HOLOMORPHIC FUNCTIONS WITH SINGULARITIES ON ALGEBRAIC SETS

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Abstract. The aim of the paper is to prove the following Theorem: Let P be a non-zero polynomial of two complex variables. Put $A := \{(z_1, z_2); P(z_1, z_2) = 0\}$, $A^1_{z_2} := \{z_1; P(z_1, z_2) = 0\}$, $A^2_{z_1} := \{z_2; P(z_1, z_2) = 0\}$. Let E_1, E_2 be two closed subsets of $\mathbb{C}$ with positive logarithmic capacities. Put $X := (E_1 \times \mathbb{C}) \cup (\mathbb{C} \times E_2)$. Let $f\colon X \setminus A \ni (z_1, z_2) \mapsto f(z_1, z_2) \in \mathbb{C}$ be a function separately holomorphic on $X \setminus A$, i.e. $f(z_1, \cdot) \in \mathcal{O}(\mathbb{C} \setminus A^2_{z_1})$ for every $z_1 \in E_1$, and $f(\cdot, z_2) \in \mathcal{O}(\mathbb{C} \setminus A^1_{z_2})$ for every $z_2 \in E_2$.

Then there exists a unique function $\tilde{f} \in \mathcal{O}(\mathbb{C}^2 \setminus A)$ with $\tilde{f} = f$ on $X \setminus A$.

If $E_1 = E_2 = \mathbb{R}$ and $P(z_1, z_2) = z_1 - z_2$, we get the result due to O. Öktem [6].

1. Introduction

The aim of this paper is to prove the following theorem.

Theorem 1.1. *Given $n \geq 2$, let E_j $(j = 1, \ldots, n)$ be a closed subset of the complex plane $\mathbb{C}$ of the positive logarithmic capacity. Put*

$$(*)\quad X := (\mathbb{C} \times E_2 \times \cdots \times E_n) \cup (E_1 \times \mathbb{C} \times E_3 \times \cdots \times E_n) \cup \cdots \cup (E_1 \times \cdots \times E_{n-1} \times \mathbb{C}).$$

Let P be a non-zero polynomial of n complex variables. Put

$$(**)\qquad A := \{z \in \mathbb{C}^n; P(z) = 0\}, \quad A^j_{z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n} := \{z_j \in \mathbb{C}; z \in A\}$$

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for $(z_1, \dots, z_{j-1}, z_{j+1}, \dots, z_n) \in \mathbb{C}^{n-1}$, $j = 1, \dots, n$. *Let* $f : X \setminus A \mapsto \mathbb{C}$ *be a function separately holomorphic on* $X \setminus A$ *in the sense that*

$$f(z_1, \dots, z_{j-1}, \cdot, z_{j+1}, \dots, z_n) \in \mathcal{O}(\mathbb{C} \setminus A^j_{z_1,\dots,z_{j-1},z_{j+1},\dots,z_n}),$$

if $z_k \in E_k (k \neq j)$, $j = 1, \dots, n$.

Then $f \in \mathcal{O}(\mathbb{C}^n \setminus A)$, *i.e. there exists a unique function* $\tilde{f} \in \mathcal{O}(\mathbb{C}^n \setminus A)$ *with* $\tilde{f} = f$ *on* $X \setminus A$.

If $n = 2$, $E_1 = E_2 = \mathbb{R}$ and $P(z_1, z_2) = z_1 - z_2$, we get the result due to O. Öktem [6]. Properties of separately holomorphic functions of the above type were used by O. Öktem ([6,6a]) to characterize the range of the exponential Radon transform (which in turn is of interest for mathematical tomography). Theorem 1.1 shows that the *Main conjecture* of paper [6a] is true at least for a class of special cases interesting for applications in mathematical tomography.

Let D_1 and D_2 be two domains in $\mathbb{C}^n$ with $D_1 \subset D_2$. In the sequel we shall say that a function f defined and holomorphic on D_1 is holomorphic on D_2, if there exists a unique function $\tilde{f}$ holomorphic on D_2 such that $\tilde{f} = f$ on D_1.

We shall need the following three known theorems.

Theorem 1.2. *Let* F_j *be a nonpolar relatively closed subset of a domain* D_j *on the complex* z_j*-plane,* $j = 1, \dots, n$. *Let* $f : X \mapsto \mathbb{C}$ *be a function of* n *complex variables separately holomorphic on the set* $X := D_1 \times F_2 \times \cdots \times F_n \cup \ldots \cup F_1 \times \cdots \times F_{n-1} \times D_n$.

Then the function f *is holomorphic on a neighborhood of the set*

$$D_1 \times (F_2)_{reg} \times \cdots \times (F_n)_{reg} \quad \cup \quad \ldots \quad \cup \quad (F_1)_{reg} \times \cdots \times (F_{n-1})_{reg} \times D_n,$$

where $(F_j)_{reg}$ *is the set of points* a *of* F_j *such that* F_j *is locally regular (in the sense of the logarithmic potential theory) at* a.

Theorem 1.3. *Let* $D \subset \mathbb{C}^m$ *(resp.* $G \subset \mathbb{C}^n$*) be a domain with a pluripolar boundary. Let* E *(resp.* F*) be a non-pluripolar relatively closed subset of* D *(resp.* G*).*

Then every function $f : X \mapsto \mathbb{C}$ *separately holomorphic on the set* $X := E \times G \cup D \times F$ *is holomorphic on* $D \times G$.

Theorems 1.2 and 1.3 are direct consequences of (e.g.) the main result of [4].

Theorem 1.4 [2]**.** *Let* A *be an analytic subset (of pure codimension 1) of the envelope of holomorphy* $\hat{D}$ *of a domain* $D \subset \mathbb{C}^n$.

Then $\hat{D} \setminus A$ *is the envelope of holomorphy of* $D \setminus A$.

2. Proof of Theorem 1.1

We shall consider three cases.

Case 1^o. $A = \emptyset$.

In this case Theorem 1.1 follows from Theorem 1.3.

Case 2^o. P depends on all variables, i.e. for every k with $1 \leq k \leq n$ the polynomial P can be written in the form

$$(\#) \qquad P(z) = \sum_{j=0}^{d_k} p_{kj}(z_1, \dots, z_{k-1}, z_{k+1}, \dots, z_n) z_k^j,$$

where $d_k \geq 1$ and $p_{kd_k} \neq 0$, $\quad k = 1, \dots, n$.

We shall show that in the case 2^o our theorem follows from the following Lemma.

Lemma 2.1. *If the polynomial P depends on all variables, then there exists a function g holomorphic on the domain $\mathbb{C}^n \setminus A$ such that $g = f$ on $F_1 \times \cdots \times F_n$, where $F_1 \times \cdots \times F_n \subset \mathbb{C}^n \setminus A$ and F_j is a non-polar subset of E_j $(j = 1, \dots, n)$.*

In order to prove Theorem 1.1 it is sufficient to show that $g = f$ on $X \setminus A$.

First we shall consider the case of $n = 2$. Fix $(a_1, a_2) \in X \setminus A$. We need to show that $g(a_1, a_2) = f(a_1, a_2)$. Without loss of generality we may assume that $a_1 \in E_1$. It is clear that $A^2_{a_1} \neq \mathbb{C}$ (because $P(a_1, a_2) \neq 0$). Therefore the set $F_2 \setminus A^2_{a_1}$ is nonpolar. Hence there exist $z_2^o \in F_2 \setminus A^2_{a_1}$ and $r > 0$ such that $B(z_2^o, r) \cap F_2$ is non-polar, and $(\{a_1\} \times B(z_2^o, r)) \cap A = \emptyset$ (equivalently, $P(a_1, z_2) \neq 0$ for all $z_2 \in B(z_2^o, r)$). We know that $g(z_1, z_2) = f(z_1, z_2)$ on $F_1 \times F_2$. Fix $z_2 \in B(z_2^o, r) \cap F_2$. The functions $g(\cdot, z_2)$ and $f(\cdot, z_2)$ are holomorphic on the domain $\mathbb{C} \setminus A^1_{z_2}$ and identical on the nonpolar subset F_1 of the domain. Hence $g(\cdot, z_2) = f(\cdot, z_2)$ on $\mathbb{C} \setminus A^1_{z_2}$. It is clear that $a_1 \in \mathbb{C} \setminus A^1_{z_2}$. By the arbitrary character of z_2 it follows that $g(a_1, z_2) = f(a_1, z_2)$ for all $z_2 \in B(z_2^o, r) \cap F_2$. The functions $g(a_1, \cdot)$ and $f(a_1, \cdot)$ are holomorphic on the domain $\mathbb{C} \setminus A^1_{a_1}$, and identical on the nonpolar subset $B(z_2^o, r) \cap F_2$ of the domain. Therefore $g(a_1, \cdot) = f(a_1, \cdot)$ on $\mathbb{C} \setminus A^2_{a_1}$; in particular $g(a_1, a_2) = f(a_1, a_2)$.

Now consider the case of $n > 2$ and assume that Theorem 1.1 is true in $\mathbb{C}^k$ with $2 \leq k \leq n - 1$. Fix $a = (a_1, \dots, a_n) \in X \setminus A$. Without loss of generality we may assume $a_1 \in E_1$. Put $a = (a_1, a')$ with $a' = (a_2, \dots, a_n)$. It is clear that $A^{(2,\dots,n)}_{a_1} := \{z' \in \mathbb{C}^{n-1}; P(a_1, z') = 0\} \neq \mathbb{C}^{n-1}$. Hence $A^{(2,\dots,n)}_{a_1}$ is pluripolar. Therefore there exist $\dot{z}' = (\dot{z}_2, \dots, \dot{z}_n) \in F_2 \times \cdots \times F_n$ and $r > 0$ such that $P(\dot{z}', r) \cap F'$ is non-pluripolar, and $(\{a_1\} \times P(\dot{z}', r)) \cap A = \emptyset$, where $P(\dot{z}', r) := (z' \in \mathbb{C}^{n-1}; |z_j - \dot{z}_j| < r (j = 2, \dots, n)\}$ and $F' := F_2 \times \cdots \times F_n$. We know that $g(z_1, z') = f(z_1, z')$ for all $(z_1, z') \in F_1 \times F'$.

Fix $z' \in F' \cap P(\dot{z}', r)$. The functions $g(\cdot, z')$ and $f(\cdot, z')$ are holomorphic on the domain $\mathbb{C} \setminus A^1_{z'}$ and $g(\cdot, z') = f(\cdot, z')$ on the nonpolar subset F_1 of the domain. Hence $g(\cdot, z') = f(\cdot, z')$ on $\mathbb{C} \setminus A^1_{z'}$. It is clear that $a_1 \in \mathbb{C} \setminus A^1_{z'}$. By the arbitrary character of $z' \in F' \cap P(\dot{z}', r)$ it follows that $g(a_1, z') = f(a_1, z')$ for all $z' \in F' \cap P(\dot{z}', r) = (F_2 \cap B(\dot{z}_2, r)) \times \cdots \times (F_n \cap B(\dot{z}_n, r))$, and $F_j \cap B(\dot{z}_j, r)$ is a nonpolar subset of E_j $(j = 2, \dots, n)$. The function $f(a_1, \cdot)$ is separately analytic on $X' \setminus A^{(2,\dots,n)}_{a_1}$, where

$$X' := \mathbb{C} \times E_3 \times \cdots \times E_n \cup \cdots \cup E_2 \times \cdots \times E_{n-1} \times \mathbb{C}.$$

The function $g(a_1, \cdot)$ is holomorphic on $\mathbb{C} \setminus A^{(2,\dots,n)}_{a_1}$, and $g(a_1, z') = f(a_1, z')$ for all $z' \in F' \cap P(\dot{z}', r)$. By the induction assumption $g(a_1, z') = f(a_1, z')$ for all $z' \in \mathbb{C}^{n-1} \setminus A^{(2,\dots,n)}_{a_1}$. In particular $g(a_1, a') = f(a_1, a')$.

Case 3^o. The polynomial P depends on k variables, where $1 \leq k \leq n - 1$. Without loss of generality we may assume that $P(z_1, \dots, z_n) \equiv p(z_1, \dots, z_k)$, where p is a polynomial of k variables $z_1, \dots, z_k$. Put $A_k := \{z \in \mathbb{C}^k; p(z) = 0\}$. Then $A = A_k \times \mathbb{C}^{n-k}$. The set A_k is pluripolar in $\mathbb{C}^k$, the set $A_k \times \mathbb{C}^{n-k}$ is pluripolar in $\mathbb{C}^n$. Therefore there exists a point $a \in ((E_1)_{reg} \times \cdots \times (E_n)_{reg}) \setminus (A_k \times \mathbb{C}^{n-k})$. Let $r > 0$ be so small that $A_k \times \mathbb{C}^{n-k}$ is disjoined with the polydisk $\bar{B}(a_1, r) \times \cdots \times \bar{B}(a_n, r)$.

The set $F_j := E_j \cap \bar{B}(a_j, r)$ has positive capacity ($j = 1, \dots, n$). The set $F_1 \times \cdots \times F_n$ is non-pluripolar.

By 2^o for each $(z_{k+1}, \dots, z_n) \in E_{k+1} \times \cdots \times E_n$ the function $f(\cdot, \dots, \cdot, z_{k+1}, \dots, z_n)$ is holomorphic on $\mathbb{C}^k \setminus A_k$, and for each $(z_1, \dots, z_k) \in F_1 \times \cdots \times F_k$ the function $f(z_1, \dots, z_k, \cdot, \dots, \cdot)$ is holomorphic on $\mathcal{O}(\mathbb{C}^{n-k})$. Hence by Theorem 1.3 the function f is holomorphic on $(\mathbb{C}^k \setminus A_k) \times \mathbb{C}^{n-k} = \mathbb{C}^n \setminus A$. This ends the proof of Theorem 1.1.

3. Proof of Lemma 2.1

Put

$$A^k \quad := \quad \{z \in \mathbb{C}^n; p_{kd_k}(z_1, \dots z_{k-1}, z_{k+1}, \dots, z_n) = 0\}, \quad k = 1, \dots, n.$$

Then the set

$$B \quad := \quad A \cup A^1 \cup \cdots \cup A^n$$

is pluripolar. We know that the set $(E_j)_{reg}$ is not polar. Therefore the cartesian product $(E_1)_{reg} \times \cdots \times (E_n)_{reg}$ is not pluripolar, and hence

$$(E_1)_{reg} \times \cdots \times (E_n)_{reg} \setminus B \neq \emptyset.$$

Fix

$$z^o = (z_1^o, \dots, z_n^o) \in (E_1)_{reg} \times \cdots \times (E_n)_{reg} \setminus (A \cup A^1 \cup \cdots \cup A^n).$$

Then there exists $r_o > 0$ such that

$$(\#\#) \qquad (\bar{B}(z_1^o, 2r_o) \times \cdots \times \bar{B}(z_n^o, 2r_o)) \cap (A \cup A^1 \cup \cdots \cup A^n) = \emptyset,$$

where $B(z_j^o, 2r_o) := \{z_j \in \mathbb{C}; |z_j - z_j^o| < 2r_o\}$. In particular, $p_{kd_k}(z_1, \dots, z_{k-1}, z_{k+1}, \dots, z_n) \neq 0$ on $\bar{B}(z_1^o, 2r_o) \times \cdots \times \bar{B}(z_{k-1}^o, 2r_o) \times B(z_{k+1}^o, 2r_o) \times \cdots \times \bar{B}(z_n^o, 2r_o)$.

We shall show that Lemma 2.1 follows from the following Proposition.

Proposition 3.1. *If $0 < \delta < \min\{1, r_o\}$ is sufficiently small then for each $k = 1, \dots, n$ there exists a sequence $\{c_{kj}\}_{j \in \mathbb{Z}}$ of functions holomorphic on the open set*

$$\Omega_k := B(z_1^o, \delta) \times \cdots \times B(z_{k-1}^o, \delta) \times \mathbb{C} \times B(z_{k+1}^o, \delta) \times \cdots \times B(z_n^o.\delta)$$

such that

(i) the series

$$f_k(z) := \sum_{j=-\infty}^{\infty} c_{kj}(z) P(z)^j$$

is locally uniformly convergent on $\Omega_k \setminus A$, so that the function f_k is holomorphic on $\Omega_k \setminus A$;

(ii) $f_k(z) = f(z)$ on the set $F_1 \times \cdots \times F_n$, where

$$F_j := E_j \cap B(z_j^o, \delta), \quad j = 1, \dots, n.$$

In order to prove Lemma 2.1 let us observe that by (##) and by (ii) $f_j = f_k = f$ on the non-pluripolar subset $F_1 \times \cdots \times F_n$ of the domain $(\Omega_j \cap \Omega_k) \setminus A$. Therefore the function

$$f_o := f_1 \cup \cdots \cup f_n$$

is well defined and holomorphic on $\Omega \setminus A$ with $\Omega := \Omega_1 \cup \cdots \cup \Omega_n$. Moreover $f_o = f$ on $F_1 \times \cdots \times F_n$. The set Ω is a Reinhardt domain with centre z^o whose envelope of holomorphy is $\mathbb{C}^n$. Therefore by Grauert-Remmert Theorem 1.4 there exists a function g holomorphic on $\mathbb{C}^n \setminus A$ such that $g = f_o$ on $\Omega \setminus A$; in particular $g = f$ on $F_1 \times \cdots \times F_n$. The proof of Lemma 2.1 is finished.

4. Proof of Proposition 3.1

Fix integer k with $1 \leq k \leq n$. Without loss of generality we may assume that $k = 1$. Let $\{a_1, \ldots, a_s\} := \{z_1 \in \mathbb{C}; P(z_1, z_2^o, \ldots, z_n^o) = 0\}$ be the zero set of the polynomial $P(\cdot, z_2^o, \ldots, z_n^o)$. By (##) the number m given by

$$2m := \min\{|p_{1d_1}(z')|; |z_j - z_j^o| \leq r_o \, (j = 2, \ldots, n)\}.$$

is positive.

Let $R_o > \max\{1, r_o\}$ be so large that $B(a_j, 2) \subset B(0, R_o) \quad (j = 1, \ldots, s)$ and

$$\text{(4.0)} \qquad |P(z)| \geq m|z_1|^{d_1} \quad \text{for all} \quad |z_1| \geq R_o, \quad |z_j - z_j^o| \leq r_o (j = 2, \ldots, n).$$

We claim that for every $R > R_o$ and for each $\epsilon, 0 < \epsilon < 1$, so small that

$$\bar{B}(z_1^o, r_o) \cap \left(\cup_{j=1}^s \bar{B}(a_j, \epsilon)\right) = \emptyset, \quad \bar{B}(a_j, \epsilon) \cap \bar{B}(a_l, \epsilon) = \emptyset (j \neq l)$$

there exists $\delta = \delta(\epsilon, R)$ such that $0 < 2\delta < r_o$ and f is bounded and holomorphic on the set

$$\{z \in \mathbb{C}^n; \epsilon < |z_1 - a_j| < \tfrac{3}{2}R \quad (j = 1, \ldots, s), \quad |z_l - z_l^o| < \delta (l = 2, \ldots, n)\}.$$

Indeed, f is separately holomorphic on the set

$$D_1 \times F_2 \times \cdots \times F_n \cup \cdots \cup F_1 \times \cdots \times F_{n-1} \times D_n$$

with $F_1 := E_1 \cap \bar{B}(z_1^o, r_o)$, $F_j := E_j \cap B(z_j^o, r_o) \quad (j = 2, \ldots, n)$, $D_1 := \mathbb{C} \setminus \left(\bar{B}(a_1, \epsilon) \cup \cdots \cup \bar{B}(a_s, \epsilon)\right)$, $D_j := B(z_j^o, r_o) \quad (j = 2, \ldots, n)$. For each j the set F_j is locally regular at z_j^o. Hence by Theorem 1.2 there exists $\delta = \delta(\epsilon, R)$ such that $O < 2\delta < r_o$ and f is holomorphic on the domain

$$(\dagger) \quad \{z \in \mathbb{C}^n; \tfrac{\epsilon}{2} < |z_1 - a_j| < 2R(j = 1, \ldots, s), \quad |z_\ell - z_\ell^o| < 2\delta \, (\ell = 2, \ldots, n)\},$$

where δ depends on ϵ and R. Moreover, we may assume that $P(z) \neq 0$ for all z in the domain $(\dagger)$.

Observe that by (#) the function

$$W(\omega, z) := \frac{P(\omega, z') - P(z_1, z')}{\omega - z_1} \equiv \sum_{l=1}^{d_1} p_{1l}(z')[\omega^{l-1} + \omega^{l-2} z_1 + \cdots + z_1^{l-1}]$$

is a polynomial of $n+1$ variables $\omega, z_1, \dots, z_n$.

It is clear that for every $j \in \mathbb{Z}$ the function

$$(\&) \qquad \Phi_j(\omega, z) := W(\omega, z)\frac{f(\omega, z')}{P(\omega, z')^{j+1}}$$

is holomorphic on the set $\{(\omega, z) \in \mathbb{C}^{n+1}; \frac{\epsilon}{2} < |\omega - a_j| < 2R \, (j = 1, \dots, s), \quad z_1 \in \mathbb{C}, \quad z' \in B(z_2^o, 2\delta) \times \cdots \times B(z_n^o, 2\delta)\}$.

Therefore the function

$$(4.1) \qquad c_{1j}(z) := \tfrac{1}{2\pi i}\int_{C(0,R)} \Phi_j(\omega, z)d\omega$$

is holomorphic on the set $\mathbb{C} \times B(z_2^o, 2\delta) \times \cdots \times B(z_n^o, 2\delta)$; here $C(0, R)$ denotes the positively oriented circle of centre 0 and radius R. Moreover, by (4.0) for every compact subset K of $\mathbb{C}$ there exist positive constants $M_1 = M_1(K, R, \epsilon)$ and $M_2 = M_2(K, R, \epsilon)$ such that

$$(4.2) \qquad |c_{1j}(z)| \leq M_1(M_2R^{d_1})^j$$

for all $j \in \mathbb{Z}$ and $z \in K \times B(z_2^o, \delta) \times \cdots \times B(z_n^o, \delta)$.

For a fixed $z' \in F_2 \times \cdots \times F_n$ with $F_j := E_j \cap B(z_j^o, \delta)$ the function $\Phi_j(\cdot, \cdot, z')$ is holomorphic on $\{\omega \in \mathbb{C}; P(\omega, z') \neq 0\} \times \mathbb{C}$. Hence, by the Cauchy residue theorem,

$$(4.3) \qquad c_{1j}(z) = \frac{1}{2\pi i}\int_{\partial D_+(z',\rho)} \Phi_j(\omega, z)d\omega, \quad z \in \mathbb{C} \times (F_2 \times \cdots \times F_n),$$

where ρ is any positive real number and

$$D_+(z', \rho) := \{z_1 \in \mathbb{C}; |P(z_1, z')| < \rho.\}$$

In the formula (4.3) the integration is taken over the positively oriented boundary of the open set $D_+(z', \rho)$ (the interior of the lemniscate on the z_1-plane).

We claim that the functions c_{1j} defined by (4.1) have the required properties. We already know that they are holomorphic on $\Omega_1 := \mathbb{C} \times B(z_2^o, \delta) \times \cdots \times B(z_n^o, \delta)$. In order to prove that they have all the required properties let us observe that, given $z' \in F_2 \times \cdots \times F_n$ and $0 < r < 1$, we have

$$f(z) = \frac{1}{2\pi i}\int_{\partial D(z',r)} \frac{f(\omega, z')}{\omega - z_1}d\omega, \quad z_1 \in D(z', r) := \{z_1 \in \mathbb{C}; r < |P(z_1, z')| < \frac{1}{r}\}.$$

Hence

$$f(z) = \frac{1}{2\pi i}\int_{\partial D_+(z',\frac{1}{r})} \frac{f(\omega, z')}{\omega - z_1}d\omega - \frac{1}{2\pi i}\int_{\partial D_-(z',r)} \frac{f(\omega, z')}{\omega - z_1}d\omega$$

for all $z_1 \in D(z', r)$, where $D_+(z', \frac{1}{r}) := \{z_1 \in \mathbb{C}; |P(z_1, z')| < \frac{1}{r}\}$, $D_-(z', r) := \{z_1 \in \mathbb{C}; |P(z_1, z')| > r\}$, $D(z', r) = D_+(z', \frac{1}{r}) \cap D_-(z', r)$.

Observe that

$$\frac{f(\omega,z')}{\omega - z_1} = \frac{P(\omega,z') - P(z_1,z')}{\omega - z_1} \cdot \frac{f(\omega,z')}{P(\omega,z') - P(z_1,z')} = \sum_{j=0}^{\infty} \Phi_j(\omega,z)P(z)^j$$

for all $\omega \in \mathbb{C}$ with $|P(\omega,z')| = \frac{1}{r}$ and all $z_1 \in D_+(z',\frac{1}{r})$, the series being uniformly convergent with respect to $\omega \in \partial D_+(z',\frac{1}{r})$.

Similarly,

$$\frac{f(\omega,z')}{\omega - z_1} = -\sum_{j=1}^{\infty} \Phi_j(\omega,z)P(z)^{-j}$$

for all $\omega \in \partial D_-(z',r)$ and all $z_1 \in D_-(z',r)$, the series being uniformly convergent with respect to $\omega \in \partial D_-(z',r)$.

By (4.3) it follows that

$$f(z) = \sum_{j=-\infty}^{\infty} c_{1j}(z)P(z)^j, \quad z_1 \in D(z',0), \quad z' \in F_2 \times \cdots \times F_n. \tag{4.4}$$

Moreover, for every $\rho > 0$, for every $z' \in F_2 \times \cdots \times F_n$, and for every compact subset K of $\mathbb{C}$ there exists $M = M(\rho,z',K) > 0$ such that

$$|c_{1j}(z)| \le M\rho^{-j}, \quad j \in \mathbb{Z}, \quad z_1 \in K, \quad z' \in F_2 \times \cdots \times F_n.$$

Hence for all $r > 0$, $\quad z_1 \in \mathbb{C}$, $\quad z' \in F_2 \times \cdots \times F_n$ one gets the inequalities

$$|c_{1j}(z_1,z')| \le M(\frac{1}{r},z',\{z_1\})r^j, \quad j \ge 0,$$

$$|c_{1j}(z_1,z')| \le M(r,z',\{z_1\})r^{|j|}, \quad j \le 1.$$

By the arbitrary character of $r > 0$ it follows that

$$\limsup_{|j|\to\infty} \frac{1}{|j|} \log |c_{1j}(z)| = -\infty, \quad z_1 \in C, \quad z' \in F_2 \times \cdots \times F_n.$$

By (4.2) the sequence $\{\frac{1}{|j|} \log |c_{1j}|\}$ is locally uniformly upper bounded on Ω_1. Put $u(z) := \limsup \frac{1}{|j|} \log |c_{1j}(z)|$, $z \in \Omega_1$. Then the upper semicontinuous regularization u^* of u is plurisubharmonic in Ω_1, and by Bedford-Taylor theorem [3] on negligible sets the set $\{z \in F_1 \times \cdots \times F_n; -\infty = u(z) = u^*(z)\}$ is non-pluripolar. Therefore $u^* \equiv -\infty$ in Ω_1.

Given a compact subset K of $\Omega_1 \setminus A$, there exists $r = r(K)$ with $0 < r < 1$ such that $r < |P(z)| < \frac{1}{r}$ for all $z \in K$. Fix $k > 0$ so large that $\frac{1}{r}e^{-k} < \frac{1}{2}$. By Hartogs Lemma there exists $j_o = j_o(k,K)$ such that

$$\frac{1}{|j|} \log |c_{1j}(z)P(z)^j| \le -k + \log \frac{1}{r}, \quad z \in K, \quad |j| > j_o,$$

i.e.

$$|c_{1j}(z)|P(z)^j| \le 2^{-|j|}, \quad z \in K, \quad |j| > j_o.$$

It follows that the series $\sum_{j=-\infty}^{\infty} c_{1j}(z)P(z)^j$ is uniformly convergent on every compact subset of $\Omega_1 \setminus A$. Its sum f_1 is holomorphic on $\Omega_1 \setminus A$. By (4.4) $f_1 = f$ on $F_1 \times \cdots \times F_n$. The proof of Proposition 3.1 is completed.

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