Equicontinuity of families of plurisubharmonic functions with bounds on their Monge-Ampère masses.

by

Sławomir Kołodziej

IMUJ PREPRINT **2001/06**

Abstract. It is proved that families of solutions of the complex Monge-Ampère equation with fixed boundary values and certain bounds on the right hand side are equicontinuous.

0. Introduction. It is a well known property of plurisubharmonic functions that if a sequence $u_j \in PSH(\Omega)$ tends to $u \in PSH(\Omega)$ in the sense of distribution theory then it is convergent in a much stronger sense; for instance, $u_j \to u$ in $L^p_{loc}(\Omega)$ for any $p \in [p, \infty)$ (see e.g. [H]). However, the Monge-Ampère measures $(dd^c u_j)^n$ may still have no limit (see [L], [C], [CK]). To be sure that $(dd^c u_j)^n$ converge as measures we need to know that $u_j \to u$ with respect to capacity [X]. In the present paper we study the convergence of plurisubharmonic functions with fixed boundary values and their Monge-Ampère measures controlled in terms of the Bedford-Taylor capacity. Then rather weak assumptions force the uniform convergence.

We shall work throughout in a strictly pseudoconvex domain $\Omega \subset \mathbb{C}^n$. In this paper we call an increasing function $h : \mathbb{R}_+ \to (1, \infty)$ *admissible* if it satisfies

$$\int_1^\infty (x h^{1/n}(x))^{-1}\, dx < \infty,$$

and if for some $a > 1$ $b > 1$ and $x_0 > 0$ we have

$$h(ax) \le b h(x) \quad \text{for} \quad x > x_0.$$

2000 *Mathematics Subject Classification.* primary 32U05, secondary 32U40.

Key words and phrases. plurisubharmonic function, complex Monge-Ampère operator.

Partially supported by KBN Grant No. PO3A 010 19

Let us define the family of nonnegative Borel measures in Ω associated to an admissible function h and a positive constant A:

$$\mathcal{F}(A,h) = \{\mu : \mu(K) \leq F(cap(K,\Omega)) \quad \text{for } F(x) = \frac{Ax}{h(x^{-1/n})} \\ \text{and any compact } K \subset \Omega\}.$$

Here *cap* stands for the Bedford-Taylor capacity (see the next section). In [K1], [K2] and [K3] it was proved that the Dirichlet problem for the complex Monge-Ampère equation in Ω

$$\begin{aligned} &u \in PSH \cap C(\overline{\Omega}), \\ &(dd^c u)^n = d\mu, \\ &\lim_{z\to x} u(z) = \varphi(x) \text{ for } x \in \partial\Omega. \end{aligned} \tag{0.1}$$

has a solution for any measure in $\mathcal{F}(A,h)$ and any given continuous boundary data φ. Furthermore, given φ and $\mathcal{F}(A,h)$ we have a uniform bound for L^∞ norm of those solutions.

For a function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\frac{\psi(x)}{x}$ increases to ∞ as $x \to \infty$ we define

$$L^\psi(c_0) = \{f \in L^1(\Omega) : f \geq 0, \int_\Omega \psi(f)\, dV \leq c_0\}.$$

We shall prove that any weakly convergent sequence in the family of functions

$$\begin{aligned} &\mathcal{P}(A,h,\psi,c_0;\varphi) \\ &= \{u \in PSH(\Omega) \cap C(\overline{\Omega}) : (dd^c u)^n \in \mathcal{F}(A,h) \cap L^\psi(c_0),\ u = \varphi \text{ on } \partial\Omega\} \end{aligned}$$

(φ continuous) is in fact uniformly convergent. As a consequence the family $\mathcal{P}(A,h,\psi,c_0;\varphi)$ is an equicontinuous family. In particular, families of solutions of (0.1) corresponding to $d\mu = f\,dV$ (dV denoting the Lebesgue measure) with $f \in L^\psi(c_0)$ for $\psi(t) = |t|(log(1+|t|))^m,\ m > n$ as well as those corresponding to $\{f : ||f||_p \leq c_0\},\ p > 1$ (where $||\cdot||_p$ denotes the norm in $L^p(\Omega)$ for $p \in [1,\infty]$) are equicontinuous.

In the last theorem we obtain a stability estimate for plurisubharmonic functions with fixed boundary values and densities of their Monge-Ampère measures uniformly bounded in $L^p(\Omega)$ norm. From [K1] we know that for $p > 1$ such a family satisfies

$$||u - v||_\infty \leq const.||f - g||_p \tag{0.2}$$

if $(dd^c u)^n = f\,dV$ and $(dd^c v)^n = g\,dV$. The constant in (0.2) depends on p and L^∞ norm of the boundary data. Here we obtain an estimate which involves also the L^q norm of $|u - v|$ for q and p conjugate. For $||f||_p$ and $||g||_p$ bounded by c_p and $\alpha \in (0,1)$ we have

$$||u - v||_\infty \leq const. c_p^{\frac{1+\alpha}{n+\alpha n+1}} ||u - v||_q^{\frac{1}{n+\alpha n+1}},$$

where *const.* depends on p, n, α, Ω.

Similar stability estimates are also valid for the solutions of the complex Monge-Ampère equation on compact Kähler manifolds. The topic is treated in a separate paper [K5]. For more information on the methods of pluripotential theory which are used in the proofs below we refer to [KL][K2][K4].

1. Preliminaries.

Given a bounded plurisubharmonic function u one can define the Monge-Ampère measure

$$(dd^c u)^n = dd^c u \wedge dd^c u \wedge ... \wedge dd^c u \quad (n \text{ terms})$$

as in [BT1]. This is a Borel nonnegative measure.

The following capacity was introduced in [BT2] for Borel sets $E \subset \Omega$:

$$cap(E, \Omega) = \sup\{\int_E (dd^c u)^n : u \in PSH(\Omega), -1 \leq u < 0\}.$$

It is now called the Bedford-Taylor capacity. It is closely related to the relative extremal function of the given compact set K:

$$u_K(z) = \sup\{u(z) : u \in PSH \cap L^\infty(\Omega), u < 0 \text{ in } \Omega, u \leq -1 \text{ on } K\},$$

Its upper semicontinuous regularization $u_K^*(z) := \overline{\lim}_{\zeta \to z} u_K(\zeta)$ is a plurisubharmonic function and, by [BT2]

$$cap(K, \Omega) = \int_K (dd^c u_K^*)^n = \int_\Omega (dd^c u_K^*)^n.$$

The following result of Bedford and Taylor will be often used in the sequel.

(Comparison principle). *[BT2]. If $u, v \in PSH \cap L^\infty(\Omega)$ and $\liminf_{z \to \partial\Omega}(u(z) - v(z)) \geq 0$, then*

$$\int_{\{u<v\}} (dd^c v)^n \leq \int_{\{u<v\}} (dd^c u)^n.$$

2. Results.

First lemma is a quantitative version of [K2, Lemma 2.3.1]

Lemma 1. *Let Ω be a strictly pseudoconvex domain in $\mathbb{C}^n$. Assume $u \in PSH \cap C(\Omega)$, and $v \in PSH \cap L^\infty(\Omega)$. Suppose that for some positive number A and an admissible function h the following inequality holds*

$$\int_K (dd^c v)^n \leq A cap(K, \Omega) h^{-1}((cap(K, \Omega))^{-1/n}) \tag{2.1}$$

for any compact set K. If the sets $U(s) := \{v+s < u\}$ are nonempty and relatively compact in Ω for $s \in [t, t+\delta]$ Then

$$\delta \leq \kappa(cap(U(t), \Omega)),$$

where

$$\kappa(s) = c(n) A^{1/n} [\int_{s^{-1/n}}^{\infty} y^{-1} h^{-1/n}(y)\, dy + h^{-1/n}(s^{-1/n})]$$

and the constant $c(n)$ depends only on n.

Proof. The estimate follows from (2.3.4) and (2.3.5) in the proof of Lemma 2.3.1 in [K2].

Theorem 1. *Let Ω be a strictly pseudoconvex domain in $\mathbb{C}^n$. Assume that $u_j \in PSH(\Omega)$ is a sequence converging weakly to $u \in PSH(\Omega)$ and for any j*

$$\liminf_{z \to \partial\Omega} (u_j - u)(z) \geq 0.$$

Suppose further that

$$(dd^c u_j)^n = f_j\, dV$$

with $f_j \in \mathcal{F}(A, h) \cap L^{\psi}(c_0)$, where $L^{\psi}(c_0)$ is defined as in Introduction with $\frac{\psi(x)}{x}$ increasing to ∞ as x goes to ∞. Then $u_j \to u$ uniformly in Ω.

Proof. Denote by $a_j(\delta) = cap(E_j(2\delta), \Omega)$ the Bedford-Taylor capacity of the set $E_j(2\delta) = \{u_j + 2\delta \leq u\}$. Since, by [K2] u and u_j are continuous the set $E_j(2\delta)$ is compact in view of our assumptions. Let us denote by v_j the relative extremal function of the set $E_j(2\delta)$. By [BT2] (see Section 1)

$$\int_{E_j(2\delta)} (dd^c v_j)^n = a_j(\delta).$$

Observe that for $V = \{u_j \leq \delta v_j + u - \delta\}$ the following inclusions hold

$$E_j(2\delta) \subset V \subset E_j(\delta).$$

Applying the comparison principle we thus get

$$\begin{aligned} a_j(\delta)\delta^n &\leq \int_{E_j(2\delta)} [dd^c(\delta v_j + u)]^n \leq \int_V (dd^c u_j)^n \\ &\leq \int_{E_j(\delta)} f_j\, dV. \end{aligned} \tag{2.2}$$

Hence for any $M > 0$ and $u_+ := \max(u, 0)$ we have

$$\begin{aligned} a_j(\delta)\delta^{n+1} &\leq \int_\Omega (u - u_j)_+ f_j\, dV \\ &= \int_{\{f_j > M\}} (u - u_j)_+ f_j\, dV + \int_{\{f_j \leq M\}} (u - u_j)_+ f_j\, dV \\ &\leq \max_\Omega (u - u_j)_+ \int_{\{f_j > M\}} f_j\, dV + M \int_\Omega (u - u_j)_+\, dV \\ &\leq \max_\Omega (u - u_j)_+ \frac{M}{\psi(M)} \int_\Omega \psi(f_j)\, dV + M \int_\Omega (u - u_j)_+\, dV \end{aligned} \tag{2.3}$$

For the last inequality we use the assumption that $\frac{\psi(x)}{x}$ is increasing. Fix $\epsilon > 0$. By [K1] there exists $c_1 > 0$ such that $v \geq -c_1$ for any $v \in \mathcal{F}(A,h)$ with $v \geq u$ on $\partial\Omega$. So, in view of our assumptions the quantities

$$\max_{\Omega}(u - u_j)_+ \int_{\Omega} \psi(f_j)\, dV$$

are uniformly bounded. Using the assumptions on ψ we can make $\frac{M}{\psi(M)}$ arbitrarily small by taking M big enough. We choose M so that the first term on the right hand side (2.3) is less than $\epsilon/2$ for any j. Since $u_j \to u$ in $L^1(\Omega)$ (see e.g. [H]) the other term is less than $\epsilon/2$ for $j > j_0$. Therefore

$$a_j(\delta) \leq \epsilon\delta^{-n-1} \quad \text{for} \ \ j > j_0.$$

Suppose for a while that $E_j(3\delta)$ were nonempty. Then, applying Lemma 1 we would get

$$\delta \leq \kappa(a_j(\delta)) \leq \kappa(\epsilon\delta^{-n-1}), \quad j > j_0.$$

Since, by the assumption on h we have $\lim_{s\to 0}\kappa(s) = 0$ the last inequality yields a contradiction if we take ϵ small enough. Thus $E_j(3\delta)$ is empty for $j > j_0$ which together with Hartogs' lemma implies the uniform convergence of the sequence u_j.

Theorem 2. *The set $\mathcal{P}(A, h, \psi, c_0; \varphi)$ is equicontinuous.*

Proof. Arguing by contradiction suppose that for some $\epsilon > 0, u_j \in \mathcal{F}(A,h) \cap L^{\psi}(c_0)$ and two sequences z_j, w_j of points in Ω we had $||z_j - w_j|| < j^{-1}$ and $u_j(z_j) - u_j(w_j) > \epsilon$. One can pick subsequences z_{j_k}, w_{j_k} converging to $z \in \overline{\Omega}$ and u_{j_k} converging in L^1 norm to $u \in PSH(\Omega) \cap C(\overline{\Omega})$. By Theorem 1 u_{j_k} converges uniformly. Thus for k large enough we have the following inequalities

$$\begin{aligned}
|u(z_{j_k}) - u(z)| &< \epsilon/4,\\
|u(w_{j_k}) - u(z)| &< \epsilon/4,\\
|u_{j_k}(z_{j_k}) - u(z_{j_k})| &< \epsilon/4,\\
|u_{j_k}(w_{j_k}) - u(w_{j_k})| &< \epsilon/4.
\end{aligned}$$

Combined they yield

$$|u_{j_k}(z_{j_k}) - u_{j_k}(w_{j_k})| < \epsilon$$

which contradicts the choice of the sequences.

Let $\psi : [0,\infty) \to [0,\infty)$ be an increasing function fulfilling the inequalities

$$2t \leq \psi(2t) \leq a\psi(t),$$

for some $a > 0$ and any $t > t_0$. Then the Orlicz space corresponding to ψ is defined by

$$L^{\psi}(\Omega) = \{f \in L^1(\Omega) : \int_{\Omega} \psi(|f|)\, dV < \infty\}.$$

Take

$$\psi(t) = |t|(log(1+|t|))^n h(log(1+|t|)),$$

for some admissible h. Then, by [K1] $L^\psi(c_0) \subset \mathcal{F}(A,h)$ for some positive A. For a continuous function φ defined on $\partial\Omega$ and ψ as above we denote by $\mathcal{P}(\psi, c_0; \varphi)$ the set of plurisubharmonic functions u in Ω which are equal to φ on $\partial\Omega$ and such that

$$(dd^c u)^n = f\, dV$$

for some $f \in L^\psi(c_0)$.

Corollary. *The functions from $\mathcal{P}(\psi, c_0; \varphi)$ are equicontinuous.*

In particular the family $\mathcal{P}(p, c_0; \varphi) := \mathcal{P}(\psi, c_0; \varphi)$ with $\psi(t) = t^p,\ p > 1$ is equicontinuous. Below we find some more precise estimates for those families.

Lemma 2. *Let u_j be a sequence converging to u in the sense of distributions theory, such that $u_j \in \mathcal{P}(p, c_0; \varphi)$ and*

$$\liminf_{z \to \partial\Omega} (u_j - u)(z) \geq 0.$$

Then for $\delta > 0$ and $a_j(\delta), E_j(2\delta)$ defined as in Theorem 1, we have

$$a_j(\delta) \leq c_0 \delta^{-n-1} ||(u - u_j)_+||_q$$

with $p^{-1} + q^{-1} = 1$.

Proof. We use (2.2) and Hölder's inequality to obtain

$$\begin{aligned} a_j(\delta)\delta^{n+1} &\leq \int_\Omega (u - u_j)_+ f_j\, dV \\ &\leq (\int f_j^p\, dV)^{1/p} (\int (u - u_j)_+^q dV)^{1/q} \leq c_0 ||(u - u_j)_+||_q. \end{aligned}$$

The proof is completed.

To estimate avereges of L^p functions over compact sets in terms of Bedford-Taylor capacity we need the following lemma (Lemma 4.3 from [K4]) which is a generalized version of Zeriahi's result [Z]. The estimate in [Z] holds on compact subsets of Ω. Zeriahi pointed out to me that the proof of Lemma 2.5.1 in [K2] (which is an ingredient of the proof of the estimate we seek to establish) has a gap. The proof is corrected in [K4, Lemma 2.8].

Lemma 3. *Suppose $u \in PSH(\Omega) \cap C(\overline{\Omega}), u = 0$ on $\partial\Omega, \int (dd^c u)^n \leq 1$. Then for any $\alpha < 2$ the Lebesgue measure $V(\Omega_s)$ of the set $\Omega_s := \{u < s\}$ is bounded from above by $c \exp(-2\pi\alpha|s|)$, where c does not depend on u.*

Now we can prove the aforementioned estimate.

Lemma 4. *Let $p > 1$, $t < \min(4\pi - 1, p-1)$ and $f \in L^p(\Omega)$. Then for any compact K in Ω*

$$\int_K f\,dV \le c(p,n,\Omega)||f||_p a(K)\exp(-ta^{-1/n}(K))$$

where we use the notation $a(K) = cap(K,\Omega)$ and the constant $c(p,n,\Omega)$ depends on p,n and the set Ω.

Proof. It is enough to show the inequality for f such that $||f||_p = 1$. Take α such that $\frac{1+t}{2\pi} < \alpha < 2$. Observe that one can pick α depending only on p since t depends only on p. Set $h(s) = e^{ts}$, $G(s) = s^n e^{ts}$ and $\psi(s) = s(s+1)^t \log^n(s+1)$. First we shall prove that

$$\int_\Omega |v|^n h(|v|) f\,dV \le A, \tag{2.4}$$

with $A = \int_\Omega \psi(f)\,dV + c\sum_1^\infty s^n e^{(t+1-2\pi\alpha)s}$, for $v \in PSH(\Omega)$ of the form $v = cap^{-1/n}(K,\Omega)u_K$, where u_K is the relative extremal function of K with respect to Ω and c is the constant from Lemma 3. Applying Young's inequality (see e.g. [H]) one obtains

$$\begin{aligned} G(|v(x)|)f(x) &\le \int_0^{f(x)} G(log(1+r))\,dr + \int_0^{G(|v(x)|)} [exp(G^{-1}(t)) - 1]\,dt \\ &\le f(x)G(log(1+f(x))) + \int_0^{|v(x)|} e^s G'(s)\,ds \\ &\le \psi(f(x)) + G(|v(x)|)e^{|v(x)|}. \end{aligned}$$

Integrating this inequality over Ω we get

$$\int G(|v|)f\,dV \le \int \psi(f)\,dV + \int G(|v|)e^{|v|}dV.$$

Observe that, by Minkowski's inequality

$$\int \psi(f)\,dV \le ||f+1||_p \le ||f||_p + (\int_\Omega dV)^{1/n}.$$

Thus to get (2.4) we need to find a suitable estimate on the second integral on the right hand side. To this end we apply Lemma 3. Then

$$\begin{aligned} &\int_\Omega G(|v(x)|)e^{|v(x)|}\,dV = \sum_{s=0}^\infty \int_{\{-s-1<v\le -s\}} G(|v(x)|)e^{|v(x)|}\,dV \\ \le &\sum_{s=0}^\infty (s+1)^n e^{(t+1)(s+1)} V(\{v < -s\}) \le c\sum_{s=0}^\infty (s+1)^n e^{(t+1)(s+1)} e^{-2\pi\alpha s} \\ \le &ce^{t+1}\sum_{s=1}^\infty (s+1)^n e^{(t+1-2\pi\alpha)s} \end{aligned}$$

and the series is convergent due to the choice of α. Finally, one infers the statement from (2.3) as follows. Since

$$|v(z)|^n = a^{-1}(K)|u_K(z)| = a^{-1}(K)$$

for $z \in K$ we get

$$\begin{aligned} A &\geq \int_\Omega |v|^n h(|v|) f\, dV \geq \int_K |v|^n h(|v|) f\, dV \\ &\geq \frac{1}{a(K)} \exp(t a^{-1/n}(K)) \int_K f\, dV, \end{aligned}$$

which is the desired estimate as we recall that $||f||_p = 1$.

We shall use the following simple proposition in the proof of Theorem 3 below.

Proposition. *Given $C_0 > 0$ and $\alpha \in (0,1)$ we have*

$$x \exp(Cx^{-\alpha}) \geq C_0$$

for any $C \geq (\alpha e)^{-1} C_0^\alpha$ and any $x > 0$.

Proof. The derivative of the function $g(x) = x\exp(Cx^{-\alpha})$ has its zero at $x_0 = (\alpha C)^{1/\alpha}$ and $g(x_0) = (\alpha e C)^{1/\alpha}$. Hence for $C \geq (\alpha e)^{-1} C_0^\alpha$ one gets $g(x) \geq [\alpha e (\alpha e)^{-1} C_0^\alpha]^{1/\alpha} = C_0$.

We are now ready to show the main estimate.

Theorem 3. *Let Ω be a strictly pseudoconvex domain in $\mathbb{C}^n$. Assume that u and v belong to $PSH(\Omega)$ and*

$$\liminf_{z \to \partial\Omega} (v - u)(z) \geq 0.$$

Suppose further that

$$(dd^c v)^n = g\, dV, \quad (dd^c u)^n = f\, dV$$

with $||f||_p, ||g||_p \leq c_p$, $p > 1$. Then for given $\alpha \in (0,1)$ there exists a constant $c' = c'(p, n, \alpha, \Omega)$ such that

$$||(u-v)_+||_\infty \leq c' c_p^{\frac{\alpha+1}{n+\alpha n+1}} ||(u-v)_+||_q^{\frac{1}{n+\alpha n+1}}.$$

where $p^{-1} + q^{-1} = 1$.

Proof. We shall apply Lemma 1 with $h(s) = e^{ts}$, where t is chosen so that $t < \min(4\pi - 1, p - 1)$. For this choice of h one has

$$\int_s^\infty y^{-1} h^{-1/n}(y)\, dy \leq \frac{n}{t} \exp(\frac{-ts}{n}). \tag{2.5}$$

Now, Lemma 4 tells us that (2.1) is satisfied with $A = c(p, n, \Omega) c_p$. Set

$$\delta = \frac{1}{2} ||(u-v)_+||_\infty.$$

Then $U(s) = \{v + s < u\}$ is nonempty for $s \in (\delta, 2\delta)$. By Lemma 1 and (2.5)

$$\delta \leq c(n) A^{\frac{1}{n}} (\frac{n}{t} + 1) \exp(-\frac{t}{n} a^{-1/n}(\delta)), \quad \text{with } \; a(\delta) = cap(U(\delta), \Omega).$$

Coupling this with the estimate from Lemma 2 one obtains

$$\text{(2.6)} \qquad \delta \leq c_1(p, n, \Omega) c_p^{\frac{1}{n}} \exp(-\frac{t}{n} (\delta/2)^{(n+1)/n} ||(u-v)_+||_q^{-1/n} c_p^{-1/n}).$$

Making use of the proposition above we infer that for $\alpha \in (0,1)$ we have

$$\delta \exp(C\delta^{-\alpha}) \geq c_1(p, n, \Omega) c_p^{1/n}$$

for $C = (\alpha e)^{-1} c_1^{\alpha}(p, n, \Omega) c_p^{\alpha/n}$. Inserting this into (2.6) and taking logarithms of both sides one obtains

$$\delta^{\alpha + \frac{n+1}{n}} \leq 4(\alpha e)^{-1} \frac{n}{t} c_1^{\alpha}(p, n, \Omega) c_p^{\frac{1+\alpha}{n}} ||(u-v)_+||_q^{\frac{1}{n}}.$$

So

$$||(u-v)_+||_\infty \leq c'(p, n, \alpha, \Omega) c_p^{\frac{1+\alpha}{n+\alpha n+1}} ||(u-v)_+||_q^{\frac{1}{n+\alpha n+1}}.$$

References

[BT1] E. Bedford and B.A. Taylor, *The Dirichlet problem for the complex Monge-Ampère operator.* Invent. Math. **37** (1976), 1-44.

[BT2] E. Bedford and B.A. Taylor, *A new capacity for plurisubharmonic functions,* Acta Math. **149** (1982), 1-40.

[C] U. Cegrell, *Discontinuité de l'opérateur de Monge-Ampère complexe.* C.R. Acad. Sci. Paris, Ser. I Math., **296** (1983), 869-871.

[CK] U. Cegrell and S. Kołodziej, *The Dirichlet for the complex Monge-Ampère operator: Perron classes and rotation invariant measures.* Michigan Math. J. **41** (1994), 563-569.

[H] L. Hörmander, *Notions of convexity,* Birkhäuser, 1996.

[KL] M. Klimek, *Pluripotential Theory,* Oxford University Press, New York, 1991.

[K1] S. Kołodziej, *Some sufficient conditions for solvability of the Dirichlet problem for the complex Monge-Ampère operator,* Ann. Polon. Math. **65** (1996), 11-21.

[K2] S. Kołodziej, *The complex Monge-Ampère equation,* Acta Math. **180** (1998),69-117.

[K3] S. Kołodziej, *A sufficient condition for solvability of the Dirichlet problem for the complex Monge-Ampère operator,* Contemporary Mathematics **222** (1998) 241-243.

[K4] S. Kołodziej, *Lecture notes in the complex Monge-Ampère equation,* www.im.uj.edu.pl/gamma/kolodziej.

[K5] S. Kołodziej, *Stability of solutions to the complex Monge-Ampère equation on compact Kähler manifolds,* preprint.

[L] P.Lelong, *Discontinuité et annulation de l'opérateur de Monge-Ampère complexe.* Sèminaire P. Lelong, P.Dolbeault, H.Skoda (Analyse) 1981/1983, Lecture Notes in Math. **1028** 219-224, Springer-Verlag, Berlin-Heidelberg-New York 1983.
[M] J. Musielak, *Orlicz spaces and Modular Spaces,* Lecture Notes in Math. **1034** Springer-Verlag, Berlin-Heidelberg-New York 1988.
[X] Y. Xing, *Continuity of the complex Monge-Ampère operator.* Proceed. of AMS **124** (1996), 457-467.
[Z] A. Zeriahi, *Volume and capacity of sublevel sets of plurisubharmonic functions in a Lelong class,* prepublication **165** Laboratoire de mathematiques E. Picard, Toulouse 1999.

postal address:
Jagiellonian University, Institute of Mathematics
Reymonta 4, 30-059 Kraków, Poland
e-mail: kolodzie@ im.uj.edu.pl