STABILITY OF THE DARBOUX EQUATION WITH POSITIVE GAUSS CURVATURE

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Abstract. J.X. Hong has recently shown that for a given Riemannian metric on the closed unit disk with positive Gauss curvature and positive geodesic curvature of the boundary there exists a solution of the corresponding Darboux equation with zero boundary data. We show that these solutions are stable.

Introduction

By $\mathcal{G}$ denote the class of Riemannian metrics g defined on the closed unit disk $\overline{\Delta}$ in $\mathbb{R}^2$ with strictly positive Gauss curvature $K = K_g$ on $\overline{\Delta}$ and such that the geodesic curvature $k = k_g$ of $\partial\Delta$ is also strictly positive. The latter assumption means exactly that $\overline{\Delta}$ is strictly convex with respect to g.

In local coordinates we write $g = (g_{ij})$ and by (g^{ij}) we will denote the inverse matrix of (g_{ij}). If $u \in C^\infty(\overline{\Delta})$ then the gradient $\nabla u = \nabla_g u$ in local coordinates is given by

$$\nabla u = \nabla_g u = g^{ij} u_{,i} \partial_j$$

and

$$|\nabla u|^2 = |\nabla_g u|_g^2 = g^{ij} u_{,i} u_{,j}.$$

The Hessian of u with respect to g in local coordinates is given by $\nabla^2 u = \nabla_g^2 u = (u^i_j)$, where

$$u^j_i = g^{jk}(u_{,ik} - \Gamma^l_{ik} u_{,l})$$

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and
$$\Gamma_{ij}^k = \frac{1}{2} g^{kl}(g_{jl,i} + g_{il,j} - g_{ij,l})$$
are the Christoffel symbols. The *Monge-Ampère operator* is defined by
$$M(u) = M_g(u) = \det \nabla^2 u.$$

It is well known that finding an embedding of $(\overline{\Delta}, g)$ into $\mathbb{R}^3$ with Euclidean metric is equivalent to finding a solution of the Darboux equation
$$M(u) = K(1 - |\nabla u|^2) \tag{1}$$
with
$$|\nabla u| < 1. \tag{2}$$
Recently J.X. Hong [Ho] has proved the following:

Theorem 1. *For any $g \in \mathcal{G}$ there exists $u \in C^\infty(\overline{\Delta})$ satisfying (1), (2) on $\overline{\Delta}$ and such that*
$$\nabla^2 u > 0 \quad on\ \overline{\Delta} \tag{3}$$
and
$$u = 0 \quad on\ \partial\Delta. \tag{4}$$

The above theorem had been proven in [De] under additional assumption that there exists a subsolution of (1), that is there is $v \in C^\infty(\overline{\Delta})$ satisfying (2), (3), (4) and such that
$$M(v) \geq K(1 - |\nabla v|^2).$$

By $\mathcal{F} = \mathcal{F}_g$ denote the class of functions $u \in C^\infty(\overline{\Delta})$ satisfying (2) and (3). One can call them admissible for the Darboux equation (1) with positive Gauss curvature. (3) means exactly that they are strictly convex with respect to g. For $u \in \mathcal{F}$ we may define
$$W(u) = W_g(u) = \frac{M(u)}{1 - |\nabla u|^2}.$$
In [De] it is called the *Waingarten operator*. The Darboux equation (1) then takes the form
$$W(u) = K.$$

Note that $\mathcal{F}$ is convex, that is if $u, v \in \mathcal{F}$ and $0 \leq t \leq 1$ then $(1-t)u + tv \in \mathcal{F}$. Therefore Theorem 17.1 from [GT] gives the following comparison principle:

Theorem 2. *If $u, v \in \mathcal{F}$ are such that $u \leq v$ on $\partial\Delta$ and $W(u) \geq W(v)$ on $\overline{\Delta}$ then $u \leq v$ on $\overline{\Delta}$.*

Theorem 2 immediately implies that the solution given by Theorem 1 is unique. We thus obtain a mapping
$$\mathcal{G} \ni g \longmapsto u_g \in C^\infty(\overline{\Delta}), \tag{5}$$
where u_g is the unique solution from Theorem 1. Our main result is the following stability of the Darboux equation:

Theorem 3. *The mapping (5) is continuous in the $C^{3,1}(\overline{\Delta})$ topology on $\mathcal{G}$ and the $L^\infty(\overline{\Delta})$ topology on $C^\infty(\overline{\Delta})$ (that is the topology of uniform convergence).*

In global coordinates on $\overline{\Delta}$ the $C^{3,1}(\overline{\Delta})$ topology on $\mathcal{G}$ is given by the norm

$$||g||_{C^{3,1}(\overline{\Delta})} = \sum_{i,j,|\alpha|\leq 4} ||\partial^\alpha g_{ij}||_{L^\infty(\overline{\Delta})},$$

which of course depends on the coordinates but the corresponding topology is independent of the choice of variables.

In fact, if one analyzes the proof of Theorem 3 below, it can be slightly improved. Instead of the $C^{3,1}$ topology on $\mathcal{G}$ one can use a slightly weaker topology: we would say that $g^k \to g$ as $k \to \infty$ in this topology if

$$||g^k - g||_{C^{1,1}(\overline{\Delta})} \to 0$$

and

$$\sup_k ||K_{g^k}||_{C^{1,1}(\overline{\Delta})} < \infty.$$

Unfortunately, Theorem 3 does not yet give stability of the corresponding embeddings. It follows from the construction of an embedding from a solution of the Darboux equation that one would have such a stability if instead of the $L^\infty(\overline{\Delta})$ topology on $C^\infty(\overline{\Delta})$ one had the $C^{1,1}(\overline{\Delta})$ topology.

Proof of Theorem 3

Fix global coordinates on $\overline{\Delta}$. In the proof of Theorem 3 we will need the following a priori estimates for the solutions u given by Theorem 1 (see [Ho]):

$$|\nabla_g u|_g \leq a < 1 \quad \text{on } \overline{\Delta}, \tag{6}$$

$$|\nabla_g^2 u| \leq C \quad \text{on } \overline{\Delta}, \tag{7}$$

where a and C are positive constants depending only on upper bounds of $||g||_{C^{1,1}(\overline{\Delta})}$, $||K||_{C^{1,1}(\overline{\Delta})}$, $\dfrac{1}{\min_{\overline{\Delta}} K}$, $\dfrac{1}{\min_{\partial\Delta} k}$.

Let $g^k \in \mathcal{G}$ be a sequence converging to $g \in \mathcal{G}$ as $k \to \infty$ in the $C^{3,1}(\overline{\Delta})$ norm on $\mathcal{G}$. Denote $u^k := u_{g^k}$, $u := u_g$. Quantities with respect to the metric g^k we will denote with the script k. From (6) and (7) it follows that

$$|\nabla_k u^k|_k \leq a < 1 \quad \text{on } \overline{\Delta}, \tag{8}$$

$$|u^k_{,i}|, |u^k_{,ij}| \leq A \quad \text{on } \overline{\Delta}, \quad i,j = 1,2, \tag{9}$$

where a and A are independent of k.

From now on, unless stated otherwise, every convergence will mean uniform convergence on $\overline{\Delta}$ as $k \to \infty$.

Since u^k and u are solutions of the Darboux equation for g^k and g respectively, we have

$$W^k(u^k) = K^k$$

and

$$W(u) = K.$$

The convergence of the metrics $g^k \to g$ in the $C^{1,1}(\overline{\Delta})$ norm gives

$$K^k \to K$$

and thus

$$W^k(u^k) \to W(u). \tag{10}$$

From (9) and since $g^k \to g$ in the $C^{0,1}(\overline{\Delta})$ norm, it easily follows that

$$|\nabla_k u^k|_k^2 - |\nabla u^k|^2 \to 0$$

and

$$M^k(u^k) - M(u^k) \to 0.$$

This coupled with (8) easily gives

$$W^k(u^k) - W(u^k) \to 0,$$

and thus by (10)

$$W(u^k) \to W(u). \tag{11}$$

We will now show with the help of Theorem 2 that (11) implies that $u^k \to u$. We have

$$\begin{aligned} W(tu) &= \frac{t^2 M(u)}{1 - t^2|\nabla u|^2} \\ &= t^2 W(u) \frac{1 - |\nabla u|^2}{1 - t^2|\nabla u|^2} \end{aligned}$$

and thus

$$W(tu) \geq t^2 W(u) > W(u) \quad \text{on} \quad \overline{\Delta}.$$

Similarly

$$W(\frac{1}{t}u) < W(u) \quad \text{on} \quad \overline{\Delta}$$

and by (10) for k big enough we get

$$W(tu) \leq W(u^k) \leq W(\frac{1}{t}u) \quad \text{on} \quad \overline{\Delta}.$$

From Theorem 2 it now follows that

$$tu \leq u^k \leq \frac{1}{t}u \quad \text{on} \quad \overline{\Delta}.$$

This coupled with (9) implies that $u^k \to u$ which completes the proof.

References

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