi_0[0,t] \subset U_0\,\} > 0$$

and $F\colon H^1_s(\Omega) \to [0,1]$ defined by

$$F(u) := \min\{\, 1, \mathrm{d}(u, A^-_{\pi_0}(N_0))\,\},$$

where $\mathrm{d}(u, A^-_{\pi_0}(N_0))$ is the distance (in $H^1_s(\Omega)$) of the point u from the set $A^-_{\pi_0}(N_0)$. Finally, we define $g^-\colon N_0 \to \mathbb{R}$ as follows:

$$g^-(u) := \sup\{\, \alpha(t)F(u\pi_0 t) \mid t \in [0, s^+(u)]\,,\ \text{if } s^+(u) < \infty \text{ and } t \in [0,\infty[\,,\ \text{if } s^+(u) = \infty\,\}.$$

Whenever $(u_n)_{n\in\mathbb{N}}$ is a sequence in N_0 with $g^-(u_n) \to 0$ as $n \to \infty$, then, by admissibility, there is a subsequence of $(u_n)_{n\in\mathbb{N}}$ converging to an element of $A^-_{\pi_0}(N_0)$. Given $a > 0$, $b > 0$ define

$$V(a,b) := \{\, u \in U_0 \mid g^-(u) < a,\ t^+(u) > b\,\}.$$

Note that $V(a,b)$ is open in $H^1_s(\Omega)$, $K_0 \subset V(a,b)$ and admissibility implies that we can choose $a_0 > 0$ and $b_0 > 0$ such that $\operatorname{cl} V(a_0,b_0) \subset U_0$.

Given $a > 0$, $b > 0$, $\epsilon > 0$ and $\alpha > 0$ define

$$V_{\epsilon,\alpha}(a,b) := \,]V(a,b)[_{\epsilon,\alpha}\,.$$

Lemma 5.2. *Fix positive numbers a, b, α, δ, η and M such that $a \leq a_0$ and $b \geq b_0$. Then*

$$K(\epsilon,\eta,a,b) := A_{\pi_\epsilon}([\operatorname{cl} V(a,b)]_{\epsilon,\eta}) \subset V_{\epsilon,\alpha}(\delta,M)$$

for all sufficiently small $\epsilon > 0$.

Proof. If the lemma is not true then there are positive numbers a, b, α, δ, η and M such that $a \leq a_0$ and $b \geq b_0$ and a sequence $(\epsilon_n)_{n\in\mathbb{N}}$ of positive numbers converging to zero such that for all $n \in \mathbb{N}$

$$K(\epsilon_n,\eta,a,b) \not\subset V_{\epsilon_n,\alpha}(\delta,M).$$

For each $n \in \mathbb{N}$, there is a solution $\sigma_n\colon \mathbb{R} \to [\operatorname{cl} V(a,b)]_{\epsilon_n,\eta}$ of π_{ϵ_n} with $\sigma_n(0) \notin V_{\epsilon_n,\alpha}(\delta,M)$. By Lemma 2.20 there exists a subsequence of $(\sigma_n)_{n\in\mathbb{N}}$ which we will denote again by $(\sigma_n)_{n\in\mathbb{N}}$ and there exists a solution $\sigma\colon \mathbb{R} \to \operatorname{cl} V(a,b) \subset N_0$ of π_0 such that for every $t \in \mathbb{R}$,

$$\big|Q_{\epsilon_n}(\sigma_n(t)) - \sigma(t)\big|_{H^1} \to 0 \text{ and } \big|(I - Q_{\epsilon_n})(\sigma_n(t))\big|_{\epsilon_n} \to 0 \tag{26}$$

as $n \to \infty$. Thus $\sigma(\mathbb{R}) \subset K_0$ and $\big|(I - Q_{\epsilon_n})(\sigma_n(0))\big|_{\epsilon_n} < \alpha$ for all n large enough. Since $V(\delta,M)$ is open in $H^1_s(\Omega)$ it follows that $Q_{\epsilon_n}(\sigma_n(0)) \in V(\delta,M)$ for all n large enough. In other words, $\sigma_n(0) \in V_{\epsilon_n,\alpha}(\delta,M)$ for all n large enough, a contradiction proving the lemma. □

Let $\epsilon > 0$ and $\nu > 0$ be arbitrary. Given $u \in\,]U_0[_{\epsilon,\nu}$, define

$$t^+_{\epsilon,\nu}(u) := \sup\{\, t \geq 0 \mid u\pi_\epsilon[0,t] \subset\,]U_0[_{\epsilon,\nu}\,\} > 0.$$

Lemma 5.3. *Fix $\nu > 0$. Let $\epsilon_n > 0$, $u_n \in \,]U_0[_{\epsilon_n,\nu}$, $n \in \mathbb{N}$, and $u \in U_0$ be such that $\epsilon_n \to 0$ and*

$$\left|u_n - u\right|_{\epsilon_n} \to 0 \text{ as } n \to \infty.$$

Then

$$t^+_{\epsilon_n,\nu}(u_n) \to t^+(u) \text{ as } n \to \infty.$$

Proof. Let $C \in \,]0,\infty[$ be such that $t^+(u) < C$. Since $u\pi_0 t^+(u) \in \partial U_0 = \partial N_0$ and N_0 is an isolating block for π_0, there exists an $s \in \mathbb{R}$ with $C > s > t^+(u)$ such that $u\pi_0 s \notin N_0$. By Corollary 2.14 we have $\left|u_n\pi_{\epsilon_n}s - u\pi_0 s\right|_{\epsilon_n} \to 0$ and so $\left|Q_{\epsilon_n}(u_n\pi_{\epsilon_n}s) - u\pi_0 s\right|_{H^1} \to 0$ as $n \to \infty$. This implies that $Q_{\epsilon_n}(u_n\pi_{\epsilon_n}s) \notin N_0$, so $u_n\pi_{\epsilon_n}s \notin [N_0]_{\epsilon_n,\nu}$ and therefore $t^+_{\epsilon_n,\nu}(u_n) < s < C$ for n large enough.

Now let $t^+(u) > C > 0$. Then $u\pi_0\,[0,C] \subset U_0$. We claim that $u_n\pi_{\epsilon_n}\,[0,C] \subset \,]U_0[_{\epsilon_n,\nu}$ for all n large enough. Suppose that this is not true. Then we may assume that for every $n \in \mathbb{N}$

$$u_n\pi_{\epsilon_n}\,[0,C] \not\subset \,]U_0[_{\epsilon_n,\nu}$$

and so there exists a $t_n \in [0,C]$ such that $u_n\pi_{\epsilon_n}t_n \notin \,]U_0[_{\epsilon_n,\nu}$. Again we can assume that there exists a $t \in [0,C]$ such that $t_n \to t$ as $n \to \infty$. Now Corollary 2.14 implies that $\left|u_n\pi_{\epsilon_n}t_n - u\pi_0 t\right|_{\epsilon_n} \to 0$ as $n \to \infty$. Thus $\left|Q_{\epsilon_n}(u_n\pi_{\epsilon_n}t_n) - u\pi_0 t\right|_{H^1} \to 0$ and $\left|(I - Q_{\epsilon_n})(u_n\pi_{\epsilon_n}t_n)\right|_{\epsilon_n} \to 0$ as $n \to \infty$. It follows that $Q_{\epsilon_n}(u_n\pi_{\epsilon_n}t_n) \in U_0$ and $\left|(I - Q_{\epsilon_n})(u_n\pi_{\epsilon_n}t_n)\right|_{\epsilon_n} < \nu$ for all n large enough.

This shows that $u_n\pi_{\epsilon_n}t_n \in \,]U_0[_{\epsilon_n,\nu}$ for all large n, contradicting the choice of the sequence $(t_n)_{n\in\mathbb{N}}$. This proves our claim, which, in turn, implies that $t^+_{\epsilon_n,\nu}(u_n) > C$ for all sufficiently large n. The lemma is proved. ∎

Definition 5.4. *Fix positive real numbers M, M', β, ν, η, ρ, a and b. For all positive numbers ϵ, α and δ define the following subsets of $H^1(\Omega)$:*

$$N_1(\alpha,\delta,\epsilon) := [\operatorname{cl} V(a,b)]_{\epsilon,\eta} \cap \operatorname{cl}_\epsilon\{\, v \mid \exists\, u \in V_{\epsilon,\alpha}(\delta,M) \text{ and } t \geq 0 \text{ such that } u\pi_\epsilon[0,t] \subset \,]U_0[_{\epsilon,\beta} \text{ and } u\pi_\epsilon t = v \,\},$$

$$\begin{aligned}
N_2(\alpha,\delta,\epsilon) &:= N_1(\alpha,\delta,\epsilon) \cap \{\, u \in \,]U_0[_{\epsilon,\nu} \mid t^+_{\epsilon,\nu}(u) \leq M' \,\},\\
\hat{E}(\epsilon) &:= [\{\, u \in U_0 \mid t^+(u) \leq 5M \,\} \cap \operatorname{cl} V(a,b)]_{\epsilon,\eta}\,,\\
\hat{C}(\epsilon,\alpha,\delta) &:= \{\, u \in \,]U_0[_{\epsilon,\nu} \mid t^+_{\epsilon,\nu}(u) \leq 4M \,\} \cap N_1(\alpha,\delta,\epsilon),\\
E(\epsilon,\alpha,\delta) &:= [\{\, u \in U_0 \mid t^+(u) \leq 3M \,\} \cap \operatorname{cl} V(\delta,M)]_{\epsilon,\alpha}\,,\\
C(\epsilon,\rho,\alpha,\delta) &:= \{\, u \in \,]U_0[_{\epsilon,\alpha} \mid t^+_{\epsilon,\alpha}(u) \leq 2M \,\} \cap \operatorname{cl}_\epsilon V_{\epsilon,\rho}(\delta,M).
\end{aligned}$$

Remark 5.5. *If $\alpha \leq \beta$, $\alpha \leq \eta$, $\delta \leq a$ and $M \geq b$ then clearly $V_{\epsilon,\alpha}(\delta,M) \subset N_1(\alpha,\delta,\epsilon)$. Moreover, if $a \leq a_0$, $b \geq b_0$ and $\eta < \nu$ then $[\operatorname{cl} V(a,b)]_{\epsilon,\eta} \subset \,]U_0[_{\epsilon,\nu}$ and whenever v lies in $K(\epsilon,\eta,a,b)$, the largest invariant set in $[\operatorname{cl} V(a,b)]_{\epsilon,\eta}$, then, by the definition of $t^+_{\epsilon,\nu}$, we have $t^+_{\epsilon,\nu}(v) = \infty$. In particular, $K(\epsilon,\eta,a,b) \cap N_2(\alpha,\delta,\epsilon) = \emptyset$.*

Proposition 5.6. *Assume that $M > b$, $M' > b$, $a \leq a_0$, $b \geq b_0$ and $\eta < \nu < \beta$.*

1. *For all sufficiently small positive α, δ and ϵ:*
   (a) *$(N_1(\alpha,\delta,\epsilon), N_2(\alpha,\delta,\epsilon))$ is a pseudo-index pair in $[\operatorname{cl} V(a,b)]_{\epsilon,\eta}$;*

(b) *the following inclusions are satisfied:*

$$E(\epsilon,\alpha,\delta)\subset\hat{C}(\epsilon,\alpha,\delta)\subset\hat{E}(\epsilon)$$

(2) *For every $\alpha>0$ and all sufficiently small positive ρ, δ and ϵ*

$$C(\epsilon,\rho,\alpha,\delta)\subset E(\epsilon,\alpha,\delta).$$

Proof. Proceeding exactly as in the proof of Lemma 12.5 in [39] we see that the sets $N_1(\alpha,\delta,\epsilon)$ and $N_2(\alpha,\delta,\epsilon)$ are closed in $H^1(\Omega)$ and $[\operatorname{cl} V(a,b)]_{\epsilon,\eta}$-positively invariant relative to π_ϵ.

Suppose that part 1 of the proposition does not hold. Then for some choice of the constants M, M', β, ν, η, a and b as above there are sequences $(\delta_n)_{n\in\mathbb{N}}$, $(\alpha_n)_{n\in\mathbb{N}}$ and $(\epsilon_n)_{n\in\mathbb{N}}$ of positive numbers converging to zero and a sequence $(u_n)_{n\in\mathbb{N}}$ such that

$$u_n\in N_1(\alpha_n,\delta_n,\epsilon_n)\cap\partial_{\epsilon_n}[\operatorname{cl} V(a,b)]_{\epsilon_n,\eta}\setminus N_2(\alpha_n,\delta_n,\epsilon_n)\text{ for all } n\in\mathbb{N}\tag{27}$$

or

$$u_n\in\hat{C}(\epsilon_n,\alpha_n,\delta_n)\setminus\hat{E}(\epsilon_n)\text{ for all } n\in\mathbb{N}\tag{28}$$

or

$$u_n\in E(\epsilon_n,\alpha_n,\delta_n)\setminus\hat{C}(\epsilon_n,\alpha_n,\delta_n)\text{ for all } n\in\mathbb{N}.\tag{29}$$

In the first two cases $u_n\in N_1(\alpha_n,\delta_n,\epsilon_n)$, so there exist a $v_n\in V_{\epsilon_n,\alpha_n}(\delta_n,M)$ and a $t_n\geq 0$ such that

$$v_n\pi_{\epsilon_n}[0,t_n]\subset\,]U_0[_{\epsilon_n,\beta}$$

and $\tilde{u}_n:=v_n\pi_{\epsilon_n}t_n$ is such that $\left|u_n-\tilde{u}_n\right|_{\epsilon_n}<2^{-n}$. Since $g^-(Q_{\epsilon_n}v_n)<\delta_n\to 0$, we may assume, taking subsequences if necessary, that $|Q_{\epsilon_n}v_n-v|_{H^1}\to 0$ for some $v\in A^-_{\pi_0}(N_0)$. Since $|(I-Q_{\epsilon_n})v_n|_{\epsilon_n}\to 0$, it follows that $|v_n-v|_{\epsilon_n}\to 0$.

We claim that there is a subsequence of $(v_n\pi_{\epsilon_n}t_n)_{n\in\mathbb{N}}$, denoted again by $(v_n\pi_{\epsilon_n}t_n)_{n\in\mathbb{N}}$, and there is a $\tilde{v}\in A^-_{\pi_0}(N_0)$ such that

$$\left|v_n\pi_{\epsilon_n}t_n-\tilde{v}\right|_{\epsilon_n}\to 0\text{ as } n\to\infty.$$

First assume that the sequence $(t_n)_{n\in\mathbb{N}}$ is bounded. By taking subsequences, if necessary, we may assume that there exists a $t\in[0,\infty[$ such that $t_n\to t$ as $n\to\infty$. It follows from Corollary 2.14 that

$$\left|v_n\pi_{\epsilon_n}t_n-v\pi_0 t\right|_{\epsilon_n}\to 0\text{ as } n\to\infty.$$

Since $A^-_{\pi_0}(N_0)$ is an N_0-positively invariant set relative to π_0, it follows that

$$\tilde{v}:=v\pi_0 t\in A^-_{\pi_0}(N_0).$$

If $(t_n)_{n\in\mathbb{N}}$ is unbounded then we may assume that $t_n\to\infty$ as $n\to\infty$. Let $\sigma_n\colon[-t_n,0]\to H^1(\Omega)$ be a solution through $v_n\pi_{\epsilon_n}t_n$ relative to π_{ϵ_n} given by

$$\sigma_n(s)=v_n\pi_{\epsilon_n}(t_n+s),\ \text{for every } s\in[-t_n,0].$$

By Lemma 2.20 there exist a subsequence of $(\sigma_n)_{n\in\mathbb{N}}$, denoted again by $(\sigma_n)_{n\in\mathbb{N}}$, and a solution $\sigma\colon\,]-\infty,0]\to H^1_s(\Omega)$ of π_0 lying in N_0 such that for each $s\in\,]-\infty,0]$,

$$\left|\sigma_n(s)-\sigma(s)\right|_{\epsilon_n}\to 0\text{ as } n\to\infty.\tag{30}$$

Let $\tilde{v}:=\sigma(0)\in A^-_{\pi_0}(N_0)$. Letting $s=0$ in (30), we obtain

$$\left|v_n\pi_{\epsilon_n}t_n-\tilde{v}\right|_{\epsilon_n}\to 0\text{ as } n\to\infty.$$

This completes the proof of our claim.

It follows that $\left|\tilde{u}_n - \tilde{v}\right|_{\epsilon_n} \to 0$ and so $\left|u_n - \tilde{v}\right|_{\epsilon_n} \to 0$ as $n \to \infty$. Thus

$$\left|(I - Q_{\epsilon_n})u_n\right|_{\epsilon_n} < \eta \text{ and } \left|(I - Q_{\epsilon_n})u_n\right|_{\epsilon_n} < \nu \text{ for all } n \text{ large enough.} \tag{31}$$

Moreover, since $u_n \in [\operatorname{cl} V(a,b)]_{\epsilon_n,\eta}$ we also have $Q_{\epsilon_n}u_n \in \operatorname{cl} V(a,b) \subset U_0$. Thus $u_n \in]U_0[_{\epsilon_n,\nu}$ for all n large enough and so Lemma 5.3 and the continuity of of t^+ imply that $t^+_{\epsilon_n,\nu}(u_n) \to t^+(\tilde{v})$ and $t^+(Q_{\epsilon_n}u_n) \to t^+(\tilde{v})$ as $n \to \infty$.

Suppose that (27) holds. Then relations (27) and (31) imply that $Q_{\epsilon_n}u_n \in \partial \operatorname{cl} V(a,b) \subset U_0$, for all n large enough and so $\tilde{v} \in \partial \operatorname{cl} V(a,b) \subset U_0$. Since $t^+_{\epsilon_n,\nu}(u_n) > M'$ for all n we also conclude that $t^+(\tilde{v}) \geq M' > b$. Since $g^-(\tilde{v}) = 0 < a$, we see that $\tilde{v} \in V(a,b)$. However $V(a,b) \cap \partial \operatorname{cl} V(a,b) = \emptyset$. This contradiction proves part 1(a).

Suppose now that (28) holds. Then $t^+_{\epsilon_n,\nu}(u_n) \leq 4M$ for all n and so $t^+(\tilde{v}) \leq 4M < 5M$. We thus conclude that $t^+(Q_{\epsilon_n}u_n) < 5M$ for all n large enough. All this clearly implies that $u_n \in \hat{E}(\epsilon_n)$ for all n large enough, a contradiction proving the second inclusion in 1(b).

Finally assume that (29) holds. Therefore for all $n \in \mathbb{N}$

$$Q_{\epsilon_n}u_n \in \{\, u \in U_0 \mid t^+(u) \leq 3M \,\} \cap \operatorname{cl} V(\delta_n, M) \text{ and } \left|(I - Q_{\epsilon_n})u_n\right|_{\epsilon_n} \leq \alpha_n. \tag{32}$$

Hence there exists a $v_n \in V(\delta_n, M)$ such that $\left|Q_{\epsilon_n}u_n - v_n\right|_{H^1} < 2^{-n}$. Since $g^-(v_n) \to 0$, we may again assume that there is a $v \in A^-_{\pi_0}(N_0)$ such that $\left|v_n - v\right|_{H^1} \to 0$ as $n \to \infty$. Therefore

$$\left|Q_{\epsilon_n}u_n - v\right|_{H^1} \to 0 \text{ as } n \to \infty.$$

Together with (32) this shows that $\left|u_n - v\right|_{\epsilon_n} \to 0$ as $n \to \infty$. For all n large enough we have $g^-(v_n) < \delta_n < a$ and $t^+(v_n) > M > b$ so $v_n \in V(a,b)$. This implies that $v \in \operatorname{cl} V(a,b) \subset U_0$. Since $t^+(Q_{\epsilon_n}u_n) \leq 3M$, the continuity of t^+ implies that $t^+(v) \leq 3M$. Recall that $u_n \in]U_0[_{\epsilon_n,\nu}$ for all n large enough. Lemma 5.3 now shows that $t^+_{\epsilon_n,\nu}(u_n) \to t^+(v)$ as $n \to \infty$. Hence we can assume that $t^+_{\epsilon_n,\nu}(u_n) \leq 4M$ for all n large enough.

Since $N_1(\alpha_n, \delta_n, \epsilon_n)$ is closed, we obtain from Remark 5.5 and Lemma 2.19 that, for all n large enough,

$$u_n \in [\operatorname{cl} V(\delta_n, M)]_{\epsilon_n,\alpha_n} = \operatorname{cl}_{\epsilon_n}]V(\delta_n, M)[_{\epsilon_n,\alpha_n} = \operatorname{cl}_{\epsilon_n} V_{\epsilon_n,\alpha_n}(\delta_n, M) \subset N_1(\alpha_n, \delta_n, \epsilon_n).$$

This implies that $u_n \in \hat{C}(\epsilon_n, \alpha_n, \delta_n)$. This contradiction completes the proof of part 1 of the proposition.

If the second part of the proposition is not true then there exist numbers $M > 0$, $\alpha > 0$ and sequences of positive numbers $(\delta_n)_{n\in\mathbb{N}}$, $(\rho_n)_{n\in\mathbb{N}}$ and $(\epsilon_n)_{n\in\mathbb{N}}$, converging to zero and a sequence $(u_n)_{n\in\mathbb{N}}$ such that

$$u_n \in C(\epsilon_n, \rho_n, \alpha, \delta_n) \text{ and } u_n \notin E(\epsilon_n, \alpha, \delta_n).$$

Therefore for every $n \in \mathbb{N}$ we have $u_n \in]U_0[_{\epsilon_n,\alpha}$, $t^+_{\epsilon_n,\alpha}(u_n) \leq 2M$ and there exists a $v_n \in V_{\epsilon_n,\rho_n}(\delta_n, M)$ such that $\left|u_n - v_n\right|_{\epsilon_n} < 2^{-n}$. Therefore $Q_{\epsilon_n}v_n \in V(\delta_n, M)$. Hence for every $n \in \mathbb{N}$,

$$g^-(Q_{\epsilon_n}v_n) < \delta_n \text{ and } t^+(Q_{\epsilon_n}v_n) > M$$

and so there exists a subsequence of $(v_n)_{n\in\mathbb{N}}$, denoted again by $(v_n)_{n\in\mathbb{N}}$, and a $v \in A^-_{\pi_0}(N_0)$ such that $\left|Q_{\epsilon_n}v_n - v\right|_{H^1} \to 0$ as $n \to \infty$.

Since $\left|(I - Q_{\epsilon_n})v_n\right|_{\epsilon_n} < \rho_n$, we conclude that $\left|v_n - v\right|_{\epsilon_n} \to 0$ and so $\left|u_n - v\right|_{\epsilon_n} \to 0$ as $n \to \infty$. For all n large enough we have $Q_{\epsilon_n} v_n \in V(a, b)$. Therefore $v \in \operatorname{cl} V(a, b) \subset U_0$. Recall that $u_n \in]U_0[_{\epsilon_n,\alpha}$. Hence Lemma 5.3 implies that $t^+_{\epsilon_n,\alpha}(u_n) \to t^+(v)$ as $n \to \infty$. Therefore $t^+(v) \le 2M$. This inequality and the continuity of t^+ imply that $t^+(Q_{\epsilon_n} u_n) \le 3M$ for all n large enough. Since $u_n \in \operatorname{cl}_{\epsilon_n} V_{\epsilon_n,\rho_n}(\delta_n, M) = [\operatorname{cl} V(\delta_n, M)]_{\epsilon_n,\rho_n}$ it follows that for all n large enough,

$$Q_{\epsilon_n} u_n \in \operatorname{cl} V(\delta_n, M) \text{ and } \left|(I - Q_{\epsilon_n})u_n\right|_{\epsilon_n} < \alpha.$$

Therefore we have proven that $u_n \in E(\epsilon_n, \alpha, \delta_n)$ for all n large enough, a contradiction. The proposition is proved. □

Lemma 5.7. *Let N_1 and N_2 be closed subsets in $H^1_s(\Omega)$ such that $N_2 \subset N_1$. Then for every $\epsilon > 0$ and $\eta > 0$ the pointed spaces N_1/N_2 and $[N_1]_{\epsilon,\eta} / [N_2]_{\epsilon,\eta}$ have the same homotopy type.*

Proof. Let $T_\epsilon\colon [N_1]_{\epsilon,\eta} / [N_2]_{\epsilon,\eta} \to N_1/N_2$ be the map induced by the projection Q_ϵ. Obviously $T_\epsilon \circ \iota = I_{N_1/N_2}$, where $\iota\colon N_1/N_2 \to [N_1]_{\epsilon,\eta} / [N_2]_{\epsilon,\eta}$ is the inclusion induced map. Let $H\colon [N_1]_{\epsilon,\eta} / [N_2]_{\epsilon,\eta} \times [0, 1] \to [N_1]_{\epsilon,\eta} / [N_2]_{\epsilon,\eta}$ be the map induced by the function

$$K\colon [N_1]_{\epsilon,\eta} \times [0, 1] \to [N_1]_{\epsilon,\eta}\,,\ (v, t) \mapsto Q_\epsilon v + (I - Q_\epsilon)(1 - t)v.$$

It is clear that H is a homotopy between $I_{[N_1]_{\epsilon,\eta}/[N_2]_{\epsilon,\eta}}$ and $\iota \circ T_\epsilon$. This concludes the proof. □

We can now complete the proof of Theorem 3.5: Choose positive numbers β, ν, η and M with $M > b_0$ and $\eta < \nu < \beta$. Now choose positive numbers $\alpha' < \eta$, $\delta' < a_0$ and ϵ' such that for all positive $\alpha \le \alpha'$, $\delta \le \delta'$ and $\epsilon \le \epsilon'$ all assertions of part 1 of Proposition 5.6 hold, where we fix $a := a_0$, $b := b_0$ and $M' := 4M$. Now choose positive numbers $\rho' < \alpha'$, $\delta'' \le \delta'$ and $\epsilon'' \le \epsilon'$ such that for all positive $\rho \le \rho'$, $\delta \le \delta''$ and $\epsilon \le \epsilon''$ part 2 of Proposition 5.6 holds, where we fix $\alpha := \alpha'$.

Notice also that $2M > b_0$, $\rho' < \alpha'$ and $\alpha' < \beta$ ($\alpha' < \eta < \nu < \beta$). Therefore we are able to apply Proposition 5.6 with M and M' replaced by $2M$, $\eta := \rho'$, $\nu := \alpha'$, $a := \delta''$, $b := M$. Thus we obtain positive numbers $\alpha''' \le \rho'$, $\delta''' < \delta''$ and $\epsilon''' \le \epsilon''$ such that for all positive $\alpha \le \alpha'''$, $\delta \le \delta'''$ and $\epsilon \le \epsilon'''$ the pair $(\tilde N_1(\alpha, \delta, \epsilon), \tilde N_2(\alpha, \delta, \epsilon))$ is a pseudo-index pair in $[\operatorname{cl} V(\delta'', M)]_{\epsilon,\rho'}$.

Here,

$$\begin{aligned}\tilde N_1(\alpha, \delta, \epsilon) = [\operatorname{cl} V(\delta'', M)]_{\epsilon,\rho'} \cap \operatorname{cl}_\epsilon\{\, v \mid \exists\, u \in V_{\epsilon,\alpha}(\delta, 2M) \text{ and } t \ge 0 \text{ such}\\ \text{that } u\pi_\epsilon[0, t] \subset\,]U_0[_{\epsilon,\beta} \text{ and } u\pi_\epsilon t = v\,\}\end{aligned}$$

and

$$\tilde N_2(\alpha, \delta, \epsilon) = \tilde N_1(\alpha, \delta, \epsilon) \cap \{\, u \in\,]U_0[_{\epsilon,\alpha'} \mid t^+_{\epsilon,\alpha'}(u) \le 2M\,\}.$$

Fix $\alpha := \alpha'''$, $\delta := \delta'''$ and write $\rho := \rho'$. We now conclude that for $\epsilon \in]0, \epsilon'''[$

$$\begin{aligned}A_1 := \tilde N_1(\alpha, \delta, \epsilon) \subset [\operatorname{cl} V(\delta'', M)]_{\epsilon,\rho} \subset A_2 := [\operatorname{cl} V(\delta'', M)]_{\epsilon,\alpha'}\\ \subset A_3 := N_1(\alpha', \delta'', \epsilon) \subset A_4 := [\operatorname{cl} V(a_0, b_0)]_{\epsilon,\eta}\,.\end{aligned}$$

$$\begin{aligned} B_1 := \tilde{N}_2(\alpha,\delta,\epsilon) &\subset [\operatorname{cl} V(\delta'',M)]_{\epsilon,\rho} \cap \{\, u \in]U_0[_{\epsilon,\alpha'} \mid t^+_{\epsilon,\alpha'}(u) \le 2M \,\} \\ &= C(\epsilon,\rho,\alpha',\delta'') \subset B_2 := E(\epsilon,\alpha',\delta'') \\ &\subset B_3 := \hat{C}(\epsilon,\alpha',\delta'') = N_2(\alpha',\delta'',\epsilon) \\ &\subset B_4 := \hat{E}(\epsilon). \end{aligned}$$

Notice that for each $i = 1,2,3,4$, $B_i \subset A_i$. Therefore we can consider the pointed spaces A_i/B_i. The two sequences of inclusions described above induce continuous maps

$$\Gamma_1 \colon A_1/B_1 \to A_2/B_2,\ \Gamma_2 \colon A_2/B_2 \to A_3/B_3, \text{ and } \Gamma_3 \colon A_3/B_3 \to A_4/B_4.$$

Using Lemma 5.2 let us now choose a positive number $\epsilon^{\mathrm{c}} \le \epsilon'''$ such that

$$K(\epsilon,\eta,a_0,b_0) \subset V_{\epsilon,\alpha'}(\delta'',M) \cap V_{\epsilon,\alpha}(\delta,2M)$$

for $\epsilon \in]0,\epsilon^{\mathrm{c}}]$. Then Remark 5.5 implies that (A_1,B_1) and (A_3,B_3) are index pairs for $K(\epsilon,\eta,a_0,b_0)$ and it follows from Theorem 9.4 in [39] that $\Gamma_2 \circ \Gamma_1$ is an isomorphism in the homotopy category of the pointed spaces. The proof of Theorem 2.3 in [39] implies that

$$(\operatorname{cl} V(\delta'',M), \operatorname{cl} V(\delta'',M) \cap \{\, u \in U_0 \mid t^+(u) \le 3M \,\})$$

and

$$(\operatorname{cl} V(a_0,b_0), \operatorname{cl} V(a_0,b_0) \cap \{\, u \in U_0 \mid t^+(u) \le 5M \,\})$$

are index pairs for K_0. Now, by Lemma 5.7, we conclude that there exist maps

$$\Gamma_4 \colon \operatorname{cl} V(\delta'',M)/(\operatorname{cl} V(\delta'',M) \cap \{\, u \in U_0 \mid t^+(u) \le 3M \,\}) \to A_2/B_2$$

and

$$\Gamma_5 \colon \operatorname{cl} V(a_0,b_0)/(\operatorname{cl} V(a_0,b_0) \cap \{\, u \in U_0 \mid t^+(u) \le 5M \,\}) \to A_4/B_4$$

which are isomorphism in the homotopy category of pointed spaces. Moreover, Theorem 9.4 in [39] allows us to conclude that $\Gamma_5^{-1} \circ \Gamma_3 \circ \Gamma_2 \circ \Gamma_4$ is also an isomorphism in the same category. Consequently $\Gamma_3 \circ \Gamma_2$ is an isomorphism. Now Lemma 12.4 in [39] implies that Γ_1, Γ_2 and Γ_3 are all isomorphisms in the homotopy category of pointed spaces. We conclude that (5) also holds for the case $K_0 \neq \emptyset$.

The proof of Theorem 3.5 is complete. □

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