POLAR QUOTIENTS AND SINGULARITIES AT INFINITY OF POLYNOMIALS IN TWO COMPLEX VARIABLES

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Abstract

Using the notion of the maximal polar quotient we characterize the critical values at infinity of polynomials in two complex variables. As application we give a necessary and sufficient condition for a family of affine plane curves to be equisingular at infinity.

Introduction

For any reduced projective curve $C \subset \mathbf{P}^2(\mathbf{C})$ given by the homogenous equation $F = 0$ and for every point $p = (a_0 : a_1 : a_2)$ not lying on C we consider the polar curve $\nabla_p C$ (possibly with multiple components) given by the equation

$$\nabla_p C = a_0 \frac{\partial F}{\partial X_0} + a_1 \frac{\partial F}{\partial X_1} + a_2 \frac{\partial F}{\partial X_2} = 0 \ .$$

Let $L \subset \mathbf{P}^2(\mathbf{C})$ be a line. Suppose that $L \not\subset C$ and fix a point $p \in L \setminus C$.

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Then L is not a component of $\nabla_p C$ and we may consider polar quotients of C with respect to L at a point $o \in C \cap L$:

$$\frac{\mathrm{ord}_{\gamma} C}{\mathrm{ord}_{\gamma} L}, \quad \gamma \text{ runs over branches of } \nabla_p C \text{ with center at } o.$$

The polar quotients do not depend on the choice of $p \in L \setminus C$. They are identical with the local plane quotients of the germ (C, o) with respect to (L, o) ([LMW], [Pł2]).

Let

$$q_o(C, L) = \sup\{q : \ q \text{ is a polar quotient of } C \text{ with respect to } L \text{ at } o\} .$$

Clearly the set of polar quotients is empty if and only if o is reqular on C and L is not tangent to C at o. In this situation $q_o(C, L) = -\infty$ for by convention $sup\,\emptyset = -\infty$. Let $\mu_o(C)$ be the Milnor number of C at o. If $q_o(C, L) \neq -\infty$ then we have

$$\frac{\mu_o(C)}{(C \cdot L)_o - 1} + 1 \leq q_o(C, L) \leq \mu_o(C) + 1$$

where $(C \cdot L)_o$ is the intersection number of C and L at o. Note that $q_o(C, L) = \mu_o(C) + 1$ provided $(C \cdot L)_o = 2$.

The maximal polar quotient $q_o(C, L)$ is a topological invariant of the germ $(C \cup L, o)$. It can be calculated explicitly in terms of the characteristics of the branches and their intersection multiplicities ([Pł2], Theorem 1.3).

Let C be a reduced projective curve and L a line such that $L \not\subset C$. Then we put

$$q(C, L) = \sup\{q_o(C, L) : \ o \in C \cap L\} .$$

Hence $q(C, L) = -\infty$ if and only if C and L meet with multiplicity 1. If $q(C, L) \neq -\infty$ then $q(C, L)$ is a rational number and

$$1 \leq q(C, L) \leq (d-1)^2 + 1$$

where $d = \deg C$.

Now, let us consider a polynomial $f = f(X, Y) \in \mathbf{C}[X, Y]$ reduced (i.e. without multiple factors) of degree $d > 0$ and let C be the projective closure of the affine curve $f(X, Y) = 0$. Let $\mathbf{P}^2(\mathbf{C}) = \mathbf{C}^2 \cup \mathbf{L}_\infty$ where $\mathbf{L}_\infty$ is the line at infinity. We call $q(C, \mathbf{L}_\infty)$ the maximal polar quotient at infinity of the affine curve $f(X,Y)=0$. If C is the projective closure of an affine curve with multiple components then we put $q(C,\mathbf{L}_\infty)=+\infty$. Throughout this paper we use usual conventions on the symbols $-\infty$ and $+\infty$.

Our purpose is to calculate the maximal polar quotient of the curves $f(X,Y)-t=0$, $t\in\mathbf{C}$ in terms of the discriminant. We use this notion to characterize the critical values at infinity. The proof of our main result is based on the notion of the Łojasiewicz exponent at infinity (see [CK1], [CK2], [H], [P1]).

1 Result

Let $f=f(X,Y)\in\mathbf{C}[X,Y]$ be a reduced polynomial of degree $d>0$ and let $F=F(X,Y,Z)\in\mathbf{C}[X,Y,Z]$ be the homogeneous form corresponding to f. Let $C_\infty=C\cap\mathbf{L}_\infty$ be the set of points at infinity of the projective curve C given by the equation $F(X,Y,Z)=0$.

We consider the projective curves C^t (possibly with multiple factors) given by equations

$$F(X,Y,Z)-tZ^d=0\ .$$

Clearly $C^t\cap\mathbf{L}_\infty=C_\infty$. Let $\mu_p^t=\mu_p(C^t)$ be the Milnor number of C^t at $p\in C_\infty$. If a multiple component of C^t passes through p then $\mu_p^t=+\infty$. Let $\mu_p^{\min}=\inf\{\mu_p^t:\ t\in\mathbf{C}\}$ and

$$\Lambda(f)=\{t\in\mathbf{C}:\ \text{there is a } p\in C_\infty \text{ such that } \mu_p^t>\mu_p^{\min}\}\ .$$

The set $\Lambda(f)$ is finite (see [B]). The elements of $\Lambda(f)$ are called critical values at infinity of the polynomial $f:\mathbf{C}^2\to\mathbf{C}$. If C^t has a multiple component then $t\in\Lambda(f)$. Different definitions of critical values at infinity are discussed in [D]. The elements of $\mathbf{C}\setminus\Lambda(f)$ are called regular values at infinity.

For every $t\in\mathbf{C}$ we put

$$\lambda^t(f)=\sum_{p\in C_\infty}(\mu_p^t-\mu_p^{\min})$$

if C^t is reduced and $\lambda^t(f)=+\infty$ if not. Note that if C and $\mathbf{L}_\infty$ meet with multiplicity 1 then $\Lambda(f)=\emptyset$ and $q(C^t,\mathbf{L}_\infty)=-\infty$ for all $t\in\mathbf{C}$. The following assumptions will be made throughout this paper:

- f is a reduced polynomial of degree $d > 0$,
- the projective closure C of the affine curve $f(X,Y) = 0$ and the line at infinity $\mathbf{L}_\infty$ meet with multiplicity > 1 at a point of $C \cap \mathbf{L}_\infty$.

Now suppose additionaly that $\deg_Y f = \deg f = d > 1$ and let T be a new variable.Let us consider the Y-discriminant $\Delta(X,T) = \mathrm{disc}_Y(f(X,Y) - T)$ of the polynomial $f(X,Y) - T \in \mathbf{C}[X,T][Y]$. Obviously $\Delta(X,t) \neq 0$ in $\mathbf{C}[X]$ if and only if the polynomial $f(X,Y) - t$ is reduced. Let us write

$$\Delta(X,T) = \Delta_0(T)X^N + \ldots + \Delta_N(T) \text{ with } \Delta_0(T) \neq 0 \, .$$

The following proposition is well-known.

Proposition 1.1 $\Lambda(f) = \{t \in \mathbf{C} : \ \Delta_0(t) = 0\}$, $\lambda^t(f) = N - \deg_X \Delta(X,t)$.

A simple, intersection-theoretic proof of Proposition 1.1 is given in [K] (see also [GP2]).

Our main result is the following.

Theorem 1.2 *With the notation introduced above*

$$q(C^t, \mathbf{L}_\infty) = \begin{cases} d - \left(\max\limits_{i=1}^{N} \dfrac{\deg \Delta_i}{i} \right)^{-1} & \text{if } \Lambda(f) = \emptyset \\ d & \text{if } \Lambda(f) \neq \emptyset \text{ and } t \notin \Lambda(f) \\ d + \left(\min\limits_{i=0}^{\lambda^t(f)-1} \dfrac{\mathrm{ord}_t \Delta_i}{\lambda^t(f) - i} \right)^{-1} & \text{if } \Lambda(f) \neq \emptyset,\ t \in \Lambda(f) \text{ and } \\ & \quad C^t \text{ is reduced} \end{cases}$$

Recall that according to our convention $q(C^t, \mathbf{L}_\infty) = +\infty$ if C^t is not reduced. The proof of Theorem 1.2 we give in Section 3 of this paper. Now, we present some applications.

Corollary 1.3

(i) *If $\Lambda(f) = \emptyset$ then $q(C^t, \mathbf{L}_\infty) = q(C, \mathbf{L}_\infty) < d$ for all $t \in \mathbf{C}$.*

(ii) *If $\Lambda(f) \neq \emptyset$ then $q(C^t, \mathbf{L}_\infty) = d$ if $t \in \mathbf{C} \setminus \Lambda(f)$ and $q(C^t, \mathbf{L}_\infty) > d$ if $t \in \Lambda(f)$.*

The above corollary shows that $q(C^t, \mathbf{L}_\infty)$ like $\lambda^t(f)$ is an upper semi-continuous function of t. It contrasts with the fact that the maximal polar quotient is neither upper nor lower semicontinuous in the family of hypersurfaces (see [T2], p. 201).

Remark 1.4 *If $\Lambda(f) \neq \emptyset$ then $d \leq q(C^t, \mathbf{L}_\infty) \leq d + \lambda^t(f)$ for all $t \in \mathbf{C}$.*

The characterization of regular values at infinity presented below is due to Neuman and Lê Vãn Thàn who used topological methods.

Corollary 1.5 (see [L] and [NL]). *A number $t_0 \in \mathbf{C}$ is a regular value at infinity of f if and only if $q(C^{t_0}, \mathbf{L}_\infty) \leq d$.*

The family of polynomials $f(X, Y) - t$, $t \in \mathbf{C}$ is said to be equisingular at infinity if $\Lambda(f) = \emptyset$. All polynomials of an equisingular family are reduced. Theorem 1.2 implies the following criterion of equisingularity.

Corollary 1.6 *Let $f = f(X, Y) \in \mathbf{C}[X, Y]$ be a reduced polynomial of degree $d > 1$. Then the following two conditions are equivalent:*

(i) *the family $f(X, Y) - t$ is equisingular,*

(ii) *$q(C, \mathbf{L}_\infty) < d$.*

To give an application of Corollary 1.6 let us recall

The Abhyankar-Moh inequality
Let C be a reduced projective plane curve of degree $d > 1$. Suppose that $o \in \mathbf{C}$ is a unibranch point of C (i.e. through o passes only one branch of C) such that the unique tangent L to C at o does not intersect C at points different from o. Then $q_0(C, L) < d$.

The original Abhyankar-Moh inequality is stated in terms of Puiseux' expansions [AM]. The formulation above is given in [GP1] (Theorem 2.2). The Abhyankar-Moh inequality and Corollary 1.6 imply

Corollary 1.7 (the Ephraim equisingularity theorem, [E], Theorem 3.4). *Let $f = f(X, Y)$ be a reduced polynomial such that the projective closure C of the affine curve $f(X, Y) = 0$ has only one branch at infinity. Then the family $f(X, Y) - t$ is equisingular.*

Here is another application of Corollary 1.6:

Corollary 1.8 *Let $f = f(X,Y)$ be a reduced polynomial of degree $d > 1$ with $d-1$ points at infinity. Let $o \in C_\infty$ be the unique point such that $(C \cdot \mathbf{L}_\infty)_o = 2$. Then the family $f(X,Y) - t$ is equisingular if and only if the Milnor number $\mu_o = \mu_o(C)$ is strictly less then $d-1$.*

To prove Corollary 1.8 it suffices to recall that the condition $(C \cdot \mathbf{L}_\infty)_o = 2$ implies $q_o(C, \mathbf{L}_\infty) = \mu_o + 1$ and apply Corollary 1.6.

2 Proof

The proof of our main result is based on some properties of the Łojasiewicz exponents. Let f, h be two polynomials of two complex variables. We assume that $h \not\equiv$ const. The Łojasiewicz exponent at infinity $\mathcal{L}_\infty(f|h)$ of the polynomial function $f|h^{-1}(0)$ is, by definition the least upper bound of the set

$$\{\Theta \in \mathbf{R} :\ \exists C, R > 0\ \forall x \in h^{-1}(0)\ |z| \geq R \Rightarrow |f(z)| \geq C|z|^\Theta\} .$$

Then $\mathcal{L}_\infty(f|h) > -\infty$ if and only if the set

$$\{(x,y) \in \mathbf{C}^2 :\ f(x,y) = h(x,y) = 0\}$$

is finite. To calculate $\mathcal{L}_\infty(f|h)$ we use meromorphic parametrisations $p(T) = (x(T), y(T))$ where $x(T), y(T) \in \mathbf{C}((T))$ are convergent in a punctured disc at origin Laurent series such that $\min\{\operatorname{ord} x(T), \operatorname{ord} y(T)\} < 0$. In the sequel we put $\operatorname{ord} p(T) = \min\{\operatorname{ord} x(T), \operatorname{ord} y(T)\}$.

Lemma 2.1

$$\mathcal{L}_\infty(f|h) = \inf\left\{\frac{\operatorname{ord}(f \circ p)}{\operatorname{ord} p} : \begin{array}{l} p \textit{ is a meromorphic parametrisation} \\ \textit{such that } h \circ p \equiv 0 \end{array}\right\} .$$

Proof. Let Γ be the image of the set $\Delta \setminus \{0\}$ where Δ is a small disc around $0 \in \mathbf{C}$ by the mapping $\Delta \setminus \{0\} \ni t \mapsto p(t) \in \mathbf{C}$. One sees easily that $\Theta_\Gamma = \operatorname{ord}(f \circ p)/\operatorname{ord} p$ is the greatest number Θ such that

$$|f(z)| \geq \text{const.}|z|^\Theta \text{ for } z \in \Gamma \text{ and } |z| \to +\infty .$$

Let $V = \{(x,y) \in \mathbf{C}^2 : \ h(x,y) = 0\}$. Then there exists a neighbourhood at infinity $\omega \subset \mathbf{C}^2$ (i.e. $\mathbf{C}^2 \setminus \omega$ is compact) and a finite sequence of meromorphic parametrisations p_i $(i = 1, \ldots, r)$ such that

$$V \cap \omega = \bigcup_{i=1}^{r} p_i(\Delta \setminus \{0\}) \text{ (disjoint union, see [CK1]).}$$

Therefore

$$\mathcal{L}_\infty(f|h) = \min_{i=1}^{n} \{\Theta_{\Gamma_i}\}, \text{ where } \Gamma_i = p_i(\Delta \setminus \{0\})$$

and the lemma follows.

A branch $\gamma \subset \mathbf{P}^2(\mathbf{C}) = \mathbf{C}^2 \cup \mathbf{L}_\infty$ is a branch at infinity if the center of γ is on $\mathbf{L}_\infty$ and $\gamma \not\subset \mathbf{L}_\infty$. We say that a meromorphic parametrisation $p(T) = (x(T), y(T))$ is a (meromorphic) parametrisation of a branch γ at infinity if γ is given in projective coordinates by $(T^k x(T) : T^k y(T) : T^k)$ where $k = -\mathrm{ord}\, p$. Any branch at infinity has a meromorphic parametrisation. We check easily that $\mathrm{ord}\, p = -\mathrm{ord}_\gamma \mathbf{L}_\infty$ and

$$\mathrm{ord}\,(f \circ p) = \mathrm{ord}_\gamma \left(\frac{F}{Z^d}\right) = \mathrm{ord}_\gamma F - d\,\mathrm{ord}_\gamma Z$$

where $d = \deg f$ and F is the homogeneous form determined by f.

Hence we get

Proposition 2.2 *With notation introduced above:*

$$\mathcal{L}_\infty(f|h) = d - \sup \left\{ \frac{\mathrm{ord}_\gamma F}{\mathrm{ord}_\gamma Z} : \begin{array}{l} \gamma \textit{ is a branch at infinity of the projective} \\ \textit{closure of } h = 0 \end{array} \right\} .$$

For every nonzero polynomial

$$R(X,T) = R_0(T)X^N + \ldots + R_N(T), \quad R(T) \neq 0$$

we define the Puiseux exponent $\pi(R)$ of R by putting

$$\pi(R) = \begin{cases} \max_{j=1}^{N} \left(\frac{\deg R_j}{j}\right) & \text{if } R_0(T) \equiv \text{const}, \\ \infty & \text{if } R_0(0) \neq 0 \text{ and } R_0(T) \not\equiv \text{const}, \\ -\min_{j=1}^{r} \left(\frac{\mathrm{ord}_0 R_j}{r+1-j}\right) & \text{if } R_0(0) = \ldots = R_r(0) = 0 \\ & \quad \text{and } R_{r+1}(0) \neq 0 \\ 0 & \text{if } R(X,0) \equiv 0 \,. \end{cases}$$

We set $1/\infty = 0$.

Lemma 2.3 *Suppose that $R(X,0) \not\equiv 0$.*

(i) *There exist positive constants A, B such that*

$$\{(x,t):\ |x| > B \text{ and } R(x,t) = 0\} \subset \{(x,t):\ |x| > B \text{ and } A|x|^{1/\pi(R)} \le |t|\},$$

(ii) *If*

$$\{(x,t):\ |x| > B \text{ and } R(x,t) = 0\} \subset \{(x,t):\ |x| > B \text{ and } A|x|^{\nu} \le |t|\}$$

for some $A, B > 0$ and $\nu \in \mathbf{R}$ then $\nu \le 1/\pi(R)$.

Proof. If $R_0(T) \equiv$ const then Lemma 2.3 reduces to Lemma 2.1 of [P1]. If $R_0(T) \not\equiv$ const then the lemma follows from Lemmas 8.2 and 8.3 of [CK2].

The following proposition is a version of the result due to Chądzyński and Krasiński [CK2, Theorem 3.1, 3.2 and 3.3].

Proposition 2.4 *Let $\deg_Y h = \deg h > 0$. Consider*

$$R(X,T) = res_Y(h(X,Y),\ f(X,Y) - T)$$

the Y-resultant of the polynomials $h(X,Y)$, $f(X,Y) - T \in \mathbf{C}[X,T][Y]$. Assume $R(X,0) \neq 0$ in $\mathbf{C}[X]$. Then

$$\mathcal{L}_\infty(f|h) = \frac{1}{\pi(R)}\ .$$

Proof. Since $\deg_Y h = \deg h > 0$ we get

$$\{(x,y):\ |x| > B \text{ and } h(x,y) = 0\} \subset \{(x,y):\ |x| > B \text{ and } |y| \le D|x|\}$$

for some constants $B, D > 0$. Therefore the inequality $|f(z)| \ge |z|^\Theta$ is equivalent on the set $\{(x,y):\ h(x,y) = 0, |x| > B\}$ to the inequality $|f(z)| \ge C_1|z|^\Theta$ with $C_1 > 0$ dependent on C and D.

Now fix $\Theta \in \mathbf{R}$. By the definition of the resultant the following two conditions are equivalent

(1) $\{(x,y) \in \mathbf{C}^2:\ h(x,y) = 0, |x| > B\} \subset \{(x,y) \in \mathbf{C}^2:\ |f(x,y)| \le C_1|x|^\Theta\}$,

(2) $\{(x,y) \in \mathbf{C}^2:\ R(x,t) = 0, |x| > B\} \subset \{(x,t) \in \mathbf{C}^2:\ |t| \le C_1|x|^\Theta\}$.

Proposition 2.4 follows from Lemma 2.3.

Remark 2.5 *The Puiseux exponent* $\pi(R)$ *can be interpreted in terms of the Newton diagram of the polynomial* $R(X,T)$ (see [GP2], Proposition 2.8 and remark to Proposition 2.8).

Remark 2.6 *Let* $R^t(X,T) = R(X,T+t)$ *for* $t \in \mathbf{C}$*. Then* $\pi(R^t) = \pi(R)$ *if* $R_0(T) \equiv$ *const,* $\pi(R^t) = \infty$ *if* $R_0(t) \neq 0$ *and* $R(T) \not\equiv$ *const. and*

$$\pi(R^t) = -\min_{j=0}^{r(t)} \left(\frac{\operatorname{ord}_t R_j}{r(t)+1-j} \right)$$

if $R_0(t) = \ldots = R_{r(t)}(t) = 0$ *and* $R_{r(t)+1}(t) \neq 0$*. Note that* $r(t)+1 = \deg_X R(X,T) - \deg R(X,t)$*.*

Now we can prove our main result.

Proof of Theorem 1.2
Assume that $\deg_Y f = \deg f = d > 1$ and put

$$C = \text{ projective closure of the curve } \left\{ \frac{\partial f}{\partial Y} = 0 \right\} = \left\{ \frac{\partial F}{\partial Y} = 0 \right\}$$

where F is the homogeneous form corresponding to f. By Proposition 2.2 and definition of $q(C, \mathbf{L}_\infty)$ we get

$$\begin{aligned} \mathcal{L}_\infty(f|\tfrac{\partial f}{\partial Y}) &= d - \sup \left\{ \frac{\operatorname{ord}_\gamma F}{\operatorname{ord}_\gamma Z} : \begin{array}{l} \gamma \text{ is a branch at infinity of the polar} \\ \partial F/\partial Y = 0 \end{array} \right\} \\ &= d - q(C, \mathbf{L}_\infty) \ . \end{aligned}$$

Applying the formula $\mathcal{L}_\infty(f|\partial f/\partial Y) = d - q(C, \mathbf{L}_\infty)$ to the polynomials $f-t$, $t \in \mathbf{C}$ we get

$$q(C^t, \mathbf{L}_\infty) = d - \mathcal{L}_\infty(f-t|\tfrac{\partial f}{\partial Y}) \ . \tag{1}$$

Let $\Delta(X,T) = \Delta_0(T)X^N + \ldots + \Delta_N(T)$, $\Delta_0(T) \not\equiv 0$ be the Y-discriminant of $f(X,Y)-T$ i.e. the Y-resultant of $f(X,Y)-T$ and $\partial f/\partial Y(X,Y)$. Let $\Delta^t(X) = \Delta(X,t)$[1]. Using Proposition 2.4 we get

$$\mathcal{L}_\infty(f-t|\tfrac{\partial f}{\partial Y}) = \frac{1}{\pi(\Delta^t)} \tag{2}$$

provided that $f-t$ is reduced. Now, Theorem 1.2 follows from (1), (2) and Remark 2.6.

[1]Consequently $\deg \Delta^t(X) = \deg_X \Delta(X,T) - \lambda^t(f)$ by Proposition 1.1

3 Uniform estimation of polar quotients

It is natural to ask how large $q_0(C,L)$ can be when $\deg C = d$ is given. For every $d > 1$ we put

$$q(d) = \sup\{q_o(C,L) : \ \deg C = d,\ o \in C \cap L \text{ and } L \not\subset C\}\ .$$

If the definition of $q(d)$ one may restrict consideration to the pairs C, L where C and L are transverse (see [Pł2], Corollary 1.4).

One sees easily that $q_o(C,L) = b/a$ where $a, b > 0$ are integers and $a \leq \deg C - 1$. Therefore the set of all polar quotients $q_o(C,L)$ where $\deg C = d$ is finite and we get

Property 3.1 *The number $q(d)$ is rational.*

Let $k(d) = \max\{k : \text{ there is a curve } C \text{ of degree } d \text{ with } A_k \text{ singularity}\}$.

Property 3.2 $k(d) + 1 \leq q(d) \leq (d-1)^2 + 1$.

Proof. If C is of degree d with $A_{k(d)}$-singularity at $o \in C$ then $\mu_o(C) = k(d)$, $\operatorname{ord}_o C = 2$ and the polar quotient of C at o with respect to a transverse line L equals to $\mu_o(C) + 1 = k(d) + 1$. Thus $k(d) + 1 \leq q(d)$.

The second estimation follows from the inequalities

$$q_o(C,L) \leq \mu_o(C) + 1 \leq (d-1)^2 + 1\ .$$

If f is a polynomial with an isolated singularity at $0 \in \mathbf{C}^2$ then its Łojasiewicz exponent $\mathcal{L}_0(f)$ is defined to be

$$\mathcal{L}_0(f) = \inf\{\Theta > 0 : \ |f(z)| \geq \text{const.}|z|^{\Theta} \text{ for small } |z|\}\ .$$

According to Teissier [T1]: $\mathcal{L}_0(f) + 1 =$ the polar quotient of the curve $f = 0$ with respect to the generic line. Thus we get

Property 3.3

$$q(d) = \sup\left\{\mathcal{L}_0(f) : \begin{array}{l} \textit{f is a polynomial of degree } d > 1 \textit{ with} \\ \textit{an isolated singularity at } 0 \in \mathbf{C}^2 \end{array}\right\} + 1\ .$$

Problem
Compare $k(d)$ and $q(d)$ for large d. In particular estimate $\operatorname{limsup}_{d\to+\infty} q(d)/d^2$ like $\operatorname{limsup}_{d\to+\infty} k(d)/d^2$ in [GZ-N].

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