# Connection matrix for Morse decompositions

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**Abstract**

A new proof of the existence of a connection matrix for Morse decompositions is presented. The matrix is constructed by extending the matrix onto larger decompositions, by using the connecting homomorphism for the attractor repeller decomposition. As a result an entire matrix is obtained, together with a family of connecting isomorphisms.

**Introduction.** For some isolated invariant sets there exists a decomposition into smaller invariant sets, with a defined partial order between them called the Morse decomposition. For any Morse decomposition there exists an index filtration, containing index pairs of all the Morse sets from the decomposition ([4]). Using the index filtration we are able to define a family of Conley indexes of the Morse sets. A connection matrix for a Morse decomposition is a coboundary operator defined on the product of the Conley indexes of the Morse sets. From the connection matrix we are able to reconstruct information concerning the indexes of the isolated invariant subsets of our isolated invariant set and determine the existence of connecting orbits between them. The ideas listed above were introduced by Franzosa who in

[3], [4], [5], [6], presented the proof of the existence of index filtrations and a connection matrix for Morse decompositions. The connection matrix theory was also extended by Richeson [7] from the theory for flows to the theory for discrete dynamical systems.

In this paper we present a new proof of the existence of a connection matrix for flows. The proof presented by Franzosa showed the existence, but did not explicitly write what this matrix should look like. In this proof we show a clear inductive method of constructing such a matrix.

# 1 Preliminaries

Throughout this paper we assume that the reader is familiar with the basic concepts of the Conley index theory as in [2]; in particular the concept of isolated invariant sets, isolating neighbourhoods, index pairs, attractor repeller pairs and the Conley index. For the definition of the Conley index we shall be using the Alexander-Spanier cohomology [1]. We will carry over the notations used in [1]. Let us recall some of them.

For a topological space $X$ and a field $\Xi$ we write

$$C^n(X) := \{\varphi \mid \varphi : X^{n+1} \to \Xi\}.$$

We define a coboundary operator $\partial^n : C^n(X) \to C^{n+1}(X)$

$$\partial^n \varphi(x_0, \ldots, x_{n+1}) := \sum_{r=0}^{n+1} (-1)^r \varphi(x_0, \ldots, \hat{x}_r, \ldots, x_{n+1}),$$

where the hat means that the corresponding $x$ is omitted. For a covering $\mathcal{U}$ of $X$ we define $\mathcal{U}^{n+1} := \bigcup_{U \in \mathcal{U}} U^{n+1}$. We say that a function $\varphi$ is locally equal zero if there exists a covering $\mathcal{U}$ of $X$, such that $\varphi$ is zero on $\mathcal{U}^{n+1}$. We define

$$C_0^n(X) := \{\varphi \mid \varphi \in C^n(X) \text{ and } \varphi \text{ is locally zero}\},$$

$$\bar{C}^n(X) := C^n(X)/C_0^n(X).$$

On $\bar{C}^n(X)$ there exists a coboundary operator $\bar{\partial}^n$ induced by $\partial^n$, $\bar{\partial}^n[\varphi] := [\partial^n \varphi]$. For $f : X \to Y$ there exists an induced function $f^\# : C^n(Y) \to C^n(X)$,

$$f^\#(\varphi)(x_0, \ldots, x_n) := \varphi(f(x_0), \ldots, f(x_n)).$$

If $f$ is a continuous function then the induced function $\bar{f}^{\#} : \bar{C}^n(Y) \to \bar{C}^n(X)$,

$$\bar{f}^{\#}[\varphi]_{C_0^n(Y)} := [f^{\#}\varphi]_{C_0^n(X)}$$

is well defined. For a topological pair $(X, A)$ we define the group of cochains on $(X, A)$ as

$$\bar{C}^n(X, A) := \ker \bar{i}^{\#},$$

where $\bar{i}^{\#}$ is induced by an inclusion $i : A \to X$. We can restrict the coboundary $\bar{\partial}^n$ to $\ker \bar{i}^{\#}$, by which we get a coboundary on $\bar{C}^n(X, A)$. We define the group of cycles $Z^n(X, A) := \ker \bar{\partial}^n$ and the group of boundaries $B^n(X, A) := \operatorname{im} \bar{\partial}^{n-1}$. Finally we define the Alexander Spanier cohomology $H^n(X, A) := Z^n(X, A)/B^n(X, A)$. The details of these definitions are discussed in [1]. In our case we shall be dealing with the cohomology over a field $\Xi$, so all the cochain and cohomology groups are vector spaces.

Let $S$ be an isolated invariant set for a locally compact metric space $X$ and a local flow $\Gamma$. For an isolated invariant set $S$ and its isolating neighborhood $N$ there exists an index pair $(P_1, P_0)$ contained in $N$. We define the Conley index of $S$ as $CH^n(S) := H^n(P_1, P_0)$. The definition is independent from the choice of $N$ and $(P_1, P_0)$ ([2]).

Let $P$ be a finite set and $<$ a partial order on $P$.

**Definition 1.1** $M = M(P, <) = \{M_p \subset S : p \in P\}$ is called a Morse decomposition of $S$ if:

1. $\forall p \in P$, $M_p$ is an isolated invariant set,
2. $\forall x \in S \setminus \bigcup M_p \quad \exists p, q \in P : \; q < p$ and $x \in C(M_p, M_q; S)$.

Where $C(M_p, M_q; S)$ is the set of connecting orbits from $M_p$ to $M_q$ in $S$.

If we take $(P, \prec)$ such that when $q < p$ then $q \prec p$, the definition of the Morse decomposition with $(P, \prec)$ is also correct, therefore we can extend $(P, <)$ to a linear order. From now on we will assume that $P = \{1, 2, \ldots, m\}$ and that $<$ is the natural order. By $A(<)$ we shall denote the set of attracting intervals, that is all the intervals $I = [1, k]$, for some $k$ in $P$. We say that the two intervals $I, J$ are adjacent, if for every $i \in I$ and $j \in J$, $i < j$ and $IJ := I \cup J$ is an interval.

For any interval $I$ in $P$ we define

$$M_I := (\bigcup_{p \in I} M_p) \cup (\bigcup_{p,q \in I} C(M_p, M_q; S)).$$

For any two adjacent intervals $I, J$ the pair $(M_I, M_J)$, is an attractor repeller decomposition of $M_{IJ}$. An immediate consequence of this fact is that for all $I \in A(<)$ the pair $(M_I, M_{P\setminus I})$ is an attractor repeller decomposition of $M_P$ . This fact inspires the following definition, which is a natural extension of the idea of the index pair.

**Definition 1.2** A collection of compact sets $\{N(I) \mid I \in A(<)\}$ is called an index filtration for $M$ if

1. for each $I \in A(<)$ $(N(I), N(\emptyset))$ is an index pair for $M(I)$,
2. for each $I_1$, $I_2$ in $A(<)$, $N(I_1 \cap I_2) = N(I_1) \cap N(I_2)$ and $N(I_1 \cup I_2) = N(I_1) \cup N(I_2)$ .

**Theorem 1.3** *For every Morse decomposition there exists an index filtration.*

The proof of this theorem is presented in [4].

For an index filtration $\{N(I) \mid I \in A(<)\}$ of $M$ we will use the following notation $N_k := N([1,k]), \quad N_0 := N(\emptyset)$.

Index filtrations have the following properties

1) $(N_k, N_{i-1})$ is an index pair for $M_{[i,k]}$,

2) $N_0 \subset N_1 \subset \ldots \subset N_n$.

We shall now introduce the definition of a connection matrix for a partially ordered Morse decomposition. First let us introduce some notations. Let

$$C\Delta(I) := \bigoplus_{p\in I} CH^*(M_p).$$

We can treat any linear function $\Delta(I) : C\Delta(I) \to C\Delta(I)$ as a matrix

$$(\Delta_{pq} : CH^*(M_q) \to CH^*(M_p) \mid p, q \in I).$$

If $\Delta(I)$ is a coboundary then by $H\Delta^*(I)$ we shall denote the cohomology group $H^*(C\Delta(I), \Delta(I))$.

We say that a matrix $(a_{ki})_{k,i\in P}$ is lower triangular if for every $k < i$, $a_{ki} = 0$.

**Definition 1.4** A lower triangular coboundary operator $\Delta(P) : C\Delta(P) \to C\Delta(P)$ is called a connection matrix for $M(P,<)$ if there exists a family of isomorphisms $\{F^*(I) : CH^*(M_I) \to H^*\Delta(I) \mid I \text{ is an interval in } P\}$ such that for any two adjacent intervals $I$,$J$ in $P$, the diagram

$$\begin{array}{ccccccccc} \dots & \to & CH^*(M_J) & \to & CH^*(M_{IJ}) & \to & CH^*(M_I) & \to & \dots \\ & & \downarrow F^*(J) & & \downarrow F^*(IJ) & & \downarrow F^*(I) & & \\ \dots & \to & H\Delta^*(J) & \to & H\Delta^*(IJ) & \to & H\Delta^*(I) & \to & \dots \end{array}$$

commutes.

# 2 Main results

Let us assume that $C_1$ and $C_2$ are cochain groups and that there exists a lower triangular coboundary operator $\partial = (\partial_{ki})_{k,i=1,2}$ on $C_1 \oplus C_2$. The sequence

$$0 \to C_2 \to C_1 \oplus C_2 \to C_1 \to 0$$

is exact, and there exists a connecting homomorphism $\delta^*$, such that the long sequence

$$\dots \to H(C_2) \to H(C_1 \oplus C_2) \to H(C_1) \stackrel{\delta^*}{\to} H(C_2) \to \dots$$

is exact. The proof of the existence of $\delta^*$ is presented in [1].

**Lemma 2.1** *For any $\varphi \in C_1 \cap \ker\partial_{11}$*

$$\delta^*[\varphi]_{\mathrm{im}\partial_{11}} = [\partial_{21}\varphi]_{\mathrm{im}\partial_{22}}$$

**Proof.**

$$\begin{array}{ccccccccc} 0 & \to & C_2 & \stackrel{\alpha}{\to} & C_1 \oplus C_2 & \stackrel{\beta}{\to} & C_1 & \to & 0 \\ & & & & \downarrow \left( \begin{array}{cc} \partial_{11} & 0 \\ \partial_{21} & \partial_{22} \end{array} \right) & & & & \\ 0 & \to & C_2 & \stackrel{\alpha}{\to} & C_1 \oplus C_2 & \stackrel{\beta}{\to} & C_1 & \to & 0 \end{array}$$

The operator $\delta^*$ constructed in [1] has the following property. If $\beta(\varphi_1 \oplus \varphi_2) = \psi$ and $\partial(\varphi_1 \oplus \varphi_2) = \alpha\gamma$ then $\delta^*[\psi] = [\gamma]$.

Take $\varphi_1 := \varphi$, $\varphi_2 := 0$, $\psi := \varphi$ and $\gamma := \partial_{21}\varphi$. We know that $\varphi \in \ker\partial_{11}$. Clearly $\beta(\varphi \oplus 0) = \varphi$ and

$$\partial(\varphi \oplus 0) = \left( \begin{array}{cc} \partial_{11} & 0 \\ \partial_{21} & \partial_{22} \end{array} \right) \left( \begin{array}{c} \varphi \\ 0 \end{array} \right) = \left( \begin{array}{c} \partial_{11}\varphi \\ \partial_{21}\varphi \end{array} \right) = 0 \oplus \partial_{21}\varphi = \alpha(\partial_{21}\varphi)$$

so $\delta^*[\varphi] = [\partial_{21}\varphi]$. ■

For an index filtration $\{N_i\}_{i\in P}$ for every $l, j$ in $P$ such that $j < l$,

$$\bar{C}^n(N_l, N_{j-1}) \cong \bigoplus_{k=j}^{l} \bar{C}^n(N_k, N_{k-1}).$$

The coboundary operator $\hat{\partial}$ on $\bigoplus_{k=j}^{l} \bar{C}^n(N_k, N_{k-1})$

$$\hat{\partial}([j,l]) = (\hat{\partial}_{ki} : \bar{C}(N_i, N_{i-1}) \to \bar{C}(N_k, N_{k-1}) \,|\, k, i \in [j,l])$$

induced from $\bar{\partial}$ takes the form of a lower triangular matrix. From now on for any Morse set $M_I$, by writing $CH^n(M_I)$ we will have in mind the cohomology

$$H^n(\bigoplus_{k\in I} \bar{C}(N_k, N_{k-1}), \hat{\partial}(I)).$$

To simplify notation, for any two intervals $L$, $J$, we shall define $\hat{\partial}_{LJ} := (\hat{\partial}_{lj})_{l\in L, j\in J}$ and $\Delta_{LJ} := (\Delta_{lj})_{l\in L, j\in J}$.

**Theorem 2.2** *Let $K, I$ be two adjacent intervals. For $\Phi_I \in \bigoplus_{i\in I} \bar{C}^n(N_i, N_{i-1})$*

$$\mathrm{pr}_{\ker\hat{\partial}_{KK}} \hat{\partial}_{KI} \Phi_I = \hat{\partial}_{KI} \mathrm{pr}_{\ker\hat{\partial}_{II}} \Phi_I.$$

The proof of this theorem as well as the proofs of the facts listed above will be presented in Section 3.

We will now present an inductive method of constructing a connection matrix. Let $(M, P)$ be a Morse decomposition and $\{N_J\}$ its index filtration. Let us assume that we already have a matrix $\Delta(I)$ for a given interval $I = [1, k-1]$, let us also assume that we have a family of cochain morphisms

$$\{F_J : \bigoplus_{i\in J} \bar{C}(N_i, N_{i-1}) \to C\Delta(J) \,|\, J \text{ is an interval in } I\}$$

which in cohomology induces the family of connecting isomorphisms $\{F_J^* : CH(M_J) \to H\Delta(J)\}$. We will show how to construct the matrix $\Delta(Ik)$.

We already have the matrix $\Delta(I) : C\Delta(I) \to C\Delta(I)$. We know that a connection matrix has to be a lower triangular matrix, so what we are looking for is a homomorphism

$$\Delta_{kI} : C\Delta(I) \to CH(M_k).$$

We know that $(M_k, M_I)$ is an attractor repeller decomposition of $M_{Ik}$. The cochain group on $M_k$ is equal $\bar{C}(N_k, N_{k-1})$ and the cochain group on $M_I$ is isomorphic to $\bigoplus_{i\in I} \bar{C}(N_i, N_{i-1})$. There exists a connecting homomorphism $\delta^*$ such that

$$\ldots \to CH^n(M_k) \to CH^n(M_{Ik}) \to CH^n(M_I) \stackrel{\delta^n}{\to} CH^{n+1}(M_k) \to \ldots$$

is exact. The main idea of this paper is to use this homomorphism to construct $\Delta_{kI}$. The idea for our construction is presented in the following diagram

$$\begin{array}{ccc} C\Delta(I) & \stackrel{\Delta_{kI}}{\cdots} & CH(M_k) \\ \downarrow & & \uparrow \delta^* \\ H\Delta(I) & \stackrel{(F_I^*)^{-1}}{\to} & CH(M_I) \end{array}$$

The operator $\Delta_{kI}$ will be defined as

$$\Delta_{kI}(X_I) := \delta^* \circ (F_I^*)^{-1}[\mathrm{pr}_{\ker\Delta(I)} X_I]_{\mathrm{im}\Delta(I)}.$$

Since we shall be dealing with various kinds of classes of abstractions and with various cochain groups, before stating our main theorems let us somewhat clarify our notations. By letters $X_J, Y_J$, we shall denote the elements of $C\Delta(J)$, by $x_j, y_j, z_j$ the elements of $CH(M_j)$, by $\Phi_J, \Psi_J$, the elements of $\bigoplus_{j\in J} \bar{C}(N_j, N_{j-1})$ and by $\varphi_j, \psi_j, \xi_j$ we will denote the elements of $\bar{C}(N_j, N_{j-1})$. The elements of $CH(M_J)$ will be denoted by $[\Phi_J], [\Psi_J]$, and the elements of $H\Delta(J)$ by $[X_J], [Y_J]$.

**Theorem 2.3** *The function*

$$\Delta(Ik) := \begin{pmatrix} \Delta(I) & 0 \\ \Delta_{kI} & 0 \end{pmatrix}$$

*is a coboundary operator and* $H\Delta(Ik) \cong CH(M_{Ik})$.

**Proof.** First we have to show that $\Delta(Ik)$ is a coboundary operator.

$$\Delta(Ik)^2 = \begin{pmatrix} \Delta(I) & 0 \\ \Delta_{kI} & 0 \end{pmatrix}^2 = \begin{pmatrix} \Delta(I)^2 & 0 \\ \Delta_{kI}\Delta(I) & 0 \end{pmatrix}$$

Since $\Delta(I)$ is a coboundary operator, $\Delta(I)^2 = 0$. We need to show that $\Delta_{kI}\Delta(I) = 0$.

$$\Delta_{kI}(\Delta(I)X_I) = \delta^*(F_I^*)^{-1}[\mathrm{pr}_{\ker\Delta(I)}\Delta(I)X_I] = \delta^*(F_I^*)^{-1}[0] = 0$$

This means that $\Delta(Ik)$ is a coboundary. Let us consider the exact sequences

$$0 \to \bar{C}^n(N_k, N_{k-1}) \xrightarrow{\rho} \bigoplus_{j \in Ik} \bar{C}^n(N_j, N_{j-1}) \xrightarrow{\iota} \bigoplus_{i \in I} \bar{C}^n(N_i, N_{i-1}) \to 0$$

$$0 \to C\Delta(k) \xrightarrow{\rho_2} C\Delta(Ik) \xrightarrow{\iota_2} C\Delta(I) \to 0.$$

We already have a connecting homomorphism $\delta^*$. There exists a connecting homomorphism $\delta_2^*$ such that the long sequences

$$\ldots \to CH(M_k) \xrightarrow{\rho^*} CH(M_{Ik}) \xrightarrow{\iota^*} CH(M_I) \xrightarrow{\delta^*} \ldots$$

$$\ldots \to H\Delta(k) \xrightarrow{\rho_2^*} H\Delta(Ik) \xrightarrow{\iota_2^*} H\Delta(I) \xrightarrow{\delta_2^*} \ldots$$

are exact. Let us write the last two parts of both of these sequences in a slightly different form. We shall also add the functions which will appear later on, which might slightly clarify the later method of our proof.

$$\begin{array}{cccccccccc}
\xrightarrow{\rho^*} & \ker\iota^* & \oplus & \frac{CH(M_{Ik})}{\ker\iota^*} & \xrightarrow{\iota^*} & \mathrm{im}\iota^* & \oplus & \frac{CH(M_I)}{\mathrm{im}\iota^*} & \xrightarrow{\delta^*} \\
 & \downarrow G & & & & \downarrow F_I^* & & & \\
\xrightarrow{\rho_2^*} & \ker\iota_2^* & \oplus & \frac{H\Delta(Ik)}{\ker\iota_2^*} & \xrightarrow{\iota_2^*} & \mathrm{im}\iota_2^* & \oplus & \frac{H\Delta(I)}{\mathrm{im}\iota_2^*} & \xrightarrow{\delta_2^*}
\end{array}$$

To show that $H\Delta(Ik) \cong CH(M_{Ik})$, we will first show that $\ker\iota^* \cong \ker\iota_2^*$. Since the sequences are exact

$$\ker\iota^* = \mathrm{im}\rho^* = \{[0 \oplus \varphi_k] \,|\, 0 \in \bigoplus_{i \in I} \bar{C}(N_i, N_{i-1}) \text{ and } \varphi_k \in \ker\hat{\partial}_{kk}\}$$

$$\ker\iota_2^* = \mathrm{im}\rho_2^* = \{[0 \oplus x_k] \,|\, 0 \in C\Delta(I) \text{ and } \exists \varphi_k \in \ker\hat{\partial}_{kk} \,:\, x_k = [\varphi_k]\}$$

Let us define a function $G : \ker\iota^* \to \ker\iota_2^*$

$$G[0 \oplus \varphi_k]_{\mathrm{im}\hat{\partial}(Ik)} := [0 \oplus [\varphi_k]]_{\mathrm{im}\Delta(Ik)}$$

We will show that $G$ is an isomorphism. First let us show that $G$ is well defined. Let us take any $[0 \oplus \varphi_k] = 0$. We have to show that $G[0 \oplus \varphi_k] = 0$.

If $[0 \oplus \varphi_k] = 0$ then there exist $\Psi_I, \psi_k$ such that $0 \oplus \varphi_k = \hat{\partial}(Ik)(\Psi_I \oplus \psi_k)$.

$$0 \oplus \varphi_k = \hat{\partial}(Ik)(\Psi_I \oplus \psi_k) = \hat{\partial}_{II}\Psi_I \oplus (\hat{\partial}_{kI}\Psi_I + \hat{\partial}_{kk}\psi_k)$$

We can see that $\hat{\partial}(I)\Psi_I = 0$. From Lemma 2.1, we know that $\delta^*[\Psi_I]_{\mathrm{im}\hat{\partial}(I)} = [\hat{\partial}_{kI}\Psi_I]_{\mathrm{im}\hat{\partial}(k)}$.

$$\begin{aligned}
[\varphi_k]_{\mathrm{im}\hat{\partial}(k)} &= [\hat{\partial}_{kI}\Psi_I + \hat{\partial}_{kk}\psi_k]_{\mathrm{im}\hat{\partial}(k)} \\
&= [\hat{\partial}_{kI}\Psi_I]_{\mathrm{im}\hat{\partial}(k)} \\
&= \delta^*[\Psi_I]_{\mathrm{im}\hat{\partial}(I)} \\
&= \delta^*(F_I^*)^{-1}[F_I\Psi_I]_{\mathrm{im}\Delta(I)}
\end{aligned}$$

Since $\hat{\partial}(I)\Psi_I = 0$ we know that $\Delta_I F_I \Psi_I = 0$, so $[\varphi_k] = \Delta_{kI}(F_I\Psi_I)$ and

$$0 \oplus [\varphi_k] = \Delta(Ik)(F_I\Psi_I \oplus 0).$$

This means that $G[0 \oplus \varphi_k]_{\mathrm{im}\hat{\partial}(Ik)} = [0 \oplus [\varphi_k]]_{\mathrm{im}\Delta(Ik)} = 0$. $G$ is well defined.

It is obvious that $G$ is an epimorphism. To show that it is a monomorphism let us take $[0 \oplus \varphi_k]$ such that $G[0 \oplus \varphi_k] = 0$. $[0 \oplus [\varphi_k]] = 0$ which means that there exist $X_I, x_k$ such that $0 \oplus [\varphi_k] = \Delta(Ik)(X_I \oplus x_k)$.

$$0 \oplus [\varphi_k] = \Delta(Ik)(X_I \oplus x_k) = \Delta(I)X_I \oplus \Delta_{kI}X_I$$

We can see that $\Delta(I)X_I = 0$.

$$\Delta_{kI}X_I = \delta^*(F_I^*)^{-1}[\mathrm{pr}_{\ker\Delta(I)}X_I] = \delta^*(F_I^*)^{-1}[X_I]$$

There exists a $\Phi_I \in \bigoplus_{i\in I}\bar{C}(N_i, N_{i-1}) \cap \ker\hat{\partial}(I)$ such that $[\Phi_I] = (F_I^*)^{-1}[X_I]$.

$$[\varphi_k] = \Delta_{kI}X_I = \delta^*(F_I^*)^{-1}[X_I] = \delta^*[\Phi_I] = [\hat{\partial}_{kI}\Phi_I]$$

This means that there exists a $\psi_k \in \bar{C}(N_k, N_{k-1})$ such that

$$\varphi_k = \hat{\partial}_{kI}\Phi_I + \hat{\partial}_{kk}\psi_k.$$

Since $\Phi_I \in \ker\hat{\partial}(I)$, it is clear that $0 \oplus \varphi_k = \hat{\partial}(Ik)(\Phi_I \oplus \psi_k)$, which means that $[0 \oplus \varphi_k] = 0$. $G$ is a monomorphism.

To show that $H\Delta(Ik) \cong CH(M_{Ik})$ we now only need to see wether

$$\frac{CH(M_{Ik})}{\ker\iota^*} \cong \frac{H\Delta(Ik)}{\ker\iota_2^*}.$$

First let us show that $\mathrm{im}\iota^* \cong \mathrm{im}\iota_2^*$. We will show that $F^*_{I|\mathrm{im}\iota^*} : \mathrm{im}\iota^* \to \mathrm{im}\iota_2^*$ is an isomorphism. From the exactness of the sequences

$$\ldots \to CH(M_k) \xrightarrow{\rho^*} CH^n(M_{Ik}) \xrightarrow{\iota^*} CH^n(M_I) \xrightarrow{\delta^*} \ldots$$

$$\ldots \to H\Delta(k) \xrightarrow{\rho_2^*} H\Delta(Ik) \xrightarrow{\iota_2^*} H\Delta(I) \xrightarrow{\delta_2^*} \ldots$$

we know that $\mathrm{im}\iota^* = \mathrm{ker}\delta^*$ and $\mathrm{im}\iota_2^* = \mathrm{ker}\delta_2^*$. We know that $F_I^*$ is an isomorphism, so what we shall need is the fact that

$$F_I^*(\mathrm{ker}\delta^*) = \mathrm{ker}\delta_2^*.$$

From Lemma 2.1 we know that $\delta_2^*[X_I] = [\Delta_{kI}X_I]$. We also know that $\Delta_{kk} = 0$. From these two facts we get

$$\delta_2^* F_I^*[\Phi_I]_{\mathrm{im}\Delta(I)} = [\Delta_{kI}F_I\Phi_I]_{\mathrm{im}\Delta_{kk}} = \Delta_{kI}F_I\Phi_I = \delta^*(F_I^*)^{-1}[F_I\Phi_I] = \delta^*[\Phi_I]$$

which means that $F_I^*[\Phi_I] \in \mathrm{ker}\delta_2^*$ if and only if $[\Phi_I] \in \mathrm{ker}\delta^*$. We have shown that $\mathrm{im}\iota^* \overset{F_I^*}{\cong} \mathrm{im}\iota_2^*$. We know that

$$\iota^*_{|CH(M_{Ik})/\mathrm{ker}\iota^*} : \frac{CH(M_{Ik})}{\mathrm{ker}\iota^*} \to \mathrm{im}\iota^*$$

$$\nu := \iota^*_{2|H\Delta(Ik)/\mathrm{ker}\iota_2^*} : \frac{H\Delta(Ik)}{\mathrm{ker}\iota_2^*} \to \mathrm{im}\iota_2^*$$

are isomorphisms so

$$\nu^{-1} \circ F_I^* \circ \iota^* : \frac{CH(M_{Ik})}{\mathrm{ker}\iota^*} \to \frac{H\Delta(Ik)}{\mathrm{ker}\iota_2^*}$$

is also an isomorphism, which concludes the proof of our theorem. For any $[\Phi_{Ik}]$ in $CH(M_{Ik})$ there exists a $[\alpha]$ in $\mathrm{ker}\iota^*$ and a $[\beta]$ in $\frac{CH(M_{Ik})}{\mathrm{ker}\iota^*}$ such that $[\Phi_{Ik}] = [\alpha] + [\beta]$. We have constructed an isomorphism

$$F : CH(M_{Ik}) \to H\Delta(Ik)$$

$$F[\Phi_{Ik}] = G[\alpha] + \nu^{-1} \circ F_I^* \circ \iota^*[\beta] \qquad \blacksquare$$

**Theorem 2.4** *The functions $F_J^* : CH(M_J) \to H\Delta(J)$ from our construction are generated by cochain morphisms $F(J) : \bigoplus_{i\in J} \bar{C}(N_i, N_{i-1}) \to C\Delta(J)$.*

$$F(J)(\bigoplus_{i\in J}\varphi_i) := \bigoplus_{i\in J}[\mathrm{pr}_{\mathrm{ker}\hat{\partial}_{ii}}\varphi_i]_{\mathrm{im}\hat{\partial}_{ii}}$$

**Proof.** Let us first show that the functions $\{F(J)\}$, are indeed cochain morphisms. We shall prove this fact by induction. For an interval $I=[1]$ we have $\Delta([1])=0$ so for $\varphi_1\in\bar{C}(N_1,N_0)$

$$\Delta([1])F([1])\varphi_1=0=[\hat{\partial}_{11}\varphi_1]_{\mathrm{im}\hat{\partial}_{11}}=[\mathrm{pr}_{\mathrm{ker}\hat{\partial}_{11}}\hat{\partial}_{11}\varphi_1]_{\mathrm{im}\hat{\partial}_{11}}=F([1])\hat{\partial}_{11}\varphi_1.$$

Let us assume that for $I=[1,k-1]$, $F(I)$ is a cochain morphism, we will show that $F(Ik)$ is also a cochain morphism. What we have to show is that $\Delta(Ik)F(Ik)=F(Ik)\hat{\partial}(Ik)$.

$$\Delta(Ik)F(Ik)=\begin{pmatrix}\Delta(I) & 0\\ \Delta_{kI} & 0\end{pmatrix}\begin{pmatrix}F(I) & 0\\ 0 & F(k)\end{pmatrix}=\begin{pmatrix}\Delta(I)F(I) & 0\\ \Delta_{kI}F(I) & 0\end{pmatrix}$$

$$F(Ik)\hat{\partial}(Ik)=\begin{pmatrix}F(I) & 0\\ 0 & F(k)\end{pmatrix}\begin{pmatrix}\hat{\partial}(I) & 0\\ \hat{\partial}_{kI} & \hat{\partial}_{kk}\end{pmatrix}=\begin{pmatrix}F(I)\hat{\partial}(I) & 0\\ F(k)\hat{\partial}_{kI} & F(k)\hat{\partial}_{kk}\end{pmatrix}$$

We know that $\Delta(I)F(I)=F(I)\hat{\partial}(I)$. For any $\varphi_k\in\bar{C}(N_k,N_{k-1})$

$$F(k)\hat{\partial}_{kk}\varphi_k=[\mathrm{pr}_{\mathrm{ker}\hat{\partial}_{kk}}\hat{\partial}_{kk}\varphi_k]_{\mathrm{im}\hat{\partial}_{kk}}=[\hat{\partial}_{kk}\varphi_k]_{\mathrm{im}\hat{\partial}_{kk}}=0$$

We need to show that $\Delta_{kI}F(I)=F(k)\hat{\partial}_{kI}$. From Theorem 2.2 we know that

$$\mathrm{pr}_{\mathrm{ker}\hat{\partial}_{kk}}\hat{\partial}_{kI}\Phi_I=\hat{\partial}_{kI}\mathrm{pr}_{\mathrm{ker}\hat{\partial}_{II}}\Phi_I.$$

$$\begin{aligned}F(k)\hat{\partial}_{kI}\Phi_I&=[\mathrm{pr}_{\mathrm{ker}\hat{\partial}_{kk}}\hat{\partial}_{kI}\Phi_I]=[\hat{\partial}_{kI}\mathrm{pr}_{\mathrm{ker}\hat{\partial}_{II}}\Phi_I]=\delta^*[\mathrm{pr}_{\mathrm{ker}\hat{\partial}_{II}}\Phi_I]=\\&=\delta^*(F_I^*)^{-1}[F(I)\mathrm{pr}_{\mathrm{ker}\hat{\partial}_{II}}\Phi_I]=\delta^*(F_I^*)^{-1}[\mathrm{pr}_{\mathrm{ker}\Delta(I)}F(I)\Phi_I]=\Delta_{kI}F(I)\Phi_I\end{aligned}$$

We have shown that $F(Ik)$ is a cochain morphism. To complete the proof we still need to show that the function $F$ constructed in the proof of the previous theorem is equal $F(Ik)^*$.

Let us take any $\bigoplus_{i\in Ik}\varphi_i\in\bigoplus_{i\in IK}\bar{C}(N_i,N_{i-1})\cap\mathrm{ker}\hat{\partial}(Ik)$. It is clear that $[0\oplus\mathrm{pr}_{\mathrm{ker}\hat{\partial}_{kk}}\varphi_k]\in\mathrm{ker}\iota^*$ and $[\bigoplus_{i\in I}\varphi_i\oplus(\varphi_k-\mathrm{pr}_{\mathrm{ker}\hat{\partial}_{kk}}\varphi_k)]\in CH(M_{Ik})/\mathrm{ker}\iota^*$.

$$\begin{aligned}F[\bigoplus_{i\in Ik}\varphi_i]&=G[0\oplus\mathrm{pr}_{\mathrm{ker}\hat{\partial}_{kk}}\varphi_k]+\nu^{-1}\circ F_I^*\circ\iota^*[\bigoplus_{i\in I}\varphi_i\oplus(\varphi_k-\mathrm{pr}_{\mathrm{ker}\hat{\partial}_{kk}}\varphi_k)]\\&=[0\oplus[\mathrm{pr}_{\mathrm{ker}\hat{\partial}_{kk}}\varphi_k]]+\nu^{-1}\circ F_I^*[\bigoplus_{i\in I}\varphi_i]\\&=[0\oplus[\mathrm{pr}_{\mathrm{ker}\hat{\partial}_{kk}}\varphi_k]]+\nu^{-1}[\bigoplus_{i\in I}[\mathrm{pr}_{\mathrm{ker}\hat{\partial}_{ii}}\varphi_i]]\\&=[0\oplus[\mathrm{pr}_{\mathrm{ker}\hat{\partial}_{kk}}\varphi_k]]+[\bigoplus_{i\in I}[\mathrm{pr}_{\mathrm{ker}\hat{\partial}_{ii}}\varphi_i]\oplus 0]\\&=[F(Ik)\bigoplus_{i\in Ik}\varphi_i]\end{aligned}$$

■

So far we have constructed our matrix $\Delta(Ik)$ and have shown that the connecting isomorphism for this matrix are generated by a family of cochain morphisms $\{F(J)\}$ defined in the previous theorem. The functions $F(J)$ are simply diagonal matrices so it is quite obvious that for any two adjacent intervals $K, L$ the diagram

$$\begin{array}{ccccccccc} 0 & \to & \bigoplus_{i\in L}\bar{C}(N_i,N_{i-1}) & \to & \bigoplus_{i\in KL}\bar{C}(N_i,N_{i-1}) & \to & \bigoplus_{i\in K}\bar{C}(N_i,N_{i-1}) & \to & 0 \\ & & \downarrow F_L & & \downarrow F_{KL} & & \downarrow F_K & & \\ 0 & \to & C\Delta(L) & \to & C\Delta(KL) & \to & C\Delta(K) & \to & 0 \end{array}$$

commutes. Passing to cohomology we obtain the following commuting diagram

$$\begin{array}{ccccccccc} \dots & \to & CH(M_L) & \to & CH(M_{KL}) & \to & CH(M_K) & \to & \dots \\ & & \downarrow F_L^* & & \downarrow F_{KL}^* & & \downarrow F_K^* & & \\ \dots & \to & H\Delta(L) & \to & H\Delta(KL) & \to & H\Delta(K) & \to & \dots \end{array}$$

Which means that we have proofed the following theorem.

**Theorem 2.5** *For a Morse decomposition $M(P,<)$, the inductively constructed matrix $\Delta(P)$ is a connection matrix. The family of connecting isomorphisms $\{F_J^*\}$ is generated by $\{F(J)\,|\,J$ is an interval in $P\}$*

$$F(J)(\bigoplus_{i\in J}\varphi_i) := \bigoplus_{i\in J}[\mathrm{pr}_{\mathrm{ker}\hat{\partial}_{ii}}\varphi_i]_{\mathrm{im}\hat{\partial}_{ii}}.$$

# 3 Properties of cochains on index filtrations

In this section we will prove some technical facts concerning the properties of cochains on index triples and on index filtrations. The main task of this section is to present the proof of Theorem 2.2.

We have defined $\bar{C}^n(X,A)$ as a kernel of $\bar{i}^\# : \bar{C}^n(X) \to \bar{C}^n(A)$, where $\bar{i}^\#$ was induced by an inclusion $i : A \to X$.

**Lemma 3.1** *For every $\bar{\varphi} \in \bar{C}^n(X,A)$ there exists a $\psi \in \mathrm{ker} i^\#$ such that $[\psi]_{C_0^n(X)} = \bar{\varphi}$.*

**Proof.** Let us remind that for $\xi \in C^n(X)$ and $(x_0, \ldots, x_n) \in A^{n+1}$

$$i^{\#}\xi(x_0, \ldots, x_n) = \xi(i(x_0), \ldots, i(x_n)) = \xi(x_0, \ldots, x_n).$$

We can see that $\ker i^{\#} = \{\xi \in C^n(X) : \xi_{|A^{n+1}} = 0\}$. Let us take any $\varphi \in C^n(X)$ such that $\bar{\varphi} = [\varphi]_{C_0^n(X)}$. We can define our function $\psi \in \ker i^{\#}$ as

$$\psi(x) := \begin{cases} 0 & x \in A^{n+1} \\ \varphi(x) & \text{otherwise} \end{cases}$$

What we need to show is that $[\varphi]_{C_0^n(X)} = [\psi]_{C_0^n(X)}$. We know that $[\varphi]_{C_0^n(X)} \in \ker \bar{i}^{\#}$ so there exists a covering $\mathcal{U}$ of $A$ such that $\varphi_{|\mathcal{U}^{n+1}} = 0$. The difference

$$(\varphi - \psi)_{|(X^{n+1} \setminus A^{n+1}) \cup \mathcal{U}^{n+1}} = 0,$$

which means that $[\varphi - \psi]_{C_0^n(X)} = 0$. ∎

**Lemma 3.2** *If $\varphi \in \ker i^{\#}$ then $\partial\varphi \in \ker i^{\#}$.*

**Proof.** $\ker i^{\#} = \{\varphi \in C^n(X) : \varphi_{|A^{n+1}} = 0\}$ so we have to check wether $(\partial\varphi)_{|A^{n+2}} = 0$. Let $(x_0, \ldots, x_{n+1}) \in A^{n+2}$

$$\partial\varphi(x_0, \ldots, x_{n+1}) = \sum_{k=0}^{n+1} (-1)^k \varphi(x_0, \ldots, \hat{x}_k, \ldots, x_{n+1})$$

For any $k$ $(x_0, \ldots, \hat{x}_k, \ldots, x_{n+1})$ is in $A^{n+1}$, so $\varphi(x_0, \ldots, \hat{x}_k, \ldots, x_{n+1}) = 0$ which means that $\partial\varphi(x_0, \ldots, x_{n+1}) = 0$. ∎

From now on, for any topological pair $(X, A)$, if we have $[\varphi] \in \bar{C}^n(X, A)$ we will assume that $\varphi \in \ker i^{\#}$.

Let $S$ be an isolated invariant set for a flow on a locally compact metric space. Let $(A, R)$ be an attractor repeller decomposition of $S$. A sequence of compact sets $(N_2, N_1, N_0)$ is called an index triple for $(A, R)$ if $(N_2, N_0)$, $(N_2, N_1)$ and $(N_1, N_0)$ are index pairs for $S, R, A$ respectively. The attractor repeller decomposition is the simplest example of a Morse decomposition. The index triple is the index filtration for this decomposition. By

$$i_{01}^{\#} : C^n(N_1) \to C^n(N_0),$$

$$i_{02}^{\#} : C^n(N_2) \to C^n(N_0),$$

$$i_{12}^{\#} : C^n(N_2) \to C^n(N_1),$$

we shall denote the natural inclusions.

**Lemma 3.3**

$$\ker i^{\#}_{01} \oplus \ker i^{\#}_{12} \cong \ker i^{\#}_{02}.$$

**Proof**. Let us define a function

$$\Psi : \ker i^{\#}_{01} \oplus \ker i^{\#}_{12} \to \ker i^{\#}_{02},$$

$$\Psi(\varphi_1 \oplus \varphi_2)(x) := \begin{cases} \varphi_1(x) & x \in N_1^{n+1} \\ \varphi_2(x) & \text{otherwise.} \end{cases}$$

We will show that $\Psi$ is an isomorphism. Since $\varphi_2 \in \ker i^{\#}_{12}$ we know that $(\varphi_2)_{N_1^{n+1}} = 0$. If $\psi(\varphi_1 \oplus \varphi_2) = 0$ then it is clear that $\varphi_1 = 0$ and $\varphi_2 = 0$. To show that $\Psi$ is an epimorphism let us take any $\varphi \in \ker i^{\#}_{02}$ and define

$$\varphi_1 := \varphi_{|N_1^{n+1}} \qquad \varphi_2(x) := \begin{cases} 0 & x \in N_1^{n+1} \\ \varphi(x) & \text{otherwise.} \end{cases}$$

Since $\varphi_{|N_0^{n+1}} = 0$ we can see that $(\varphi_1)_{|N_0^{n+1}} = 0$ so $\varphi_1 \in \ker i^{\#}_{01}$ and from the definition $\varphi_2 \in \ker i^{\#}_{12}$. It is clear that $\Psi(\varphi_1, \varphi_2) = \varphi$. ■

It is easy to show that the inverse function $\Psi^{-1}$ is

$$(\Psi^{-1})(\varphi) := i^{\#}_{12}(\varphi) \oplus \mathrm{pr}_{\ker i^{\#}_{12}}(\varphi) = \varphi_1 \oplus \varphi_2,$$

where $\varphi_1$ and $\varphi_2$ are the functions defined at the end of the proof.

**Lemma 3.4**

$$\bar{C}^n(N_1, N_0) \oplus \bar{C}^n(N_2, N_1) \cong \bar{C}^n(N_2, N_0).$$

**Proof**. We will show that the function

$$\Psi^* : \bar{C}^n(N_1, N_0) \oplus \bar{C}^n(N_2, N_1) \to \bar{C}^n(N_2, N_0),$$

$$\Psi^*([\varphi_1]_{C_0^n(N_1)} \oplus [\varphi_2]_{C_0^n(N_2)}) := [\Psi(\varphi_1 \oplus \varphi_2)]_{C_0^n(N_2)}$$

is an isomorphism. From Lemma 3.1 and the fact that $\Psi$ is an epimorphism we know that $\Psi^*$ is also an epimorphism. If $[\Psi(\varphi_1 \oplus \varphi_2)] = 0$ then there exists a covering $\mathcal{U}$ of $N_2$ such that $\Psi(\varphi_1 \oplus \varphi_2)_{|\mathcal{U}^{n+1}} = 0$. Since $\varphi_2 \in \ker i^{\#}_{12}$ we know that $(\varphi_2)_{|N_1^{n+1}} = 0$ , so

$$(\varphi_1)_{|\mathcal{U}^{n+1} \cap N_1} = 0 \quad \text{and} \quad (\varphi_2)_{|\mathcal{U}^{n+1}} = 0.$$

This means that $[\varphi_1]=0$ and $[\varphi_2]=0$. ■

We will now define a coboundary $\tilde{\partial}$ on $\mathrm{ker} i^{\#}_{01} \oplus \mathrm{ker} i^{\#}_{12}$ induced from the operator $\partial$,

$$\tilde{\partial} := \Psi^{-1} \circ \partial \circ \Psi.$$

We can look at $\tilde{\partial}$ as a two by two matrix

$$\tilde{\partial} = (\tilde{\partial}_{lk} : \mathrm{ker} i^{\#}_{k-1\,k} \to \mathrm{ker} i^{\#}_{l-1\,l} \,|\, k,l \in \{1,2\}).$$

This coboundary induces a coboundary $\hat{\partial}$ on $\bar{C}^n(N_1,N_0) \oplus \bar{C}^n(N_2,N_1)$

$$\hat{\partial}([\varphi_1] \oplus [\varphi_2]) := [\tilde{\partial}_{11}\varphi_1 + \tilde{\partial}_{12}\varphi_2] \oplus [\tilde{\partial}_{21}\varphi_1 + \tilde{\partial}_{22}\varphi_2],$$

where $\varphi_1 \in \mathrm{ker} i^{\#}_{01}$ and $\varphi_2 \in \mathrm{ker} i^{\#}_{12}$. We have two isomorphic cochain complexes, $(\bar{C}^n(N_2,N_0),\bar{\partial}^n)$ and $(\bar{C}^n(N_1,N_0) \oplus \bar{C}^n(N_2,N_1),\hat{\partial}^n)$. The coboundary $\hat{\partial}^n$ is a two by two matrix

$$\hat{\partial}^n = (\hat{\partial}^n_{lk} : \bar{C}^n(N_k,N_{k-1}) \to \bar{C}^{n+1}(N_l,N_{l-1}) \,|\, l,k \in \{1,2\}).$$

**Lemma 3.5** *$(\hat{\partial}_{kl})_{l,k\in\{1,2\}}$ is a lower triangular matrix.*

**Proof**. We have to show that $\hat{\partial}_{12}=0$. We will first show that $\tilde{\partial}_{12}=0$. From Lemma 3.2 we know that if $\varphi_2 \in \mathrm{ker} i^{\#}_{12}$ then $\partial\varphi_2 \in \mathrm{ker} i^{\#}_{12}$ so

$$\begin{aligned} \tilde{\partial}(0 \oplus \varphi_2) &= \Psi^{-1} \circ \partial \circ \Psi(0 \oplus \varphi_2) \\ &= \Psi^{-1} \circ \partial\varphi_2 \\ &= i^{\#}_{12}(\partial\varphi_2) \oplus \mathrm{pr}_{\mathrm{ker} i^{\#}_{12}}(\partial\varphi_2) \\ &= 0 \oplus \partial\varphi_2 \end{aligned}$$

which means that $\tilde{\partial}_{12}=0$ (and also $\tilde{\partial}_{22}\varphi_2 = \partial\varphi_2$). We can now write the following equation

$$\begin{aligned} \hat{\partial}([0] \oplus [\varphi_2]) &= [\tilde{\partial}_{11}0 + \tilde{\partial}_{12}\varphi_2] \oplus [\tilde{\partial}_{21}0 + \tilde{\partial}_{22}\varphi_2] \\ &= [\tilde{\partial}_{12}\varphi_2] \oplus [\tilde{\partial}_{22}\varphi_2] \\ &= [0] \oplus [\tilde{\partial}_{22}\varphi_2] \end{aligned}$$

which gives us $\hat{\partial}_{12}[\varphi_2]=0$. ■

**Lemma 3.6** *Let $(x_1,\ldots,x_{n+2}) \in N_1^{n+2}$ and $\varphi_1 \in \mathrm{ker} i^{\#}_{01}$. If there exists $x_0 \in N_2 \setminus N_1$ such that $\tilde{\partial}_{22}\tilde{\partial}_{21}\varphi_1(x_0,\ldots,x_{n+2})=0$ then $\tilde{\partial}_{11}\varphi_1(x_1,\ldots,x_{n+2})=0$.*

**Proof.** At the beginning of our last proof we have shown that

$$(1) \quad \text{if } \varphi_2 \in \ker i_{12}^{\#} \text{ then } \tilde{\partial}_{22}\varphi_2 = \partial\varphi_2.$$

We also know that $\tilde{\partial}_{21}\varphi_1 \in \ker i_{12}^{\#}$ which means that

$$(2) \quad \text{if } x \in N_1^{n+2} \text{ then } \tilde{\partial}_{21}\varphi_1(x) = 0.$$

$\ker i_{12}^{\#} = \{\xi \in C^n(N_2) \,:\, \xi_{|N_1^{n+1}} = 0\}$ so for any $\varphi \in C^n(N_2)$

$$(3) \quad \text{if } x \notin N_1^{n+1} \text{ then } (\mathrm{pr}_{\ker i_{12}^{\#}}\varphi)(x) = \varphi(x).$$

We know that $x_0 \in N_2 \setminus N_1$. Let us take any $k \neq 0$.

$$\begin{aligned}
&(\partial \circ \Psi)(\varphi_1 \oplus 0)(x_0, \ldots, \hat{x}_k, \ldots, x_{n+2}) \\
&\quad = \sum_{l<k} (-1)^l \Psi(\varphi_1 \oplus 0)(x_0, \ldots, \hat{x}_l, \ldots, \hat{x}_k, \ldots, x_{n+2}) \\
&\qquad + \sum_{l>k} (-1)^{l-1} \Psi(\varphi_1 \oplus 0)(x_0, \ldots, \hat{x}_k, \ldots, \hat{x}_l, \ldots, x_{n+2})
\end{aligned}$$

We know that for $l \neq 0$ $(x_0, \ldots, \hat{x}_l, \ldots, \hat{x}_k, \ldots, x_n) \notin N_1^{n+1}$ so

$$\Psi(\varphi_1 \oplus 0)(x_0, \ldots, \hat{x}_l, \ldots, \hat{x}_k, \ldots, x_{n+2}) = 0$$

This means that for $k \neq 0$ we have

$$\begin{aligned}
& (\partial \circ \Psi)(\varphi_1 \oplus 0)(x_0, \ldots, \hat{x}_k, \ldots, x_{n+2}) \\
(4) \quad &= \Psi(\varphi_1 \oplus 0)(\hat{x}_0, x_1, \ldots, \hat{x}_k, \ldots, x_{n+2}) \\
&= \varphi_1(\hat{x}_0, x_1, \ldots, \hat{x}_k, \ldots, x_{n+2})
\end{aligned}$$

From the points (1), (2), (3) and (4) we get

$$\begin{aligned}
0 &= \tilde{\partial}_{22}\tilde{\partial}_{21}\varphi_1(x_0, \ldots, x_{n+2}) \\
&\overset{(1)}{=} \partial\tilde{\partial}_{21}\varphi_1(x_0, \ldots, x_{n+2}) \\
&= \sum_{k=0}^{n+2} (-1)^k \tilde{\partial}_{21}\varphi_1(x_0, \ldots, \hat{x}_k, \ldots, x_{n+2}) \\
&\overset{(2)}{=} \sum_{k \neq 0} (-1)^k \tilde{\partial}_{21}\varphi_1(x_0, \ldots, \hat{x}_k, \ldots, x_{n+2})
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k\neq 0}(-1)^k(\mathrm{pr}_{\ker i_{12}^{\#}} \circ \partial \circ \Psi)(\varphi_1 \oplus 0)(x_0, \ldots, \hat{x}_k, \ldots, x_{n+2}) \\
&\overset{(3)}{=} \sum_{k\neq 0}(-1)^k(\partial \circ \Psi)(\varphi_1 \oplus 0)(x_0, \ldots, \hat{x}_k, \ldots, x_{n+2}) \\
&\overset{(4)}{=} \sum_{k\neq 0}(-1)^k\varphi_1(\hat{x}_0, x_1, \ldots, \hat{x}_k, \ldots, x_{n+2}) \\
&= -\tilde{\partial}_{11}\varphi_1(\hat{x}_0, x_1, \ldots, x_{n+2}).
\end{aligned}
$$

■

**Lemma 3.7** *For any $\bar{\varphi} \in \bar{C}^n(N_1, N_0)$, if $\hat{\partial}_{22}\hat{\partial}_{21}\bar{\varphi} = 0$ then*

$$\hat{\partial}_{21} \circ \mathrm{pr}_{\ker\hat{\partial}_{11}}\bar{\varphi} = \hat{\partial}_{21}\bar{\varphi}.$$

**Proof.** By Lemma 3.1 there exists a $\varphi \in \ker i_{01}^{\#}$ such that $\bar{\varphi} = [\varphi]$. We know that $\hat{\partial}_{22}\hat{\partial}_{21}[\varphi] = 0$ so there exists a covering $\mathcal{U}$ of $N_2$ such that $\tilde{\partial}_{22}\tilde{\partial}_{21}\varphi = 0$ on $\mathcal{U}^{n+3}$. Let us define a family of sets

$$\mathcal{A} := \{V \cap N_1 : V \in \mathcal{U} \text{ and } V \cap N_2 \neq \emptyset\}.$$

For any $(x_1, \ldots, x_{n+2}) \in \mathcal{A}^{n+2}$ there exists an $x_0 \notin N_1$ such that

$$(x_0, x_1, \ldots, x_{n+2}) \in \mathcal{U}^{n+3}.$$

We know that $\tilde{\partial}_{22}\tilde{\partial}_{21}\varphi(x_0, \ldots, x_{n+2}) = 0$ so from Lemma 3.6 we know that $\tilde{\partial}_{11}\varphi(x_1, \ldots, x_{n+2}) = 0$. We have shown that $\tilde{\partial}_{11}\varphi = 0$ on $\mathcal{A}^{n+2}$ which means that for $x \in \mathcal{A}^{n+1}$

$$(\mathrm{pr}_{\ker\tilde{\partial}_{11}}\varphi)(x) = \varphi(x).$$

Let us now show that for any $\psi \in \ker i_{01}^{\#}$ such that $\psi = 0$ on $\mathcal{A}^{n+1}$

$$\hat{\partial}_{21}[\psi] = 0.$$

We will show that for any $(x_0, \ldots, x_{n+1}) \in \mathcal{U}^{n+2}$

$$\tilde{\partial}_{21}\psi(x_0, \ldots, x_{n+1}) = 0.$$

If $(x_0, \ldots, x_{n+1}) \in N_1^{n+2} \cap \mathcal{U}^{n+2}$ then since $\tilde{\partial}_{21}\psi \in \ker i_{12}^{\#}$ we have

$$\tilde{\partial}_{21}\psi(x_0, \ldots, x_{n+1}) = 0.$$

Let us now take $(x_0, \ldots, x_{n+1}) \notin N_1^{n+2}$.

$$\tilde{\partial}_{21}\psi(x_0, \ldots, x_{n+1}) = \sum_{k=0}^{n+1}(-1)^k \Psi(\psi \oplus 0)(x_0, \ldots, \hat{x}_k, \ldots, x_{n+1}).$$

If for every $k$ $(x_0, \ldots, \hat{x}_k, \ldots, x_{n+1}) \notin N_1^{n+1}$ then

$$\Psi(\psi \oplus 0)(x_0, \ldots, \hat{x}_k, \ldots, x_{n+1}) = 0,$$

which means that $\tilde{\partial}_{21}\psi(x_0, \ldots, x_{n+1}) = 0$. If there exists a $k$ such that $x_k \notin N_1$ and $(x_0, \ldots, \hat{x}_k, \ldots, x_{n+1}) \in N_1^{n+1}$ then since $(x_0, \ldots, x_{n+1}) \in \mathcal{U}^{n+2}$

$$(x_0, \ldots, \hat{x}_k, \ldots, x_{n+1}) \in \mathcal{A}^{n+1}.$$

We have assumed that $\psi = 0$ on $\mathcal{A}^{n+1}$ so

$$\Psi(\psi \oplus 0)(x_0, \ldots, \hat{x}_k, \ldots, x_{n+1}) = \psi(x_0, \ldots, \hat{x}_k, \ldots, x_{n+1}) = 0.$$

For $j \neq k$ $(x_0, \ldots, \hat{x}_j, \ldots, x_{n+1}) \notin N_1^{n+1}$ so $\Psi(\psi \oplus 0)(x_0, \ldots, \hat{x}_j, \ldots, x_{n+1}) = 0$. This means that

$$\tilde{\partial}_{21}\psi(x_0, \ldots, x_{n+1}) = \sum_{i=0}^{n+1}(-1)^i \Psi(\psi \oplus 0)(x_0, \ldots, \hat{x}_i, \ldots, x_{n+1}) = 0.$$

We have shown that for all $x \in \mathcal{U}^{n+2}$ $\tilde{\partial}_{21}\psi(x) = 0$ which gives us $\hat{\partial}_{21}[\psi] = 0$.

From the beginning of the proof we know that

$$(\mathrm{pr}_{\ker\tilde{\partial}_{11}}\varphi)(x) = \varphi(x) \text{ for } x \in \mathcal{A}^{n+1}.$$

This means that

$$\hat{\partial}_{21}[\mathrm{pr}_{\ker\tilde{\partial}_{11}}\varphi - \varphi] = 0$$

$$\hat{\partial}_{21}\mathrm{pr}_{\ker\hat{\partial}_{11}}[\varphi] - \hat{\partial}_{21}[\varphi] = 0 \qquad \blacksquare$$

**Lemma 3.8** *For* $\bar{\varphi} \in \bar{C}^n(N_1, N_0)$ *if* $\bar{\varphi} \in \ker\hat{\partial}_{11}$ *then* $\hat{\partial}_{21}\bar{\varphi} \in \ker\hat{\partial}_{22}$.

**Proof.** We know that $\hat{\partial}$ is a coboundary operator so $\hat{\partial}^2 = 0$.

$$0 = \begin{pmatrix} \hat{\partial}_{11} & 0 \\ \hat{\partial}_{21} & \hat{\partial}_{22} \end{pmatrix}^2 = \begin{pmatrix} \hat{\partial}_{11}^2 & 0 \\ \hat{\partial}_{21}\hat{\partial}_{11} + \hat{\partial}_{22}\hat{\partial}_{21} & \hat{\partial}_{22}^2 \end{pmatrix}$$

$$\hat{\partial}_{22}\hat{\partial}_{21}\bar{\varphi} = -\hat{\partial}_{21}\hat{\partial}_{11}\bar{\varphi} = -\hat{\partial}_{21}0 = 0 \qquad \blacksquare$$

Since $\bar{C}^n(N_1, N_0)$ is a vector space we know that there exists a subspace $(\ker\hat{\partial}_{11})'$ of $\bar{C}^n(N_1, N_0)$ such that $\bar{C}^n(N_1, N_0) = \ker\hat{\partial}_{11} \oplus (\ker\hat{\partial}_{11})'$.

**Lemma 3.9** *If* $\bar{\varphi} \in (ker\hat{\partial}_{11})'$ *then* $\mathrm{pr}_{\mathrm{ker}\hat{\partial}_{22}} \circ \hat{\partial}_{21}\bar{\varphi} = 0$.

**Proof.** Let us take $\bar{\psi} \in (\hat{\partial}_{21})^{-1}(\{\mathrm{pr}_{\mathrm{ker}\hat{\partial}_{22}} \circ \hat{\partial}_{21}\bar{\varphi}\})$. Since $\bar{\varphi} \in (\mathrm{ker}\hat{\partial}_{11})'$ we can chose $\bar{\psi}$ so that

$$\bar{\psi} \in (\mathrm{ker}\hat{\partial}_{11})'.$$

It is clear that $\hat{\partial}_{22}\hat{\partial}_{21}\bar{\psi} = 0$, so from Lemma 3.7 we know that $\hat{\partial}_{21}\circ\mathrm{pr}_{\mathrm{ker}\hat{\partial}_{11}}\bar{\psi} = \hat{\partial}_{21}\bar{\psi}$. Since $\bar{\psi} \in (\mathrm{ker}\hat{\partial}_{11})'$

$$\mathrm{pr}_{\mathrm{ker}\hat{\partial}_{22}} \circ \hat{\partial}_{21}\bar{\varphi} = \hat{\partial}_{21}\bar{\psi} = \hat{\partial}_{21} \circ \mathrm{pr}_{\mathrm{ker}\hat{\partial}_{11}}\bar{\psi} = \hat{\partial}_{21}0 = 0. \qquad \blacksquare$$

The lemmas proved above have been a preparation to the proof of Theorem 2.2. The following lemma is a restriction of the theorem for index filtrations to the case of an index triple.

**Lemma 3.10** *For any* $\bar{\varphi} \in \bar{C}^n(N_1, N_0)$

$$\mathrm{pr}_{\mathrm{ker}\hat{\partial}_{22}}(\hat{\partial}_{21}\bar{\varphi}) = \hat{\partial}_{21}(\mathrm{pr}_{\mathrm{ker}\hat{\partial}_{11}}\bar{\varphi})$$

**Proof.** If $\bar{\varphi} \in \mathrm{ker}\hat{\partial}_{11}$ then from Lemma 3.8

$$\hat{\partial}_{21}(\mathrm{pr}_{\mathrm{ker}\hat{\partial}_{11}}\bar{\varphi}) = \hat{\partial}_{21}\bar{\varphi} = \mathrm{pr}_{\mathrm{ker}\hat{\partial}_{22}}(\hat{\partial}_{21}\bar{\varphi})$$

If $\bar{\varphi} \in (\mathrm{ker}\hat{\partial}_{11})'$ then from Lemma 3.9 we get

$$\mathrm{pr}_{\mathrm{ker}\hat{\partial}_{22}}(\hat{\partial}_{21}\bar{\varphi}) = 0 = \hat{\partial}_{21}(\mathrm{pr}_{\mathrm{ker}\hat{\partial}_{11}}\bar{\varphi}) \qquad \blacksquare$$

We know that for any two adjacent intervals $K, I$ the pair $(M_K, M_I)$ is a atractor repeller decomposition of $M_{KI}$. By using this fact, Lemma 3.4 and induction we can easyly show that for every $l, j$ in $P$ such that $j < l$,

$$\bar{C}^n(N_l, N_{j-1}) \cong \bigoplus_{k=j}^{l} \bar{C}^n(N_k, N_{k-1}).$$

By using Lemma 3.5 and induction we can show that the coboundary $\hat{\partial}$ on $\bigoplus_{k=j}^{l} \bar{C}^n(N_k, N_{k-1})$ is a lower triangular matrix. From Lemma 3.10 we also get the proof of Theorem 2.2.

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