A REMARK ON THE DIFFERENTIABILITY OF GERMS

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IMUJ Preprint 2000/26

Abstract

The notion of continuous differentiability for functions from **R** to to **R** is described in a somewhat non-typical language.

The remark presented in this paper is - if one looks from purely mathematical point of view - very simple. We present it, since this remark concerns something, which is so fundamental as the definition of differentiability. We will restrict ourselves to the simplest possible situation, i.e. to the differentiation of functions from **R** to **R**, although some more general observations are also true.

1 Quasialgebras

First we will adapt some known notions to a slightly new situation. Let us recall, that an algebra $\mathcal{A}$ (over $\mathbf{R}$) is an algebraic structure with three operations: addition, multiplication and scalar multiplication. Since it occurs, that the multiplication is not commutative, then to have an algebra one must assume two distribution laws. Namely

$$(i): \ a \cdot (b + c) = a \cdot b + a \cdot c$$

and

$$(ii): \ (b+c)\cdot a = b\cdot a + c\cdot a.$$

((i) will be said the left and (ii) the right distribution law).

Definition 1.1. A *quasialgebra* is an algebraic structure $\mathcal{A}$, which satisfies all the axioms needed in the case of a non-commutative algebra with unit, without perhaps the left distribution law.

Clearly, each algebra is a quasialgebra. Some essential examples of quasialgebras will be presented below. We introduce this non-typical notion only to have the possibility of using the language used in the case of "true" algebras. In particular:

a). It is clear what one will mean by *"subquasialgebra"* of a given quasialgebra;
b). It is clear what one will mean by *"homomorphism of quasialgebras"*;
c). Since we have a unit - say $\mathbf{e}$ - in $\mathcal{A}$, we may define the regular elements in $\mathcal{A}$. Namely

Definition 1.2. An element $x \in \mathcal{A}$ will be said to be *regular* if there exists an element $y \in \mathcal{A}$ such that $x \cdot y = y \cdot x = \mathbf{e}$.
If x is not regular, it will be said to be *singular*.

d). Simiraly, as in the case of algebras, we have
Definition 1.3. If $x \in \mathcal{A}$ then the *spectrum* of x is the set

$$\mathbf{Sp}(x) = \{\lambda \in \mathbf{R} : x - \lambda \cdot \mathbf{e} \text{ is singular}\}.$$

Clearly,the elements of $\mathbf{Sp}(x)$ will be said the *eigenvalues* of x.

2 Different spaces of germs

First - for establish the notations - we will recall some standard definitions. Let $\mathcal{F}$ denote the family of all real functions defined in a neighborhood of 0 in $\mathbf{R}$. If $f \in \mathcal{F}$, $g \in \mathcal{F}$ then the relation :

$$f \sim g \Longleftrightarrow \exists V : V \text{is a neighbourhood of 0 and } f|_V = g|_V$$

is an equivalence relation in $\mathcal{F}$.

The quotient $\mathcal{F}/\sim$ will be denoted by $\widehat{\mathcal{F}}$ and the equivalence class of $f \in \mathcal{F}$ will be denoted by $\widehat{f}$ and called the germ of f.

There is a natural way to introduce the addition and the multiplication to $\widehat{\mathcal{F}}$, namely

$$\widehat{\mathcal{F}} = \mathrm{limind}_{V \to \{0\}} \mathcal{F}(V),$$

where V is a neighbourhood of the origin and

$$\mathcal{F}(V) = \{f : V \to \mathbf{R}\}$$

is the algebra of real functions on V with natural addition and multiplication.

We will say that a germ $\widehat{f}$ is *bounded* if there exists a bounded function g such that $f \sim g$. Similarly, we will say that a germ $\widehat{f}$ is *continuous* (resp. *differentiable*) it there exists a continuous (resp. differentiable) at 0 function g, such that $g \sim f$. Finally, we will say that that a germ $\widehat{f}$ is *lipschitzian* if there exist a function g satisfying the Lipschitz condition and equivalent to f. The subspaces (of $\widehat{\mathcal{F}}$) of bounded, continuous, differentiable, or lipschitian germs will be denoted by $\widehat{\mathcal{O}}$, $\widehat{\mathcal{C}}$, $\widehat{\mathcal{D}}$ and $\widehat{\mathcal{L}}$ respectively.

The value of a germ $\widehat{f}$ at 0 (denoted by $\widehat{f}(0)$) is by definition the common value at 0 of all functions equivalent to f. If a germ $\widehat{f}$ is differentiable then in an analogous way one defines the value $\widehat{f}'(0)$.

Let

$$\widehat{\mathcal{C}}_o = \{\widehat{f} \in \widehat{\mathcal{C}} : \widehat{f}(0) = 0\}.$$

We define $\widehat{D_o}$ and $\widehat{L}_o$ as the suitable subspaces of $\widehat{\mathcal{C}}$.

Let us look now at the space $\widehat{C}_o$. As a subspace of $\widehat{C}$ with standard multiplication, this is an ideal, but for our purposes it is essential to observe that one can define the *composition* of germs in $\widehat{\mathcal{C}}_o$. Indeed, let $\widehat{f}$ and $\widehat{g}$ be two germs from $\widehat{\mathcal{C}}_o$ and let $g_1 : W \to R$ and f_1 be the functions equivalent to g and f respectively. Since f_1 is continuous then there exists a neighbourhood V, such that $f_1(V) \subset W$. Hence one can compose $f_1|V$ with g_1 and the germ of this composition (i.e. $\widehat{g_1 \circ f_1}$) is by definition the composition of germs $\widehat{g} \circ \widehat{f}$. One can easily check that this definition is independent on the choice of f_1 and g_1.

Consider now the set $\widehat{\mathcal{C}}_o$ together with the operations of addition, scalar multiplication and compositions of germs. With respect to the addition and scalar multiplication $\widehat{\mathcal{C}}_o$ is a vector space. The operation of composition of germs is clearly associative, non commutative, but with the neutral element $\widehat{i}$ ($\widehat{i}$ the germ of the identity). However we cannot state that $(\widehat{\mathcal{C}}_o, +, \circ, \cdot)$ is an algebra, because the distribution law does not hold. More exactly, since the composition of germs is not commutative, we must consider left, and right distribution laws, namely:

$$(\widehat{f} + \widehat{g}) \circ \widehat{h} = \widehat{f} \circ \widehat{h} + \widehat{g} \circ \widehat{h}$$

and

$$\widehat{h} \circ (\widehat{f} + \widehat{g}) = \widehat{h} \circ \widehat{f} + \widehat{h} \circ \widehat{g}.$$

It is easy to see, that the first of this properties is fullfiled (the right distribution property) while the second (the left distribution property) does not hold in general. Taking into account the definition 1.2. we may state that $\widehat{\mathcal{C}}_o$ is a quasialgebra with composition and that $\widehat{\mathcal{D}}_o$ (and $\widehat{\mathcal{L}}_o$) are the subquasialgebras of $\widehat{\mathcal{C}}_o$, since the composition of differentiable (lipschitzian) germs is a differentiable (lipschitzian) germ.

3 A remark

In this section we will work with the quasialgebra $\widehat{\mathcal{L}}_o$. For a germ $\widehat{f} \in \widehat{\mathcal{L}}_o$ we set (f is an element of of $\widehat{f}$)

$$I_f(x, y) = \frac{f(x) - f(y)}{x - y}.$$

Let us observe the following

Lemma 3.1. For each $\widehat{f} \in \widehat{\mathcal{L}}_o$ we have: $\widehat{f}$ is singular if and only if there exist sequences $x_n \to 0, y_n \to 0, x_n \neq y_n$ such that

$$I_f(x_n, y_n) \to 0.$$

Proof: Suppose that $\widehat{f}$ is regular. This means that there exist a neighbourhoods of 0 - say V and W - and the Lipschitz functions

$f : V \to W$ (f is an element of $\widehat{f}$) and $g : W \to V$, which are inverse to each other. Clearly we may assume that $x_n \in V, y_n \in V$ and then there exist $u_n \in W, v_n \in W$ such that $g(u_n) = x_n$ and $g(v_n) = y_n$.

Hence, by the definition of I_f and from our assumption we obtain

$$\frac{u_n - v_n}{g(u_n) - g(v_n)} \to 0,$$

but this is impossible since g satisfy the Lipschitz condition.

To prove the converse, let us observe that if U is a connected neighbourhood of 0 then

$$I_f(U \times U \setminus \{(x,x) : x \in U\})$$

is connected in $\mathbf{R}$. Hence we may suppose (without loss of generality) that there exist a neighbourhood U of 0 and and a real $\delta > 0$ such that

$$\frac{f(x) - f(y)}{x - y} > \delta,$$

for all $x \in U, y \in U, x \neq y$. But this means that f is strictly increasing in U and then has an inverse $g : W \to U$ for a moment in $\widehat{\mathcal{C}}_o$. But from the inequality above it follows that

$$0 < g(u) - g(v) < \delta^{-1} \cdot (u - v),$$

for all $u \in W, v \in W, u \neq v$, which means that the germ $\widehat{g}$ is the inverse of $\widehat{f}$ in $\widehat{\mathcal{L}}_o$. This ends the proof of 3.1.

Lemma 3.2:Let us set for $\widehat{f} \in \widehat{\mathcal{L}}_o$

$$a(\widehat{f}) = \liminf_{x \to 0, y \to 0, x \neq y} \frac{f(x) - f(y)}{x - y}$$

and

$$b(\widehat{f}) = \limsup_{x \to 0, y \to 0, x \neq y} \frac{f(x) - f(y)}{x - y}.$$

Then $\mathbf{Sp}(\widehat{f}) = [a(\widehat{f}), b(\widehat{f})]$.

Proof. It follows from the remark on the connectedness made above, that for each point from $[a(\widehat{f}), b(\widehat{f})]$ - say α - there exist sequences $x_n \to 0, y_n \to 0, x_n \neq y_n$ such that $I_f(x_n, y_n) \to \alpha$. But by 3.1. this means that the germ of the function $x \to f(x) - \alpha \cdot x$ is singular.

On the other hand if $c < a(\widehat{f})$ (when $b(\widehat{f}) < c$ the proof is similar) and $\delta = 2^{-1}(a(\widehat{f}) - c)$ then there exists a neighbourhood U of 0 such that

$$I_f(x, y) - a(\widehat{f}) > \delta,$$

for all $x \in U, y \in U$. But this means that the germ of the function $x \to f(x) - c \cdot x$ is regular in $\widehat{\mathcal{L}}_o$. This ends the proof of 3.2.

Now we can restate 3.2. in a following form

Theorem 3.3. If a germ $\widehat{f} \in \widehat{\mathcal{L}}_o$ has only one eigenvalue, then it is differentiable.

Proof.It follows directly from 3.2.

Example 3.4. The converse in 3.3. does not hold. The function $f(x) = x^2 \cdot \sin \frac{1}{x}$ (and $f(0) = 0$) is differentiable , its derivative is bounded (then f is lipschitzian) and the spectrum of the germ of f is $[-1, 1]$.

4 Further remarks

We will come back now to the notion of spectrum of an element. For $t \in \mathbf{R}$ we denote by τ_t the translation $\mathbf{R} \ni x \to x + t \in \mathbf{R}$. If $f : (a, b) \to \mathbf{R}$ is a function satisfing locally the Lipschitz condition and $x \in (a, b)$ then we set:

$$\widehat{f}_x = (\widehat{\tau_{-f(x)} \circ f} \circ \tau_x).$$

Clearly, for each $x \in (a, b)$ the germ $\widehat{f}_x$ is in $\widehat{\mathcal{L}}_o$. It is not hard to observe that the spectrum of each germ in $\widehat{\mathcal{L}}_o$ is not empty, but in general this spectrum may have more than one element.

Definition 4.1. We will say that a function $f : (a, b) \to \mathbf{R}$ is *S-differentiable* if it satisfies locally the Lipschitz condition and for each $x \in (a, b)$ the spectrum of the germ $\widehat{f}_x$ (in $\widehat{\mathcal{L}}_o$) has exactly one element.

This definition is justified by the following

Theorem 4.2. A function $f : (a, b) \to \mathbf{R}$ is S-differentiable if and only if is of class C_1 on (a, b).

Proof: Suppose that $f \in C_1(a, b)$. Then f is locally lipschitzian. Let us fix $x_o \in (a, b)$. We will show that $\mathbf{Sp}(\widehat{f}_{x_o}) = \{f'(x_o)\}$. Indeed, let

$$\widehat{g_{x_o}} = \widehat{f}_{x_o} - f'(x_o) \cdot \widehat{\mathbf{i}}.$$

Clearly, $g'_{x_o}(0) = 0$ and then by 3.1. $\widehat{g}_{x_o}$ is singular, which means that $f'(x_o) \in sp(\widehat{f}_{x_o})$.

Moreover, for a real $\delta \neq 0$ we set $\widehat{h}_\delta = \widehat{f}_{x_o} - (f'(x_o) + \delta) \cdot \widehat{i}$. Then for some function h_δ we have:

$$h_\delta(x) = (f(x + x_o) - f(x_o) - f'(x_o) \cdot x) - \delta \cdot x$$

and we see that h_δ is C_1 in a neighbourhood of x_o and $h'_\delta(0) = -\delta$. But this means that $\widehat{h}_\delta$ is regular in $\widehat{\mathcal{L}}_o$ and then $f'(x_o) + \delta \notin \mathbf{Sp}(\widehat{f}_{x_o})$.

To prove the converse, suppose that spectrum $\widehat{f}_{x_o}$ =$\{a\}$. Consider the germ of the function:$g(x) = f(x + x_o) - f(x_o) - a \cdot x$. This germ satisfies 3.1. hence $g'(0) = 0$ and in consequence $f'(x_o) = a$. To prove the continuity of f' it is sufficient to show that

$$\lim_{x \to x_o, y \to x_o, x \neq y} \frac{f(x) - f(y)}{x - y} = f'(x_o).$$

It is clear that if this limit exists, it equals $f'(x_o)$, hence suppose that there exist the sequences $\{x_n\}_1^\infty, \{y_n\}_1^\infty, \{u_n\}_1^\infty, \{v_n\}_1^\infty$ and the reals b and c, such that $b \neq c$,

$$\lim_{n \to \ infty} x_n = \lim_{n \to \infty} y_n = \lim_{n \to \infty} u_n = \lim_{n \to \infty} v_n = x_o$$

and

$$\lim_{n \to \infty} \frac{f(x_n) - f(y_n)}{x_n - y_n} = b$$

and

$$\lim_{n \to \infty} \frac{f(u_n) - f(v_n)}{u_n - v_n} = c.$$

Consider now the germ of the function

$$h(x) = f(x + x_o) - f(x_o) - b \cdot x.$$

Setting $\tilde{x_n} = x_n - x_o$ and $\tilde{y_n} = y_n - x_o$ we obtain

$$h(\tilde{x_n}) - h(\tilde{y_n}) = f(x_n) - f(y_n) - b \cdot (x_n - y_n).$$

Hence

$$\frac{h(\tilde{x_n}) - h(\tilde{y_n})}{\tilde{x_n} - \tilde{y_n}} = \frac{f(x_n) - f(y_n)}{x_n - y_n} - b$$

since $\tilde{x_n} - \tilde{y_n} = x_n - y_n$.

Thus the derivative of h at 0 is 0, and in consequence h is singular. But this means that b is in the spectrum of $\widehat{f}_{x_o}$. Since the same is true for c we obtain a contradiction with our assumption on the spectrum of the germ $\widehat{f}_{x_o}$.

5 A comment

Roughly speaking the differentiation of germs is a projection from an quasialgebra of germs (exactly those which are differentiable) onto the (quasi)algebra of germs of linear functions, and this projection is also a homomorphism of quasialgebras. Our theorem shows that at least in $\widehat{\mathcal{L}}_o$ the subquasialgebra $\widehat{\mathcal{D}}_o \cap \widehat{\mathcal{L}}_o$ is uniquely determined by these two conditions i.e. "projection" and "homomorphism".