# ON THE POLAR QUOTIENTS OF AN ANALYTIC PLANE CURVE

Janusz Gwoździewicz and Arkadiusz Płoski

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Abstract. We give explicit formulae for the polar quotients of a plane many–branched singularity

## Introduction

We use standard notations: $\mathbb{C}\{X,Y\}$ is the ring of convergent power series, $\operatorname{ord} f$ is the order of $f = f(X,Y) \in \mathbb{C}\{X,Y\}$, $(f,\phi)_0$ denotes the intersection multiplicity of $f$ with $\phi$. Recall that $(f,\phi)_0 \geq (\operatorname{ord} f)(\operatorname{ord}\phi)$ with equality if and only if $f$, $\phi$ are transverse.

Let $f = f(X,Y) \in \mathbb{C}\{X,Y\}$ be a reduced power series (i.e. without multiple factors) and let $t = t(X,Y) \in \mathbb{C}\{X,Y\}$ be a regular parameter (i.e. a series of order 1) such that $t$ does not divide $f$. We consider the set of polar quotients of $f$ with respect to $t$:

$$Q(f,t) = \left\{ \frac{(f,\phi)_0}{(t,\phi)_0} : \phi \text{ is an irreducible factor of} \frac{\partial(t,f)}{\partial(X,Y)} \right\}.$$

The polar quotients were studied by many authors ([T], [M], [E], [Eph], [Ca], [LMW1], [LMW2], [Ga], [D], [LP]). Recently, an explicit formula for the maximal polar quotient $q(f,t) = \sup Q(f,t)$ has been given ([P], Theorem 1.3). In this note

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we give similar formulae for all polar quotients. Like in [P] our main tool is the Kuo–Lu lemma ([KL], Lemma 3.3 and Section 2 of this paper). Instead of using Puiseux' trees (see [KL], [E]) we make our calculations by means of Puiseux' date (Section 3). The notion of symmetric power explained in [Wh], Appendix V turns out very useful.

Let $A$ be a nonempty set. Then $A^p_{\mathrm{sym}}$ is the set of all $p$–tuples regarded as unordered. If $\alpha = \langle a_1, \dots, a_p\rangle \in A^p_{\mathrm{sym}}$ then $|\alpha| = \{a_1, \dots, a_p\}$ is the set corresponding to $\alpha$. We put $\deg \alpha = p$. If $\alpha = \langle a_1, \dots, a_p\rangle$ and $\beta = \langle b_1, \dots, b_q\rangle$ then $\alpha \oplus \beta = \langle a_1, \dots, a_p, b_1, \dots, b_q\rangle$. Clearly $|\alpha \oplus \beta| = |\alpha| \cup |\beta|$ and $\deg \alpha \oplus \beta = \deg \alpha + \deg \beta$.

Instead of $\langle \underbrace{a_1, \dots, a_1}_{m_1 \text{ times}}, \dots, \underbrace{a_p, \dots, a_p}_{m_p \text{ times}} \rangle$ we write $\langle a_1 : m_1, \dots, a_p : m_p\rangle$.

## 1. Main result

The semigroup $\Gamma_0(f)$ of an irreducible power series $f \in \mathbb{C}\{X, Y\}$ is the set of all intersection numbers $(f, \phi)_0$ where $\phi$ runs over all power series $\phi \in \mathbb{C}\{X, Y\}$ such that $f$ does not divide $\phi$. Let $t = t(X, Y)$ be a regular parameter. Suppose that $\operatorname{ord} f > 1$ and put $p = (t, f)_0$. Let $\bar b_0, \dots, \bar b_h$ be the $p$-minimal system of generators of $\Gamma_0(f)$ determined by conditions

(i) $\bar b_0 = p$,

(ii) $\bar b_k = \min(\Gamma_0(f) \setminus (\mathbb{N}\bar b_0 + \dots + \mathbb{N}\bar b_{k-1}))$ for $k = 1, \dots, h$,

(iii) $\Gamma_0(f) = \mathbb{N}\bar b_0 + \dots + \mathbb{N}\bar b_h$.

Let $B_k = \mathrm{GCD}(\bar b_0, \dots, \bar b_k)$ be the greatest common divisor of $\bar b_0, \dots, \bar b_k$.

**Proposition 1.1.** ([M], [Eph]). *Let $f \in \mathbb{C}\{X, Y\}$ be an irreducible power series with* $\operatorname{ord} f > 1$. *Then*

$$Q(f, t) = \left\{ \frac{B_{k-1}\bar b_k}{\bar b_0} : k = 1, \dots, h \right\}.$$

The above formula was given by Merle ([M], Théorème 3.1) in transverse case. Then Ephraim ([Eph], Lemma 1.6) observed that Merle's result holds also for polar quotients with respect to a regular parameter $t$. In Section 4 we give a new proof of (1.1). Recall that the sequence $B_{k-1}\bar b_k/\bar b_0$ is strictly increasing.

If $\operatorname{ord} f = 1$ then $Q(f, t) = \emptyset$ if $f$, $t$ are transverse and $Q(f, t) = \{1\}$ if $f$, $t$ are not.

*Remark 1.2.* Using the inversion formulae for plane branches (see [P], proof of Proposition 1.1) one can calculate the polar quotients of a branch $f = 0$ in terms of the minimal system of generators $\bar\beta_0, \dots, \bar\beta_g$ of $\Gamma_0(f)$. Let $e_k = \mathrm{GCD}(\bar\beta_0, \dots, \bar\beta_k)$ for $k = 0, 1, \dots, g$. Then

$$Q(f, t) = \left\{ \frac{e_{k-1}\bar\beta_k}{\bar\beta_0} : k = 1, \dots, g \right\} \cup O(f, t)$$

where $O(f, t) = \emptyset$ if $(f, t)_0 = \bar\beta_0$ or $(f, t)_0 = \bar\beta_1$ and $O(f, t) = \{\operatorname{ord} f\}$ otherwise.

Let $\phi$, $\psi \in \mathbb{C}\{X,Y\}$ be irreducible power series. The contact coefficient (in the sense of Hironaka) with respect to a regular parameter $t \in \mathbb{C}\{X,Y\}$ is the rational number

$$h(\phi,\psi;t) = \frac{(\phi,\psi)_0}{(t,\psi)_0}$$

In [LMW1] this notion is introduced in the case where $t$ and $\psi$ are transverse. Then $h(\phi,\psi;t) = (\phi,\psi)_0/\operatorname{ord}\psi$ and we write $h(\phi,\psi)$ instead of $h(\phi,\psi;t)$.

Let us consider a reduced power series $f = f_1\cdots f_r$ with $r>1$ irreducible factors. Fix $i \in \{1,\dots,r\}$. For every real number $\tau > 0$ we put
$J_i(\tau) = \{j \in \{1,\dots,r\} : h(f_i,f_j;t) \le \tau\}$ and $J_i(\tau)^c = \{1,\dots,r\}\setminus J_i(\tau)$.

By convention $h(f_i,f_i;t) = +\infty$, hence $i \in J_i(\tau)^c$. We put

$$q_i(\tau) = \frac{1}{(t,f_i)_0}\left\{\Big(\sum_{j\in J_i(\tau)^c}(t,f_j)_0\Big)\tau + \sum_{j\in J_i(\tau)}(f_i,f_j)_0\right\} \qquad \text{for } \tau>0.$$

To write the above formula more explicitly suppose $i=1$ and assume
$h(f_1,f_r;t) \le h(f_1,f_{r-1};t) \le \cdots \le h(f_1,f_2;t) < h(f_1,f_1;t) = +\infty$. Then

$$q_1(\tau) = \begin{cases} \frac{(t,f_1\cdots f_j)_0}{(t,f_1)_0}\tau + \frac{(f_1,f_{j+1}\cdots f_r)_0}{(t,f_1)_0}, & h(f_1,f_{j+1};t) \le \tau < h(f_1,f_j;t), j<r \\ \frac{(t,f)_0}{(t,f_1)_0}\tau, & \tau < h(f_1,f_r;t) \end{cases}.$$

It is easy to see that the functions $q_i$ are piecewise linear, continuous and strictly increasing. Note that $q_i$ is determined by the intersection multiplicities of $i$-th branch $f_i=0$ with the remaining branches and with the smooth curve $t=0$. Let $H_i(f,t) = \{h(f_i,f_j;t) : j \ne i\}$ be the set of contact coefficients of the branch $f_i = 0$ with branches $f_j = 0$ $(j\ne i)$. The main results of this paper is

**Theorem 1.3.** *Let $f = f_1\cdots f_r$ be a reduced power series with $r>1$ irreducible factors. Then*

$$Q(f,t) = \bigcup_{i=1}^{r} q_i(Q(f_i,t)\cup H_i(f,t)).$$

We give the proof of (1.3) in Section 4. We put $Q_i(f,t) = q_i(Q(f_i,t)\cup H_i(f,t))$ and call the elements of $Q_i(f,t)$ polar quotients associated with the branch $f_i=0$.

A polar quotient can be associated with more than one branch. From Theorem 1.3 it follows that the polar quotients of the curve $f=0$ with respect to a regular parameter $t$ depend only on the equisingularity type of the curve $tf=0$.

If $t$ and $f$ are transverse then the polar quotients of $f=0$ do not depend on the regular parameter $t$ and are determined by the equisingularity type of the curve $f=0$. Then we write $Q_i(f)$ instead of $Q_i(f,t)$ and $Q(f)$ instead of $Q(f,t)$.

Using (1.3) we obtain easily the formula for the maximal polar quotient
$q(f,t) = \sup Q(f,t)$ given in [P]. Obviously $q(f,t) = \max_{i=1}^{r}\{\max Q_i(f,t)\}$, hence it suffices to calculate the maximal polar quotients associated to branches.

**Corollary 1.3.**

$$\max Q_i(f,t) = \max\{\, q(f_i,t), \max_{j\neq i}\{h(f_i,f_j;t)\}\,\} + \frac{1}{(t,f_i)_0}\sum_{j\neq i}(f_i,f_j)_0$$

*Proof.* Let $\tau_i = \max(Q(f_i,t) \cup H_i(f,t)) = \max\{\, q(f_i,t), \max_{j\neq i}\{\, h(f_i,f_j,t)\,\}\,\}$. Then $J_i(\tau_i) = \{1,\dots,r\}\setminus\{i\}$ and $q_i(\tau_i) = \tau_i + \frac{1}{(t,f_i)_0}\sum_{j\neq i}(f_i,f_j)_0$. We get then $\max Q_i(f,t) = q_i(\tau_i)$ for $q_i$ is an increasing function.

*Remark 1.4.* For every $i \in \{1,\dots,r\}$ the rational numbers $q(f_i,t)+\frac{1}{(t,f_i)_0}\sum_{j\neq i}(f_i,f_j)_0$ (provided that $q(f_i,t) \neq -\infty$) and $\max_{j\neq i}\{\, h(f_i,f_j;t)\,\} + \frac{1}{(t,f_i)_0}\sum_{j\neq i}(f_i,f_j)_0$ are polar quotients of $f=0$.

**Corollary 1.5.** *If $f = f_1\cdots f_r$ $(r>1)$ with transverse factors $f_i$, $f_j$ for $i\neq j$ then*

$$Q(f) = \bigcup_{i=1}^{r}\{\, \tau_i + \operatorname{ord} f - \operatorname{ord} f_i : \tau_i \in Q(f_i)\,\} \cup \{\operatorname{ord} f\}$$

*Proof.* Let us calculate $Q_1(f)$. It is easy to see that $J_1(\tau) = \{\, j \in \{2,\dots,r\} : \operatorname{ord} f_1 \leq \tau\,\}$. Hence $J_1(\tau) = \emptyset$ if $\tau < \operatorname{ord} f_1$ and $J_1(\tau) = \{2,\dots,r\}$ if $\tau \geq \operatorname{ord} f_1$. Obviously $H_1(f) = \{\operatorname{ord} f_1\}$. Therefore we get

$$q_1(\tau) = \begin{cases} \tau + \operatorname{ord} f - \operatorname{ord} f_1, & \tau \geq \operatorname{ord} f_1 \\ \frac{\operatorname{ord} f}{\operatorname{ord} f_1}\tau, & \tau < \operatorname{ord} f_1 \end{cases}$$

For every $\tau_1 \in Q(f_1)$ one gets $\tau_1 \geq \operatorname{ord} f_1$. Consequently $Q_1(f) = \{\, q_1(\tau_1) : \tau_1 \in Q(f_1)\,\} \cup q_1(\operatorname{ord} f_1) = \{\, \tau_1 + \operatorname{ord} f - \operatorname{ord} f_1 : \tau_1 \in Q(f_1)\,\} \cup \{\operatorname{ord} f\}$ and the corollary follows.

To give another application of the main results we calculate polar quotients of "diagonal curves" introduced by Delgado [D].

**Corollary 1.6.** *Let $f = f_1\cdots f_r$ be a reduced power series. Suppose that $\Gamma_0(f_i)$ and $(t,f_i)_0$ do not depend on $i \in \{1,\dots,r\}$, $\Gamma_0(f_i) = \langle \bar b_0, \bar b_1, \dots, \bar b_h\rangle$ and $(f_i,f_j)_0 = B_{h-1}\bar b_h + c$ with $c>0$, for $i\neq j$. Then*

$$Q(f,t) = \left\{ r\frac{B_{k-1}\bar b_k}{\bar b_0} : k = 1,\dots,h \right\} \cup \left\{ r\frac{B_{h-1}\bar b_h + c}{\bar b_0}\right\}.$$

*Proof.* By Proposition 1.1 we get $Q(f_i,t) = \{\, B_{k-1}\bar b_k/\bar b_0 : k = 1,\dots,h\,\}$ on the other hand $H_i(f,t) = \{\, (B_{h-1}\bar b_h + c)/\bar b_0\,\}$. It is easy to check that $q_i(\tau) = r\tau$ if $\tau < (B_{h-1}\bar b_h + c)/\bar b_0$ and $q_i(\tau) = \tau + (r-1)(B_{h-1}\bar b_h + c)/\bar b_0$ if $\tau \geq (B_{h-1}\bar b_h + c)/\bar b_0$. Now we get (1.6) from Theorem 1.3.

## 2. Zeros of a derivative

Let $\mathbb{C}\{X\}^* = \bigcup_{n\geq 1}\mathbb{C}\{X^{1/n}\}$ be the ring of Puiseux series. If $f = f(X,Y) \in \mathbb{C}\{X,Y\}$ is a power series $Y$-regular of order $p > 0$ i.e. such that $\operatorname{ord} f(0,Y) = p$ then the equation $f(X,Y) = 0$ has in $\mathbb{C}\{X\}^*$ $p$ roots $y_1 = y_1(X), \dots, y_p = y_p(X)$ of positive order counted with multiplicities. We put $\operatorname{Zer} f = \langle y_1,\dots,y_p\rangle$. Let $S \subset \mathbb{C}\{X\}^*$ be a set of Puiseux series. If $y_1, \dots, y_q \in S$ and $y_{q+1}, \dots, y_p \notin S$ then we put $\operatorname{Zer} f \cap S = \langle y_1,\dots,y_q\rangle$. Let us recall

**The Kuo–Lu lemma.** ([KL], lemma 3.3) *Suppose that $f = f(X,Y)$ is $Y$-regular of order $p > 1$. Let $\operatorname{Zer} f = \langle y_1,\dots,y_p\rangle$ and $\operatorname{Zer}\frac{\partial f}{\partial Y} = \langle z_1,\dots,z_{p-1}\rangle$. Then for every $i \in \{1,\dots,p\}$:*

$$\langle \operatorname{ord}(z_1-y_i),\dots,\operatorname{ord}(z_{p-1}-y_i)\rangle = \langle \operatorname{ord}(y_1-y_i),\dots,\widehat{\operatorname{ord}(y_i-y_i)},\dots,\operatorname{ord}(y_p-y_i)\rangle.$$

It is convenient to prove the lemma for formal power series with coefficients in $\mathbb{C}\{X\}^*$. Replacing $f(X,Y)$ by $f(X,Y+y_i)$ we may assume $y_i = 0$. Then the series $f(X,Y)/Y$ and $\frac{\partial f}{\partial Y}$ have the same Newton diagram (see [Wa], Chapter IV) and the lemma follows from the fact that the Newton diagram determines the order of roots and the number of roots of given order. For more details see [GP1].

For every $u \in \mathbb{C}\{X\}^*$ and $r > 0$ we put
$S(u,r) = \{\, v \in \mathbb{C}\{X\}^* : \operatorname{ord}(u-v) = r \,\}$,
$B(u,r) = \{\, v \in \mathbb{C}\{X\}^* : \operatorname{ord}(u-v) > r \,\}$,
$\bar{B}(u,r) = \{\, v \in \mathbb{C}\{X\}^* : \operatorname{ord}(u-v) \geq r \,\}$.

According to the given definitions $\deg(\operatorname{Zer} f \cap S)$ is equal to the number of roots counted with multiplicities of the equation $f(X,Y) = 0$ lying in the set $S$ of Puiseux series. If $|\operatorname{Zer} f| \cap S = \emptyset$ then we put $\deg(\operatorname{Zer} f \cap S) = 0$.

**Lemma 2.1.** *Suppose that $f = f(X,Y)$ is $Y$-regular of order $p > 1$. Then for every $y \in \operatorname{Zer} f$ and for every $r > 0$*

*(i)* $\deg(\operatorname{Zer} f \cap S(y,r)) = \deg(\operatorname{Zer}\frac{\partial f}{\partial Y} \cap S(y,r))$,

*(ii)* $\deg(\operatorname{Zer} f \cap B(y,r)) = \deg(\operatorname{Zer}\frac{\partial f}{\partial Y} \cap B(y,r)) + 1$,

*(iii)* $\deg(\operatorname{Zer} f \cap \bar{B}(y,r)) = \deg(\operatorname{Zer}\frac{\partial f}{\partial Y} \cap \bar{B}(y,r)) + 1$.

*Proof.* The part (i) is a reformulation of the Kuo–Lu lemma, (ii) and (iii) follow from (i).

**Proposition 2.2.** *Under assumptions of Lemma 2.1:*

*(i) for every $z \in |\operatorname{Zer}\frac{\partial f}{\partial Y}|$ there exist $y_1, y_2 \in \operatorname{Zer} f$, $y_1 \neq y_2$ such that* $\operatorname{ord}(z-y_1) = \operatorname{ord}(y_1-y_2) \geq \operatorname{ord}(z-y)$ *for all $y \in |\operatorname{Zer} f|$,*

*(ii) for every $y_1, y_2 \in \operatorname{Zer} f$, $y_1 \neq y_2$ there exists $z \in |\operatorname{Zer}\frac{\partial f}{\partial Y}|$ such that* $\operatorname{ord}(z-y_1) = \operatorname{ord}(y_1-y_2) \geq \operatorname{ord}(z-y)$ *for all $y \in |\operatorname{Zer} f|$.*

*Proof.*

(i) Fix $z \in |\operatorname{Zer}\frac{\partial f}{\partial Y}|$. Let $y_1 \in |\operatorname{Zer} f|$ be such that $\operatorname{ord}(z-y_1) = \max\{\operatorname{ord}(z-y) : y \in |\operatorname{Zer} f|\,\}$. By the Kuo–Lu lemma there exists $y_2 \in |\operatorname{Zer} f|$ such that $\operatorname{ord}(z-y_1) = \operatorname{ord}(z-y_2)$ and we are done.

(ii) Suppose $y_1 \neq y_2$ are given roots of the equation $f(X,Y) = 0$. Let $\bar{B} = \bar{B}(y_1, \operatorname{ord}(y_1-y_2))$. Let $B_1, \dots B_s$ be a sequence of pairwise disjoint balls such that $\bigcup_{i=1}^{s} B_i = \bigcup\{\, B(y, \operatorname{ord}(y_1-y_2)) : y \in |\operatorname{Zer} f| \cap \bar{B}\,\}$.

Note that

1) $s \geq 2$ for $B(y_1, \operatorname{ord}(y_1 - y_2)) \cap B(y_2, \operatorname{ord}(y_1 - y_2)) = \emptyset$,
2) $B_i \subset \bar{B}$ $(i = 1, \ldots, s)$ because the center of $B_i$ belongs to $\bar{B}$ and $B_i$, $\bar{B}$ have the same radius.
3) $\bigcup_{i=1}^{s}(B_i \cap |\operatorname{Zer} f|) = \bar{B} \cap |\operatorname{Zer} f|$ for $B_i$ cover $\bar{B} \cap |\operatorname{Zer} f|$ and are pairwise disjoint.

Using lemma 2.1 we calculate $\deg\big((\bar{B} \setminus \bigcup_{i=1}^{s} B_i) \cap \operatorname{Zer} \frac{\partial f}{\partial Y}\big) = \deg(\bar{B} \cap \operatorname{Zer} \frac{\partial f}{\partial Y}) - \sum_{i=1}^{s} \deg(B_i \cap \operatorname{Zer} \frac{\partial f}{\partial Y}) = (\deg(\bar{B} \cap \operatorname{Zer} f) - 1) - \sum_{i=1}^{s}(\deg(B_i \cap \operatorname{Zer} f) - 1) = \deg(\bar{B} \cap \operatorname{Zer} f) - \sum_{i=1}^{s} \deg(B_i \cap \operatorname{Zer} f) + (s-1) = s - 1 > 0$.

Therefore the set $(\bar{B} \setminus \bigcup_{i=1}^{s} B_i) \cap |\operatorname{Zer} \frac{\partial f}{\partial Y}|$ is nonempty.
Let $z \in (\bar{B} \setminus \bigcup_{i=1}^{s} B_i) \cap |\operatorname{Zer} \frac{\partial f}{\partial Y}|$. Then $\operatorname{ord}(z - y_1) \geq \operatorname{ord}(y_1 - y_2) \geq \operatorname{ord}(z - y)$ for all $y \in |\operatorname{Zer} f| \cap \bar{B}$. In particular $\operatorname{ord}(z - y_1) = \operatorname{ord}(y_1 - y_2)$. If $y \in |\operatorname{Zer} f| \setminus \bar{B}$ then $\operatorname{ord}(y - y_1) < \operatorname{ord}(y_1 - y_2) = \operatorname{ord}(z - y_1)$. Hence we get $\operatorname{ord}(z - y_1) \geq \operatorname{ord}(z - y)$ for $y \in |\operatorname{Zer} f|$.

For given $y_1$, $y_2 \in |\operatorname{Zer} f|$, $y_1 \neq y_2$ we put

$$L_{y_1,y_2} = |\operatorname{Zer} f| \cap \bar{B}(y_1, \operatorname{ord}(y_1 - y_2)),$$

$$L^c_{y_1,y_2} = |\operatorname{Zer} f| \setminus L_{y_1,y_2},$$

$$l_{y_1,y_2} = \sum_{y \in L^c_{y_1,y_2}} \operatorname{ord}(y_1 - y) + (\# L_{y_1,y_2}) \operatorname{ord}(y_1 - y_2).$$

**Theorem 2.3.** *Let $f = f(X,Y)$ be reduced $Y$-regular with $\operatorname{ord} f(0,Y) > 1$. Then* $\{\, l_{y_1,y_2} : y_1, y_2 \in |\operatorname{Zer} f|, y_1 \neq y_2 \,\} = \{\, \operatorname{ord} f(X, z(X)) : z \in |\operatorname{Zer} \frac{\partial f}{\partial y}| \,\}$.

*Proof.* For every $z = z(X) \in \mathbb{C}\{X\}^*$ without constant term one has

$$(1)\quad \operatorname{ord} f(X, z(X)) = \sum_{y \in |\operatorname{Zer} f|} \operatorname{ord}(z-y) = \sum_{y \in L{y_1,y_2}^c} \operatorname{ord}(z-y) + \sum_{y \in L{y_1,y_2}} \operatorname{ord}(z-y).$$

Let $y_1$, $y_2 \in \operatorname{Zer} f$ and $z \in \left|\operatorname{Zer} \frac{\partial f}{\partial Y}\right|$ be such that $\operatorname{ord}(z - y_1) = \operatorname{ord}(z - y_2) \geq \operatorname{ord}(z - y)$ for all $y \in |\operatorname{Zer} f|$. We will check that

$$(2)\qquad \operatorname{ord}(z - y) = \begin{cases} \operatorname{ord}(y - y_1), & y \in L^c_{y_1,y_2} \\ \operatorname{ord}(y_2 - y_1), & y \in L_{y_1,y_2} \end{cases}$$

Indeed, if $y \in L^c_{y_1,y_2}$ then $\operatorname{ord}(z - y) = \operatorname{ord}(y - y_1)$ for $\operatorname{ord}(y - y_1) < \operatorname{ord}(y_2 - y_1) = \operatorname{ord}(z - y_1)$.

If $y \in L_{y_1,y_2}$ then
$\operatorname{ord}(z - y) \geq \min\{\operatorname{ord}(z - y_1), \operatorname{ord}(y_1 - y)\} = \min\{\operatorname{ord}(y_2 - y_1), \operatorname{ord}(y_1 - y)\} = \operatorname{ord}(y_2 - y_1)$. On the other hand $\operatorname{ord}(z - y) \leq \operatorname{ord}(z - y_1)$ for all $y \in |\operatorname{Zer} f|$ and $\operatorname{ord}(z - y) \leq \operatorname{ord}(y_2 - y_1)$. Hence $\operatorname{ord}(z - y) = \operatorname{ord}(y_2 - y_1)$ for $y \in L_{y_1,y_2}$. Now, Theorem 2.3 follows from (1), (2) and Proposition 2.2.

## 3. Puiseux' date

We use the notations and definitions from [P], Section 3. Let $f = f(X,Y) \in \mathbb{C}\{X,Y\}$ be an irreducible power series $Y$-regular of order $p > 0$. Let $(b_0,\dots,b_h)$ be the characteristic of $f$. We put $B_k = \mathrm{GCD}(\bar{b}_0,\dots,\bar{b}_k)$ for $k = 0, 1, \dots, h$ and $\bar{b}_k = b_k + \frac{1}{B_{k-1}}\sum_{i=1}^{k-1}(B_{i-1}-B_i)b_i$ for $k = 0, 1, \dots, h$. Then the sequence $\bar{b}_0 = p$, $\bar{b}_1, \dots, \bar{b}_h$ is the $p$-minimal system of generators of the semigroup $\Gamma_0(f)$ (see [P], Proposition 3.2). We put $P(f) = \langle \frac{b_1}{b_0} : B_0 - B_1, \dots, \frac{b_h}{b_0} : B_{h-1} - B_h\rangle$ and call $P(f)$ Puiseux' date of the branch $f = 0$. Clearly $|P(f)| = \{\, \frac{b_1}{b_0}, \dots, \frac{b_h}{b_0}\,\}$.

**Proposition 3.1.** *For every $i \in \{1,\dots,p\}$:*

$$\langle \mathrm{ord}(y_1 - y_i), \dots, \widehat{\mathrm{ord}(y_i - y_i)}, \dots, \mathrm{ord}(y_p - y_i)\rangle = P(f).$$

*Proof.* It is a reformulation of Proposition 3.1 from [P].

The following notation will be useful. Let $\alpha \in \mathbb{R}^p_{\mathrm{sym}}$ and let $K > 0$ be a real number. Write $\alpha = \langle a_1,\dots,a_q,a_{q+1},\dots,a_p\rangle$ where $a_1, \dots, a_q < K$ and $K \le a_{q+1}, \dots, a_p$. Then we put $\alpha_K = \langle a_1,\dots,a_q,K,\dots,K\rangle \in \mathbb{R}^{p+1}_{\mathrm{sym}}$ with $K$ repeated $p-q+1$ times. We let $\alpha_{(+\infty)} = \alpha$.

Let $g = g(X,Y) \in \mathbb{C}\{X,Y\}$ be an irreducible $Y$-regular power series. Let $\mathrm{Zer}\, g = \langle z_1(X),\dots,z_q(X)\rangle$. Then the order of contact of $f$ and $g$ is defined to be

$$\mathrm{cont}(f,g) = \max\{\, \mathrm{ord}(y_i(X) - z_j(X)) : 1 \le i \le p, 1 \le j \le q\,\}.$$

Note that for every $K > 0$

$$P(f)_K = \langle \frac{b_1}{b_0} : B_0 - B_1, \dots, \frac{b_{k-1}}{b_0} : B_{k-2} - B_{k-1}, K : B_{k-1}\rangle$$

where $k > 0$ is the smallest integer such that $K \le \frac{b_k}{b_0}$.

**Proposition 3.2.** *Let $z = z(X) \in |\,\mathrm{Zer}\, g|$. Then*

$$\langle \mathrm{ord}(z - y_1), \dots, \mathrm{ord}(z - y_p)\rangle = P(f)_{\mathrm{cont}(f,g)}.$$

*Proof.* It is easy to check that $\max\{\, \mathrm{ord}(z-y_1),\dots,\mathrm{ord}(z-y_p)\,\} = \mathrm{cont}(f,g)$. We may assume that $\mathrm{cont}(f,g) = \mathrm{ord}(z - y_p)$. For every $i \in \{1,\dots,p\}$ $\mathrm{ord}(z-y_i) < \mathrm{ord}(z-y_p)$ and then $\mathrm{ord}(z-y_i) = \mathrm{ord}(y_i - y_p)$ or $\mathrm{ord}(z-y_i) = \mathrm{ord}(z-y_p)$. Therefore (3.2) follows from (3.1).

For every $\alpha = \langle a_1,\dots,a_p\rangle \in \mathbb{R}^p_{\mathrm{sym}}$ we put $\sum\alpha = \sum_{i=1}^p a_i$. Clearly $\sum(\alpha \oplus \beta) = \sum\alpha + \sum\beta$.

**Proposition 3.3.** *Let $f = f(X,Y) \in \mathbb{C}\{X,Y\}$ and $g = g(X,Y) \in \mathbb{C}\{X,Y\}$ be irreducible $Y$-regular power series. Suppose $f$ is of characteristic $(b_0,\dots,b_h)$. Then*

*(i)* $\sum P(f)_{b_k/b_0} = \frac{B_{k-1}\bar{b}_k}{b_0}$,

*(ii)* $\sum P(f)_{\mathrm{cont}(f,g)} = \frac{(f,g)_0}{(X,g)_0}$,

*(iii) if* $K \leq \operatorname{cont}(f,g)$ *then* $\sum P(g)_K = \frac{(X,g)_0}{(X,f)_0} \sum P(f)_K$.

*Proof.* Property (i) follows from the definitions. To check (ii) recall that $\frac{(f,g)_0}{(X,g)_0} = \operatorname{ord} f(X, z(X))$ where $z = z(X) \in |\operatorname{Zer} g|$.
Hence $\frac{(f,g)_0}{(X,g)_0} = \sum_{i=1}^{p} \operatorname{ord}(z(X) - y_i(X)) = \sum P(f)_{\operatorname{cont}(f,g)}$ by Proposition 3.2. Now fix $K \leq \operatorname{cont}(f,g)$. Let $(b'_0, \ldots, b'_{h'})$ be the characteristic of $g$.
Let $B'_k = \operatorname{GCD}(b'_0, \ldots, b'_k)$ and let $l > 0$ be the smallest integer such that $K \leq b'_l/b'_0$. Then

$$\sum P(g)_K = \sum_{i=1}^{l-1} \frac{b'_i}{b'_0}(B'_{i-1} - B'_i) + B'_{l-1}K$$

Let $k > 0$ be the smallest integer such that $\operatorname{cont}(f,g) \leq b_k/b_0$. Then $k \leq h'$, $\operatorname{cont}(f,g) \leq b'_k/b'_0$ and $b_i/b_0 = b'_i/b'_0$, $B_i/b_0 = B'_i/b'_0$ for $i < k$ and $i < l$ for $l \leq k$ (recall that $K \leq \operatorname{cont}(f,g)$).

Hence we get

$$\sum P(g)_K = \Big(\sum_{i=1}^{l-1} \frac{b_i}{b_0}(B_{i-1} - B_i) + B_{l-1}K\Big)\frac{b'_0}{b_0} = \frac{(X,g)_0}{(X,f)_0}\sum P(f)_K.$$

Consider a reduced power series $f = f_1 \cdots f_r$ with $r > 1$ factors. For every $i \in \{1, \ldots, r\}$ we put $K_{ij} = \operatorname{cont}(f_i, f_j)$ if $i \neq j$ and $K_{ii} = +\infty$. We define Puiseux' date $P_i(f)$ of $f$ with respect to the irreducible factor $f_i$ by putting

$$P_i(f) = P(f_1)_{K_{i1}} \oplus \cdots \oplus P(f_r)_{K_{ir}}.$$

**Proposition 3.4.**

$$|P_i(f)| = |P(f_i)| \cup \{ K_{i1}, \ldots, K_{ii}, \ldots, K_{ir} \}$$

*Proof.* We have $|P_i(f)| = \bigcup_{j=1}^{r} |P(f_j)_{K_{ij}}|$. Let $b_0, \ldots, b_h$ be the characteristic of the branch $f_i$. Then $|P(f_i)| = \{ b_1/b_0, \ldots, b_h/b_0 \}$
and $|P(f_j)_{K_{ij}}| = \{ b_1/b_0, \ldots, b_{k(j)-1}/b_0, \operatorname{cont}(f_i, f_j) \}$ where $k(j)$ is the smallest integer $k > 0$ such that $\operatorname{cont}(f_i, f_j) \leq b_k/b_0$.

**Proposition 3.5.** *Let* $y_i \in |\operatorname{Zer} f_j|$ *for some* $j \in \{1, \ldots, r\}$. *Then*
$\langle \operatorname{ord}(y_i - y_1), \ldots, \operatorname{ord}\widehat{(y_i - y_i)}, \ldots, \operatorname{ord}(y_i - y_p)\rangle = P_j(f)$

*Proof.* Proposition 3.5 follows from (3.1), (3.2) and form the definition of $P_j(f)$.

## 4. Proof

A local isomorphism $\Phi$ is a pair of power series without constant term such that $\operatorname{Jac}\Phi(0,0) \neq 0$. It is easy to check the following

**Lemma 4.1.** *Let* $\Phi$ *be a local isomorphism. A rational number* $q$ *is a polar quotient of* $f$ *with respect to a regular parameter* $t$ *if and only if* $q$ *is a polar quotient of* $f \circ \Phi$ *with respect to* $t \circ \Phi$.

Therefore to prove (1.1) and (1.3) it suffices to consider the case $t = X$.

**Proposition 4.2.** *Let $f_1 \cdots f_r$ be a reduced power series with irreducible factors $f_i$. Suppose that $f$ is $Y$-regular of order $p > 0$ and let $p_i : \mathbb{R}_+ \to \mathbb{R}$ be given by the formula*

$$p_i(K) = \sum P_i(f)_K \qquad \text{for } K > 0.$$

*Then $Q(f, X) = \bigcup_{i=1}^{r} p_i(|P_i(f)|)$.*

*Proof.* Let $\operatorname{Zer} f = \langle y_1, \dots, y_p \rangle$ and $\operatorname{Zer} \frac{\partial f}{\partial Y} = \langle z_1, \dots, z_{p-1} \rangle$. It is easy to see that $Q(f, X) = \{ \operatorname{ord} f(X, z_1(X)), \dots, \operatorname{ord} f(X, z_{p-1}(X)) \}$. Indeed, if $\phi$ is an irreducible factor of $\frac{\partial f}{\partial Y}$ and $z_i = z_i(X) \in \operatorname{Zer} \phi$ then $\frac{(f,\phi)_0}{(X,\phi)_0} = \operatorname{ord} f(X, z_i(X))$.

Let $l_{y_i,y_j}$ $(i \neq j)$ be the quantities introduced in Section 3 of this paper. It is easy to see that

$$l_{y_i,y_j} = \sum \langle \operatorname{ord}(y_i - y_1), \dots, \widehat{\operatorname{ord}(y_i - y_i)}, \dots, \operatorname{ord}(y_i - y_p) \rangle_{\operatorname{ord}(y_i - y_j)}$$

for $i \neq j$.

Now suppose that $y_i \in |\operatorname{Zer} f_1|$ and let $K_1 = \operatorname{ord}(y_i - y_j)$ for a $j \neq i$. By Proposition 3.5 we get

$$l_{y_i,y_j} = \sum P_1(f)_{K_1} = p_1(K_1)$$

and consequently $\{ l_{y_i,y_j} : j \neq i \} = \{ p_1(K_1) : K_1 \in |P_1(f)| \} = p_1(|P_1(f)|)$.

Similarly $\{ l_{y_i,y_j} : j \neq i \} = p_2(|P_2(f)|)$
if $y_i \in |\operatorname{Zer} f_2|$, ..., $\{ l_{y_i,y_j} : j \neq i \} = p_r(|P_r(f)|)$ if $y_i \in |\operatorname{Zer} f_r|$. Therefore by Theorem 2.3 we get $Q(f, X) = \{ l_{y_i,y_j} : j \neq i \} = p_1(|P_1(f)|) \cup \cdots \cup p_r(|P_r(f)|)$.

Now we can give

*Proof of Proposition 1.1.* We assume $t = X$. Let $f$ be an irreducible power series $Y$-regular of order $p > 1$. Let $(b_0, b_1, \dots, b_h)$ be the characteristic of $f$. According to Proposition 4.2 we have $Q(f, X) = p(|P(f)|)$ where $p : \mathbb{R}_+ \to \mathbb{R}$ is given by the formula $p(K) = \sum P(f)_K$ for $K > 0$. Recall that $|P(f)| = \{ b_1/b_0, \dots, b_h/b_0 \}$ and $p(b_k/b_0) = \sum P(f)_{b_k/b_0} = \frac{B_{k-1}\bar{b}_k}{\bar{b}_0}$ by Proposition 3.3(i). Therefore
$Q(f, X) = \{ \frac{B_{k-1}\bar{b}_k}{\bar{b}_0} : k = 1, \dots, h \}$.

*Proof of Theorem 1.3.* We assume $t = X$. We need to calculate $p_i(|P_i(f)|)$ for $i = 1, \dots, r$. Without restriction of generality we assume $i = 1$. Let $K > 0$. By definition of Puiseux' date we get

$$P_1(f)_K = P(f_1)_{\min(K,K_{11})} \oplus \cdots \oplus P(f_r)_{\min(K,K_{1r})}$$

Therefore
(1)

$$p_1(K) = \sum P_1(f)_K = \sum_{j=1}^{r} \Big( \sum P(f_j)_{\min(K,K_{1j})} \Big) = \sum_{j=1}^{r} \frac{(X, f_j)_0}{(X, f_1)_0} \Big( \sum P(f_1)_{\min(K,K_{1j})} \Big)$$

by Proposition 3.3(iii).

It is easy to see that $K_{1j} \le K$ if and only if $\sum P(f_1)_{K_{1j}} \le \sum P(f_1)_K$. On the other hand by Proposition 3.3(ii) we get

$$\sum P(f_1)_{K_{1j}} = \frac{(f_1, f_j)_0}{(X, f_j)_0}.$$

Let $p_{11}(K) = \sum P(f_1)_K$. Thus $K_{1j} \le K$ if and only if $\frac{(f_1,f_j)_0}{(X,f_j)_0} \le p_{11}(K)$. By definition of the set $J_1(\tau)$ we see that $K_{1j} \le K$ if and only if $j \in J_1(p_{11}(K))$. Then, we can rewrite (1) as follows

$$p_1(K) =$$

$$\sum_{j \in J_1^c(p_{11}(K))} \frac{(X, f_j)_0}{(X, f_1)_0}\Big(\sum P(f_1)_{\min(K, K_{1j})}\Big) + \sum_{j \in J_1(p_{11}(K))} \Big(\sum P(f_1)_{\min(K, K_{1j})}\Big) =$$

$$\sum_{j \in J_1^c(p_{11}(K))} \frac{(X, f_j)_0}{(X, f_1)_0} p_{11}(K) + \sum_{j \in J_1(p_{11}(K))} \frac{(X, f_j)_0}{(X, f_1)_0} \frac{(f_1, f_j)_0}{(X, f_j)_0} =$$

$$\Big(\sum_{j \in J_1^c(p_{11}(K))} \frac{(X, f_j)_0}{(X, f_1)_0}\Big) p_{11}(K) + \sum_{j \in J_1(p_{11}(K))} \frac{(f_1, f_j)_0}{(X, f_1)_0} = q_1(p_{11}(K)).$$

Recall that $|P_1(f)| = |P(f_1)| \cup \{ K_{12}, \dots, K_{1r} \}$. Thus $p_{11}(|P_1(f)|) = Q(f_1, X) \cup H_1(f, X)$. Therefore $p_1(|P_1(f)|) = q_1(p_{11}(|P_1(f)|)) = q_1(Q(f_1, X) \cup H_1(f, X))$ and the theorem follows from Proposition 4.2.

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A.P. Technical University, Al. 1000 LPP 7, 25–314 Kielce, Poland
*E-mail address*: matap@tu.kielce.pl

J.G. Technical University, Al. 1000 LPP 7, 25–314 Kielce, Poland
*E-mail address*: matjg@tu.kielce.pl