INTERIOR $C^{1,1}$ REGULARITY OF THE DEGENERATE MONGE-AMPÈRE EQUATION

Zbigniew Błocki

IMUJ PREPRINT 2001/02

Abstract. We study interior $C^{1,1}$ regularity of generalized solutions of the Monge-Ampére equation $\det D^2u = \psi$, $\psi \geq 0$, on a bounded convex domain Ω in $\mathbb{R}^n$ with $u = \varphi$ on $\partial\Omega$. We prove in particular that $u \in C^{1,1}(\Omega)$ if either i) $\varphi = 0$ and $\psi^{1/(n-1)} \in C^{1,1}(\Omega)$ or ii) Ω is $C^{1,1}$ strongly convex, $\varphi \in C^{1,1}(\overline{\Omega})$, $\psi^{1/(n-1)} \in C^{1,1}(\overline{\Omega})$ and $\psi > 0$ on $U \cap \Omega$, where U is a neighborhood of $\partial\Omega$. The main tool is an improvement of a well known Pogorelov's $C^{1,1}$ estimate so that it can be applied to the degenerate case.

1. Introduction

For an arbitrary convex function u one can well define a nonnegative Borel measure $M(u)$ such that

$$M(u) = \det D^2u \, d\lambda$$

for smooth, and even $W^{2,n}_{loc}$ functions (see [RT] for details). The Dirichlet problem for M is solvable in a fairly general situation: let Ω be an arbitrary bounded convex domain in $\mathbb{R}^n$ and $\varphi \in C(\partial\Omega)$ be such that it is convex on any line segment in $\partial\Omega$ (we will call such a φ *admissible*). Then for any nonnegative Borel measure μ with $\mu(\Omega) < \infty$ the Dirichlet problem

$$(1.1) \qquad \begin{cases} u \text{ continuous and convex on } \overline{\Omega} \\ M(u) = \mu \text{ in } \Omega \\ u = \varphi \text{ on } \partial\Omega \end{cases}$$

Partially supported by KBN Grant #2 P03A 028 19

has a unique solution. (This was proven for example in [RT] for strictly convex Ω where all continuous φ are admissible, and the general case easily follows from this - see Proposition 2.1 below.)

We will primarily consider measures μ with continuous, nonnegative densities ψ in Ω. Unless otherwise stated, u will always denote the solution of (1.1) (with $\mu = \psi d\lambda$), whereas v will be the solution of the corresponding homogeneous problem:

$$\begin{cases} v \text{ continuous and convex on } \overline{\Omega} \\ M(v) = 0 \text{ in } \Omega \\ v = \varphi \text{ on } \partial\Omega. \end{cases} \tag{1.2}$$

Below we list several known regularity results for solutions of (1.1):

1) $\phi = 0$, $\psi \in C^\infty(\Omega)$, $\psi > 0 \Rightarrow u \in C^\infty(\Omega)$ (Cheng, Yau [CY1], see also [CY2]);
2) Ω is $C^{1,1}$ strongly convex, $\varphi \in C^{1,1}(\overline{\Omega}) \Rightarrow v \in C^{0,1}(\overline{\Omega}) \cap C^{1,1}(\Omega)$ (Trudinger, Urbas [TU]);
3) Ω is $C^{1,1}$ strongly convex, $\varphi \in C^{1,1}(\overline{\Omega})$, $\psi \in C^{1,1}(\overline{\Omega})$, $\psi > 0 \Rightarrow u \in C^{1,1}(\Omega)$ (Trudinger, Urbas [TU]);
4) Ω is C^∞ strongly convex, $\varphi \in C^\infty(\partial\Omega)$, $\psi \in C^\infty(\overline{\Omega})$, $\psi > 0 \Rightarrow u \in C^\infty(\overline{\Omega})$ (Krylov [Kr] and Caffarelli, Nirenberg, Spruck [CNS]);
5) Ω is $C^{3,1}$ strongly convex, $\varphi \in C^{3,1}(\overline{\Omega})$, $\psi^{1/(n-1)} \in C^{1,1}(\overline{\Omega}) \Rightarrow u \in C^{1,1}(\overline{\Omega})$ (Guan, Trudinger, Wang [GTW]).

Pogorelov had claimed to prove 1) earlier but his proof had many gaps. However, his arguments were enough to get $u \in C^{1,1}(\Omega)$ in 1). The following results on the local regularity of the Monge-Ampère operator M are also known:

6) u is strictly convex, $\psi \in C^\infty$, $\psi > 0 \Rightarrow u \in C^\infty$ (this is in fact equivalent to 1));
7) Either $u \in W^{2,p}_{loc}$ for some $p > n(n-1)/2$ or $u \in C^{1,\alpha}$ for some $\alpha > 1 - 2/n$, $\psi \in C^\infty$, $\psi > 0 \Rightarrow u \in C^\infty$ (Urbas [Ur]).

In this paper we prove two more regularity results:

Theorem 1.1. *If $\varphi = 0$ and $\psi^{1/(n-1)} \in C^{1,1}(\Omega)$ then $u \in C^{1,1}(\Omega)$.*

Theorem 1.2. *Suppose that Ω is a $C^{1,1}$ strongly convex domain, $\varphi \in C^{1,1}(\overline{\Omega})$ and $\psi^{1/(n-1)} \in C^{1,1}(\overline{\Omega})$. Assume moreover that*

$$\psi > 0 \ \text{on } U \cap \Omega \tag{1.3}$$

for some neighborhood U of $\partial\Omega$. Then $u \in C^{1,1}(\Omega)$.

Note that in Theorem 1.1 Ω is an arbitrary bounded convex domain in $\mathbb{R}^n$. By 7), Theorem 1.1 is a generalization of 1). We also immediately get the following local regularity which generalizes 6):

Theorem 1.3. *If u is strictly convex and $\psi^{1/(n-1)} \in C^{1,1}$ then $u \in C^{1,1}$.*

This paper was mostly motivated by recent articles [Gu] and [GTW], where global a priori estimates for second derivatives of u depending on $||\psi^{1/(n-1)}||_{C^{1,1}(\overline{\Omega})}$

were established. The main tool in proving Theorems 1.1 and 1.2 will be a corresponding interior a priori estimate (Theorem 3.1). The importance of the exponent $1/(n-1)$ is that, by [Wa, Example 3], it is optimal in all of the above results.

We believe that the assumption (1.3) in Theorem 1.2 is superfluous. However, at the end of section 4 we give an example which shows that the application of our methods only does not allow to drop this assumption. Namely, what we really prove is that under the assumptions of Theorem 1.2 except for (1.3), we have $u \in C^{1,1}(\{u < v\})$ (Theorem 4.2). Moreover, $\{u < v\} = \Omega$ if (1.3) is satisfied (Proposition 4.1), but without (1.3) it may happen that $\{u < v\} \neq \Omega$.

Terminology and notation. We say that a convex domain Ω (resp. convex function u) is *strictly convex* if $\partial\Omega$ (resp. graph u) contains no line segment. A convex function v in Ω is called *maximal* if for any $D \Subset \Omega$ and u continuous on $\overline{D}$, convex in D, we have that $u \leq v$ on ∂D implies $u \leq v$ in D. This condition is in fact equivalent to $M(v) = 0$ (see [RT]) . A function u is called *strongly convex* if locally there exists $\lambda > 0$ such that $u - \lambda|x|^2$ is convex. We say that a bounded domain Ω is *strongly convex* if there exists w, a strongly convex function in a neighborhood of $\overline{\Omega}$ such that $\Omega = \{w < 0\}$ and $Dw \neq 0$ near $\partial\Omega$ (that is there exists $\varepsilon > 0$ such that for any supporting hyperplane $H = \operatorname{graph} L$ of w at a point near $\partial\Omega$ we have $|\nabla L| \geq \varepsilon$). A domain is called $C^{k,\alpha}$ strongly convex if one can find a $C^{k,\alpha}$ defining function w. If we write $f \in C^{k,1}(\overline{\Omega})$ then we mean that $f \in C^{k,1}(\Omega)$ and $|D^{k+1}f|$ is globally bounded in Ω. Then

$$||f||_{C^{k,1}(\overline{\Omega})} = \sum_{i=0}^{k+1} \sup_{\Omega} |D^i f|$$

and the values of f at $\partial\Omega$ are uniquely determined. Finally, $\overline{B}(x, r)$ will denote a closed ball centered at x with radius r and $[x, y]$ will stand for the line segment joining x and y.

2. Existence of generalized solutions

In this section we will show how to generalize the solution of the Dirichlet problem (1.1) in [RT] from strictly convex to arbitrary convex domains.

Proposition 2.1. *Assume that Ω is an arbitrary bounded convex domain in $\mathbb{R}^n$. Let $\varphi \in C(\partial\Omega)$ be admissible (that is φ is convex on any line segment in $\partial\Omega$) and μ be a nonnegative Borel measure in Ω with $\mu(\Omega) < \infty$. Then the problem (1.1) has a unique solution.*

Proof. The uniqueness follows from the comparison principle (see e.g. [RT]). If Ω is strictly convex then the proof can be found for example in [RT]. We shall use this as well as other results from [RT] in the proof.

We first solve the homogeneous problem (1.2) using the Perron method. Define

$$v := \sup\{w \text{ convex and continuous on } \overline{\Omega}, w \leq \varphi \text{ on } \partial\Omega\}.$$

Then v is convex in Ω and we have to show that v continuously extends to $\partial\Omega$ and equals φ there. Fix $x_0 \in \partial\Omega$ and $\varepsilon > 0$. Since φ is convex on any line segment contained in $\partial\Omega$ and passing through x_0, we can find an affine function L such that $L(x_0) \geq \varphi(x_0) - \varepsilon$ and $L \leq \varphi$ on $\partial\Omega$. Let h be a solution of the classical Dirichlet problem

$$\begin{cases} h \text{ continuous on } \overline{\Omega} \\ h \text{ harmonic in } \Omega \\ h = \varphi \text{ on } \partial\Omega. \end{cases}$$

Then we have $L \leq v \leq h$ in Ω and

$$\varphi(x_0) - \varepsilon \leq \liminf_{x \to x_0} v(x) \leq \limsup_{x \to x_0} v(x) \leq \varphi(x_0).$$

This shows that v is continuous on $\overline{\Omega}$ and $v = \varphi$ on $\partial\Omega$. The definition of v implies that v is maximal and thus $M(v) = 0$ there. This solves (1.2).

Next we solve (1.1) with $\varphi = 0$. This has already been done in [Bł, Theorem 4.1] but we include it for the convenience of the reader. Let Ω_j be strictly convex domains such that $\Omega_j \uparrow \Omega$ as $j \uparrow \infty$. Let u_j be continuous and convex on $\overline{\Omega}_j$ and such that $M(u_j) = \mu$ in Ω_j and $u_j = 0$ on $\partial\Omega_j$. By the comparison principle the sequence u_j is decreasing. Moreover, [RT, Lemma 3.5] gives

$$(-u_j(x))^n \leq c_n (\operatorname{diam}\Omega)^{n-1} \operatorname{dist}(x, \partial\Omega) \mu(\Omega), \quad x \in \Omega_j, \tag{2.1}$$

where c_n is a constant depending only on n. This implies that u_j converges to a convex $\widetilde{u}$ uniformly on $\overline{\Omega}$. By the continuity theorem for M ([RT, Proposition 3.1]) we have $M(\widetilde{u}) = \mu$ in Ω. This solves (1.1) for $\varphi = 0$.

Now let μ and φ be arbitrary. Again, we approximate Ω by strictly convex domains Ω_j from inside. There we can find u'_j, continuous and convex on $\overline{\Omega}_j$, such that $M(u'_j) = \mu$ in Ω_j and $u'_j = v$ on $\partial\Omega_j$. [RT, Proposition 3.3] gives $M(\widetilde{u}+v) \geq \mu$ and thus

$$\widetilde{u} + v \leq u'_{j+1} \leq u'_j \leq v \quad \text{in } \Omega_j$$

by the comparison principle. It now easily follows that u'_j decreases to a function u which solves (1.1). $\square$

3. A priori estimate for second derivatives

In this section we will generalize the Pogorelov estimate (see [Po, p. 73-76]) so that it can be applied to the degenerate case. We will modify Pogorelov's method using some ideas from [Gu]. We shall prove:

Theorem 3.1. *Let $u \in C^4(\Omega) \cap C^{1,1}(\overline{\Omega})$ be a strongly convex solution of*

$$\det D^2 u = \psi \tag{3.1}$$

in a bounded domain Ω in $\mathbb{R}^n$. Assume that w is a C^2 convex function in Ω such that $u \leq w$ in Ω and $\lim_{x \to \partial\Omega} (w(x) - u(x)) = 0$. Then in Ω we have

$$(w - u)^\alpha |D^2 u| \leq C,$$

where $\alpha = \alpha(n)$ *is any number such that*

$$\alpha(n) \begin{cases} \geq n-1 & \text{for } n \geq 3, \\ > 1 & \text{for } n = 2, \end{cases}$$

and C *depends only on* n, α *and on upper bounds of* $\operatorname{diam}\Omega$, $||\psi^{1/(n-1)}||_{C^{1,1}(\overline{\Omega})}$ *and* $||w-u||_{C^{0,1}(\overline{\Omega})}$.

Proof. We may assume that $\Omega \subset B(0,R)$, where $R \leq \operatorname{diam}\Omega$. We will use the standard notation: $u_i = \partial u/\partial x_i$, $(u_{ij}) = D^2u$ and $(u^{ij}) = (D^2u)^{-1}$. First, we differentiate the logarithm of both sides of (3.1) twice with respect to x_p. We get

$$u^{ij}u_{pij} = (\log\psi)_p \tag{3.2}$$

$$u^{ij}u_{ppij} = (\log\psi)_{pp} + u^{ik}u^{jl}u_{pij}u_{pkl}. \tag{3.3}$$

Consider the auxiliary function

$$h = (w-u)^\alpha e^{\beta|x|^2/2}|D^2u|,$$

where $\alpha, \beta > 0$ will be specified later. By the assumptions on u and w, h attains a maximum at some $y \in \Omega$. After rotation of variables we may assume that (u_{ij}) is diagonal at y. Since $|D^2u|$ is equal to the maximal eigenvalue of D^2u, we may also assume that at y we have $|D^2u| = u_{11}$. Thus the function

$$\widetilde{h} = (w-u)^\alpha e^{\beta|x|^2/2}u_{11}$$

also attains a maximum at y and $\widetilde{h}(y) = h(y)$.

We may assume that $w < u$ at y. We will use the fact that $D\widetilde{h} = 0$ and $D^2\widetilde{h} \leq 0$ at y. Therefore at y we have

$$0 = (\log\widetilde{h})_i = \alpha\frac{w_i - u_i}{w-u} + \beta x_i + \frac{u_{11i}}{u_{11}} \tag{3.4}$$

for every $i = 1, \ldots, n$ and

$$u^{ii}(\log\widetilde{h})_{ii} \leq 0.$$

Now we analyze the term where the third and fourth derivatives of u appear. Using (3.2), (3.3), (3.4) and denoting $g := \psi^{1/(n-1)}$ we compute at y

$$\begin{aligned}
u^{ii}(\log u_{11})_{ii} &= \frac{1}{u_{11}}\sum_i \frac{u_{11ii}}{u_{ii}} - \frac{1}{(u_{11})^2}\sum_i \frac{(u_{1ii})^2}{u_{ii}} \\
&= \frac{(\log\psi)_{11}}{u_{11}} + \frac{1}{u_{11}}\sum_{i,j}\frac{(u_{1ij})^2}{u_{ii}u_{jj}} - \frac{1}{(u_{11})^2}\sum_i \frac{(u_{1ii})^2}{u_{ii}} \\
&\geq \frac{(\log\psi)_{11}}{u_{11}} + \frac{1}{u_{11}}\sum_{i\geq 2}\frac{(u_{1ii})^2}{(u_{ii})^2} + \frac{1}{(u_{11})^2}\sum_{i\geq 2}\frac{(u_{11i})^2}{u_{ii}} \\
&\geq \frac{(\log\psi)_{11}}{u_{11}} + \frac{1}{(n-1)u_{11}}\left((\log\psi)_1 - \frac{u_{111}}{u_{11}}\right)^2 + \frac{1}{(u_{11})^2}\sum_{i\geq 2}\frac{(u_{11i})^2}{u_{ii}} \\
&\geq \frac{(n-1)g_{11}}{g\,u_{11}} - \frac{2g_1u_{111}}{g(u_{11})^2} + \frac{1}{(u_{11})^2}\sum_{i\geq 2}\frac{(u_{11i})^2}{u_{ii}} \\
&= \frac{(n-1)g_{11}}{g\,u_{11}} + \frac{2g_1}{g\,u_{11}}\left(\alpha\frac{w_1-u_1}{w-u} + \beta x_1\right) + \sum_{i\geq 2}\frac{1}{u_{ii}}\left(\alpha\frac{w_i-u_i}{w-u} + \beta x_i\right)^2.
\end{aligned}$$

We thus obtain at y

$$
\begin{aligned}
0 &\geq u^{ii}(\log\widetilde{h})_{ii}\\
&= \frac{\alpha}{w-u}\sum_i \frac{w_{ii}}{u_{ii}} - \frac{\alpha\, n}{w-u} - \frac{\alpha}{(w-u)^2}\sum_i \frac{(w_i-u_i)^2}{u_{ii}} + \beta\sum_i \frac{1}{u_{ii}}\\
&\quad + u^{ii}(\log u_{11})_{ii}\\
&\geq -\frac{\alpha\, n}{w-u} - \frac{\alpha(w_1-u_1)^2}{(w-u)^2 u_{11}} + \frac{(n-1)g_{11}}{g\, u_{11}} + \frac{2g_1}{g\, u_{11}}\left(\alpha\frac{w_1-u_1}{w-u} + \beta x_1\right)\\
&\quad + \beta\sum_i \frac{1}{u_{ii}} + \sum_{i\geq 2}\frac{1}{u_{ii}}\left(\frac{\alpha(\alpha-1)(w_i-u_i)^2}{(w-u)^2} + \frac{2\alpha\beta(w_i-u_i)}{w-u} + \beta^2 x_i^2\right)\\
&\geq -\frac{\alpha\, n}{w-u} - \frac{\alpha(w_1-u_1)^2}{(w-u)^2 u_{11}} + \frac{(n-1)g_{11}}{g\, u_{11}} + \frac{2g_1}{g\, u_{11}}\left(\alpha\frac{w_1-u_1}{w-u} + \beta x_1\right)\\
&\quad + \left(\beta - \frac{\beta^2 R^2}{\alpha-1}\right)\sum_{i\geq 2}\frac{1}{u_{ii}}.
\end{aligned}
$$

An optimal choice for β is therefore $(\alpha-1)/(2R^2)$. Multiplying both sides of the obtained inequality by $4R^2(w-u)^n g\, u_{11}/(n-1)$ and observing that the inequality between arithmetic and geometric means gives

$$\sum_{i\geq 2}\frac{1}{u_{ii}} \geq \frac{(n-1)(u_{11})^{1/(n-1)}}{g},$$

we get at y

$$(\alpha-1)\left((w-u)^{n-1}u_{11}\right)^{n/(n-1)} - C_1(w-u)^{n-1}u_{11} - C_2 \leq 0,$$

where C_1, C_2 are positive constants under control. From this the desired estimate easily follows. $\square$

Note, that in fact in Theorem 3.1 we could replace $||\psi^{1/(n-1)}||_{C^{1,1}(\overline{\Omega})}$ with $||\psi^{1/(n-1)}||_{C^{0,1}(\overline{\Omega})}$ and the maximal eigenvalue of $-D^2(\psi^{1/(n-1)})$ in Ω.

4. $C^{1,1}$ regularity of generalized solutions

First we prove Theorem 1.1:

Proof of Theorem 1.1. We may assume that $\psi \in C^{1,1}(\overline{\Omega})$ - otherwise shrink Ω a little. Fix $\varepsilon > 0$. Let Ω_j be a sequence of C^∞ strongly convex domains such that $\Omega_j \uparrow \Omega$ as $j \uparrow \infty$. We can find $\psi_j \in C^\infty(\overline{\Omega}_j)$ such that $\psi_j > 0$, ψ_j tends uniformly to ψ in Ω and

$$||\psi_j^{1/(n-1)}||_{C^{1,1}(\overline{\Omega}_j)} \leq C_1.$$

(By $C_1, C_2, \dots$ we will denote constants independent of j.) By 4) one can find $u_j \in C^\infty(\overline{\Omega}_j)$, convex in Ω_j such that $\det D^2 u_j = \psi_j$ in Ω_j and $u_j = 0$ on $\partial\Omega_j$.

From the comparison principle we can easily get that u_j tends uniformly to u. In particular,

$$\sup_{\Omega_j} |u_j| \leq C_2.$$

Since u_j is convex,

$$|Du_j(x)| \leq \frac{-u_j(x)}{\operatorname{dist}(x, \partial\Omega_j)}, \quad x \in \Omega_j.$$

Thus

$$|Du_j| \leq C_3 \quad \text{in } \{u_j < -\varepsilon\}.$$

Theorem 3.1 now gives for some α

$$(\varepsilon - u_j)^\alpha |D^2 u_j| \leq C_4 \quad \text{in } \{u_j < -\varepsilon\}.$$

Hence

$$|D^2 u_j| \leq C_5 \quad \text{in } \{u_j < -2\varepsilon\}$$

and $u \in C^{1,1}(\{u < -2\varepsilon\})$. $\square$

Theorem 1.2 will be a direct consequence of 2) and the next two results.

Proposition 4.1. *Assume that ψ is continuous and let φ be admissible and such that $v \in C^{1,1}(\Omega)$. Then $u < v$ on the domain consisting of $\{\psi > 0\} \cap \Omega$ and the connected components of $\{\psi = 0\}$ which are relatively compact in Ω.*

Proof. Since v is a maximal convex function, it is enough to show that $u < v$ on $\psi > 0$. As the problem is now purely local, we may assume that $\psi \geq a > 0$ and $|D^2 v| \leq M < \infty$ on $\overline{B}(x_0, r)$. We will proceed in the same way as in [TU, p. 329]. For $\varepsilon > 0$ define

$$w(x) := \varepsilon(|x - x_0|^2 - r^2).$$

Then

$$\det D^2(v + w) \leq c_n \sum_{i=1}^{n} \varepsilon^i M^{n-i} \leq a$$

if ε is small enough. From the comparison principle we get $u \leq v + w$ on $\overline{B}(x_0, r)$. $\square$

Remark. If $n = 2$ then the assumption that $v \in C^{1,1}(\Omega)$ in Proposition 4.1 is superfluous. For by an old result of Aleksandrov [Al] u is then strictly convex in $\{\psi > 0\} \cap \Omega$. If $x_0 \in \{\psi > 0\} \cap \Omega$ and $H = \operatorname{graph} L$ is a supporting hyperplane of $\operatorname{graph} v$ at x_0 then by [TU, Lemma 2.1] x_0 belongs to the convex hull of the set $\{v = L\} \cap \partial\Omega$. We can thus find $y_1, y_2 \in \partial\Omega$ such that v is affine on $[y_1, y_2]$ and $x_0 \in [y_1, y_2]$. From the strict convexity of u near x_0 it now follows that $u(x_0) < v(x_0)$ which proves the claim.

However, the following example due to Pogorelov shows that this assumption cannot be dropped if $n \geq 3$. Set

$$u(x) = (1 + x_1^2)(x_2^2 + \cdots + x_n^2)^{1-1/n}.$$

Then in a neighborhood of the origin u is convex (if $n \geq 3$), $M(u) \in C^\infty$, $M(u) > 0$. But if Ω is a small ball centered at the origin, we will always have $u = v = 0$ on $\{x_2 = \cdots = x_n = 0\} \cap \Omega$.

We shall now show that to get $u \in C^{1,1}(\{u < v\})$ one needs only very mild assumptions on Ω and φ:

Theorem 4.2. *Assume that Ω is strongly convex and let $\varphi \in C(\partial\Omega)$ be such that $v \in C^{0,1}(\overline{\Omega})$. Then, if $\psi^{1/(n-1)} \in C^{1,1}(\overline{\Omega})$, we have $u \in C^{1,1}(\{u < v\})$.*

Proof. Regularizing the defining function for Ω we get C^∞ strongly convex domains $\Omega_j = \{w_j < 0\}$ such that

$$D^2 w_j \geq \frac{1}{C_1} I \quad \text{on } \overline{\Omega}_j, \tag{4.1}$$

and, since convex functions are locally Lipschitz,

$$|Dw_j| \leq C_2, \quad \text{on } \partial\Omega_j, \tag{4.2}$$

where $C_1, C_2, \dots$ are positive constants independent of j and I is the unitary matrix. Regularizing v we get $v_j \in C^\infty(\overline{\Omega}_j)$, convex, converging locally uniformly to v, and such that

$$|Dv_j| \leq C_3 \quad \text{on } \overline{\Omega}_j. \tag{4.3}$$

We can also find $\psi_j \in C^\infty(\overline{\Omega}_j)$, $\psi_j > 0$, ψ_j tending uniformly to ψ in Ω and such that

$$||\psi_j^{1/(n-1)}||_{C^{1,1}(\overline{\Omega}_j)} \leq C_4.$$

Let $u_j \in C^\infty(\overline{\Omega}_j)$ be convex in Ω_j and such that $\det D^2 u_j = \psi_j$ in Ω_j, $u_j = v_j$ on $\partial\Omega_j$. We could choose ψ_j so that $u_j < v_j$ in Ω_j. Of course u_j tends uniformly to u. By (4.1)

$$v_j + C_5 w_j \leq u_j \leq v_j \quad \text{on } \overline{\Omega}_j.$$

By (4.2) and (4.3) we thus get

$$|Du_j| \leq C_6 \quad \text{on } \overline{\Omega}_j.$$

We may now use Theorem 3.1 to get

$$(v_j - u_j)^\alpha |D^2 u_j| \leq C_7 \quad \text{on } \overline{\Omega}_j.$$

The required result follows if we let $j \uparrow \infty$. $\square$

Finally, we want to show that if we drop the assumption (1.3) in Theorem 1.2 then it may happen that $\{u = v\} \cap \Omega \neq \emptyset$. Let Ω be the unit ball B and set $\varphi(x) := x_1^2$ for $x \in \partial B$, so that $v(x) = x_1^2$ for $x \in \overline{B}$. Let ψ be such that $\psi^{1/(n-1)}$ is smooth and $\operatorname{supp}\psi \subset \{x_1 > 0\} \cap B$. For $\varepsilon > 0$ let u_ε denote the solution of (1.1) with $\mu = \varepsilon\psi\, d\lambda$. Then $u_\varepsilon \uparrow v$ uniformly as $\varepsilon \downarrow 0$. For ε sufficiently small we thus have $u_\varepsilon \geq 0$ on $\operatorname{supp}\psi$ and then one can easily show that $u_\varepsilon = v$ on $\{x_1 \leq 0\} \cap B$.

References

[Al] A.D. Aleksandrov, *Smoothness of a convex surface of bounded Gaussian curvature*, Dokl. Akad. Nauk SSSR **36** (1942), 195-199.

[Bł] Z. Błocki, *Smooth exhaustion functions in convex domains*, Proc. Amer. Math. Soc. **125** (1997), 477-484.

[CNS] L. Caffarelli, L. Nirenberg, J. Spruck, *The Dirichlet problem for nonlinear second order elliptic equations I: Monge-Ampère equation*, Comm. Pure Appl. Math. **37** (1984), 369-402.

[CY1] S.-Y. Cheng, S.-T. Yau, *On the regularity of the Monge-Ampère equation* $\det(\partial^2 u/\partial x^i \partial x^j) = F(x,u)$, Comm. Pure Appl. Math. **33** (1977), 41-68.

[CY2] S.-Y. Cheng, S.-T. Yau, *The real Monge-Ampère equation and affine flat structures*, Proc. Symp. Diff. Geom. Diff. Eq. (Beijing, 1980) ed. S.S. Chern and W.T. Wu, vol. 1, Science Press Beijing / Gordon and Breach, New York, 1982, pp. 339-370.

[Gu] P. Guan, C^2 *a priori estimates for degenerate Monge-Ampère equations*, Duke Math. J. **86** (1997), 323-346.

[GTW] P. Guan, N.S. Trudinger, X.-J. Wang, *On the Dirichlet problem for degenerate Monge-Ampère equations*, Acta Math. **182** (1999), 87-104.

[Kr] N.V. Krylov, *Boundedly inhomogeneous elliptic and parabolic equations in a domain*, Izv. Acad. Nauk SSSR **47** (1983), 75-108 (Russian); English translation, Math. USSR-Izv. **22** (1984), 67-98.

[Po] A.V. Pogorelov, *The Minkowski Multidimensional Problem*, Wiley, New York, 1973.

[RT] J. Rauch, B. A. Taylor, *The Dirichlet problem for the multidimensional Monge-Ampère equation*, Rocky Mountain Math. J. **7** (1977), 345-364.

[TU] N.S. Trudinger, J. Urbas, *On second derivative estimates for equations of Monge-Ampère type*, Bull. Austral. Math. Soc. **28** (1983), 321-334.

[Ur] J. Urbas, *Regularity of generalized solutions of the Monge-Ampère equation*, Math Z. **197** (1988), 365-393.

[Wa] X.-J. Wang, *Some counterexamples to the regularity of Monge-Ampère equations*, Proc. Amer. Math. Soc. **123** (1995), 841-845.

Jagiellonian University, Institute of Mathematics, Reymonta 4, 30-059 Kraków, Poland, Currently at the Mathematical Institute of the Polish Academy of Sciences

E-mail address: blocki@im.uj.edu.pl