Stability of solutions to the complex Monge-Ampère equation on compact Kähler manifolds.

by

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Abstract. We prove a stability theorem for generalized solutions of the complex Monge-Ampère equation. The uniqueness and equicontinuity of families of solutions follow from it.

0.Introduction

We continue our study of the complex Monge-Ampère equation on compact Kähler manifolds initiated in [K2]. Let M be a compact n-dimensional Kähler manifold with the fundamental form ω given in local coordinates by

$$\omega = \frac{i}{2} \sum_{k,j} g_{k\bar{j}} dz^k \wedge d\bar{z}^j.$$

We normalize the metric by

$$\int_M \omega^n = 1.$$

In this paper a continuous function on M will be called *ω-plurisubharmonic (ω-psh* in short) if $dd^c\varphi + \omega \geq 0$, where $d = \partial + \overline{\partial}$, $d^c = i(\overline{\partial} - \partial)$. The set of those functions we denote by $PSH(\omega)$. Elsewhere (e.g. [A2] [T]) such functions are often called admissible if they are also smooth and satisfy strict inequality $dd^c\varphi + \omega > 0$.

We study ω-psh solutions φ of the complex Monge-Ampère equation

$$(\omega + dd^c\varphi)^n = f\omega^n, \tag{0.1}$$

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where $f \in L^1(M), f \geq 0, \int_M f\omega^n = 1$ is a given function. The wedge product of positive currents on the left is defined as in [BT1].

The geometrical meaning of the equation (0.1) (with smooth positive f) is that given the volume form $f\omega^n$ on M we seek for the Kähler metric (represented by the fundamental form $\omega + dd^c\varphi$) which yields this volume form. More interestingly, as a short calculation shows (see [A1][A2] [T] [Y]), equation (0.1) arises when given a closed $(1,1)$ form τ representing the first Chern class of M we want to find a Kähler form ω' such that $\tau = Ricc(\omega')$ (the Ricci form of ω') and ω' lives in the same Chern class as τ. E. Calabi conjectured that this is always possible. He also proved in [C] the uniqueness of such ω' which is equivalent to the fact that any two solutions of (0.1) differ by a constant. The Calabi conjecture was confirmed by S.-T. Yau [Y] who proved the following theorem.

Theorem 0.1. *Let $f > 0, f \in C^k(M), k \geq 3$. Then there exists a solution to* (0.1) *belonging to Hölder class $C^{k+1,\alpha}(M)$ for any $0 \leq \alpha < 1$.*

The existence of generalized solutions of (0.1) was shown in [K2]. The result may be rephrased as follows (see Section 4 for details). Consider a family of functions

$$\mathcal{F}(F) = \{f \in L^1(M) : f \geq 0, \int_M f\omega^n = 1,$$
$$\int_E f\omega^n \leq F(cap_\omega(E)) \ \text{ for any Borel set } E \subset M\},$$

where cap_ω is a capacity defined in Section 1 and $F(x) = \frac{Ax}{h(x^{-1/n})}$, with $A > 0$ and $h : \mathbb{R}_+ \to [1, \infty)$ which is nondecreasing and satisfies

$$\int_1^\infty y^{-1} h^{-1/n}(y)\, dy < \infty.$$

Then for any $f \in \mathcal{F}(F)$ there exists a continuous solution of (0.1). Furthermore, if we normalize those solutions by $\max_M \varphi = 0$ then there exists a constant $a(F) > 0$ such that $\varphi > -a(F)$ for any φ solving (0.1) with $f \in \mathcal{F}(F)$.

In particular, for any $p > 1$ and any $f \in L^p(M)$ such that $f \geq 0$ and $\int_M f\omega^n = 1$ the Monge-Ampère equation is solvable. Despite the fact that we deal with nonsmooth solutions the result may be useful for geometers (cf. [TZ]).

The primary objective of the present paper is to establish a stability result for those generalized solutions (Theorem 4.1). For the solutions of

$$(\omega + dd^c\varphi)^n = f\omega^n, \quad (\omega + dd^c\psi)^n = g\omega^n$$

(suitably normalized) we find a bound on sup norm of $|\varphi - \psi|$ in terms of $||f - g||_{L^1(M)}$ when $f, g \in \mathcal{F}(F)$. Such an estimate is also new for classical solutions. In the proof of Theorem 0.1 stability estimates are established between norms in $C^{k,\alpha}$ spaces for $k \geq 3$.

As an application of the main result one obtains the following explicit estimate in the case the right hand side is bounded in L^p norm $p > 1$. For any $\epsilon > 0$ and $c_0 > 0$

there exists a constant $c(\epsilon, c_0, p)$ such that for all f, g with $||f||_{L^p} < c_0$, $||g||_{L^p} < c_0$ we have

$$\sup|\varphi - \psi| \le c(\epsilon, c_0, p)||f - g||_{L^p}^{1/(n+1+\epsilon)}.$$

Having such an estimate one can deduce the equicontinuity of solutions of (0.1) when f belongs to the set

$$\mathcal{F}_p(c) = \{f \in L^p(M) : f \ge 0, \int_M f\,dV = 1, \int_M f^p dV \le c\},$$

for $p > 1$ and $c > 0$. The main stability result also implies that for $f \in \mathcal{F}(F)$ the solution of the Monge-Ampère equation is unique up to an additive constant which is a generalization of Calabi's theorem. It would be interesting to know if the uniqueness holds in general for ω-psh functions. A particular case $M = \mathbb{P}^n$ was treated in [BT3].

In the proofs we use the methods of pluripotential theory. In Section 2 the comparison principle for ω-psh on compact Kähler manifolds is shown which might be of independent interest. For more backround in pluripotential theory we refer to [K][K2][K3].

Notation: We shall denote by $||\cdot||_p$ the norm in $L^p(M)$ for $p \in [1, \infty]$.
For the sake of brevity we shall write

$$\omega_\varphi = \omega + dd^c\varphi.$$

1. Preliminaries

We work throughout on a compact Kähler manifold M with a fundamental form ω and assume

$$\int_M \omega^n = 1.$$

If in an open subset of M there exists a potential function v satisfying $\omega = dd^c v$ then for ω-psh φ the function $v + \varphi$ is a true plurisubharmonic function. Thus such properties of plurisubharmonic functions as Hartogs' lemma or the theorem saying that the weak convergence implies convergence in L^p_{loc} (see e.g. [H]) hold also for ω-psh functions. The same goes for the convergence theorems, proved by Bedford and Taylor [BT2], which say that the Monge-Ampère measures associated to a convergent monotone sequence of plurisubharmonic functions converge. For a Borel set $E \subset M$ one can define a capacity

$$cap_\omega(E) = \sup\{\int_E \omega_\varphi^n : \varphi \in PSH(\omega), 0 \le \varphi \le 1\}.$$

This is modeled on Bedford-Taylor capacity introduced in [BT2].

Let us consider two open finite coverings $\{V_s\}$, $\{V'_s\}$, $s = 1, 2, ..., N$ of M such that $V'_s \subset V_s$ and in each strictly pseudoconvex V_s there exists $v_s \in PSH(V_s)$ with $dd^c v_s = \omega$ and $v_s = 0$ on ∂V_s. Given a compact set $K \subset M$ define $K_s = K \cap \overline{V'_s}$.

We are going to show that $cap_\omega(K)$ is comparable with $cap'_\omega(K) = \sum_s cap(K_s, V_s)$, where $cap(K,V)$ denotes the Bedford-Taylor capacity. From [BT2] we know that

$$cap(K,V) = \int_K (dd^c u_{K,V})^n$$

with

$$u_{K,V} = \sup\{u \in PSH(V) : u < 0, u \leq -1 \quad \text{on} \quad K\}.$$

For fixed s put $\varphi_s = u_{K_s,V_s} - v_s$. Then $\varphi_s \geq -1$ on K_s and $\varphi_s = 0$ on ∂V_s. One can find $\psi_s \in PSH(\omega) \cap C^\infty(M)$ such that $\psi_s = 0$ outside V_s and $\psi_s \leq -3\delta < 0$ on V'_s with the same δ for all s. (To see this just take any smooth ψ'_s which is equal to 0 outside V_s and negative on V'_s and choose $\epsilon > 0$ so small that $\psi_s = \epsilon\psi'_s$ is ω-psh.) Take χ_s which is equal to $\max(\delta\varphi_s - \delta, \psi_s)$ on V_s and equal to 0 elsewhere on M. Note that this function is ω-psh and equal to $\delta\varphi_s - \delta$ on a neighbourhood of K_s. Therefore

$$\begin{aligned}\int_{K_s} \omega_{\chi_s}^n &= \int_{K_s} [\delta\omega_{\varphi_s} + (1-\delta)\omega]^n \\ &\geq \delta^n \int_{K_s} \omega_{\varphi_s}^n = \delta^n \int_{K_s} (dd^c u_{K_s,V_s})^n = \delta^n cap(K_s, V_s).\end{aligned}$$

So

$$cap_\omega(K) \geq cap_\omega(K_s) \geq \delta^n cap(K_s, V_s)$$

By the definitions,

$$cap_\omega(K) \leq (C_1+1)^n \sum_s cap(K_s, V_s),$$

where C_1 is chosen so that $v_s \geq -C_1$ for any s. Indeed, let $\varphi \in PSH(\omega)$ with $-1 < \varphi \leq 0$. Then $(C_1+1)^{-1}(\varphi+v_s)$ is a competitor in the definition of $cap(K_s, V_s)$ (see [BT2]). So

$$\begin{aligned}(C_1+1)^{-n} \int_K \omega_\varphi^n &\leq (C_1+1)^{-n} \sum_s \int_{K_s} \omega_\varphi^n \\ &\leq \sum_s cap(K_s, V_s)\end{aligned}$$

which proves the preceding inequality. Thus we finally obtain

$$\text{(1.1)} \qquad \frac{\delta^n}{N} cap'_\omega(K) \leq cap_\omega(K) \leq (C_1+1)^n cap'_\omega(K).$$

A sequence φ_j of functions defined in M is said to converge with respect to capacity to φ if for any $t > 0$ and a Borel $K \subset M$

$$\lim_{j\to\infty} cap(K \cap \{|\varphi - \varphi_j| > t\}, \Omega) = 0.$$

The following lemma is taken from Demailly's paper [D1] who attributes it to Richberg [R] and Green, Wu [GW]

Lemma 1.1. *If $\varphi \in PSH(\omega)$ and γ is a continuous $(1,1)$ form on M such that $dd^c\varphi \geq \gamma$ (as currents) then given $\delta > 0$ one can find a smooth function ψ satisfying $\varphi < \psi < \varphi + \delta$ and $dd^c\psi \geq \gamma - \delta\omega$ on M.*

Next lemma is well known for smooth forms. We need a more general version (cf [D2]).

Lemma 1.2. *Suppose $g \in L^1(M)$ and $\varphi, \psi \in PSH(\omega)$ satisfy*

$$\omega_\varphi^n \geq g\omega^n, \quad \omega_\psi^n \geq g\omega^n.$$

Then $\omega_\varphi^k \wedge \omega_\psi^{n-k} \geq g\omega^n$.

Proof. The statement is local and so it is equivalent to the following one:
For $u, v \in PSH(B) \cap C(\overline{B})$ (B - a ball in $\mathbb{C}^n$) satisfying

$$(dd^c u)^n \geq g\, dV, \quad (dd^c v)^n \geq g\, dV$$

we have $(dd^c u)^k \wedge (dd^c v)^{n-k} \geq g\, dV$, where dV denotes the Lebesgue measure and $g \in L^1(B)$.

For smooth u, v and $g > 0$ it is a well known matrix inequality which follows from concavity of the mapping $A \to \log\det^{1/n} A$ defined on the set of positive definite Hermitian matrices (see Corollary 7.6.9 in [HJ]). Next we shall prove it for $g \in L^2(B)$. Let g_j be a sequence of smooth positive functions, positive on $\overline{B}$ and tending in $L^2(B)$ to g. Fix also two sequences f_j, h_j of smooth functions on ∂B such that $f_j \to u$ and $h_j \to v$ uniformly on ∂B. Applying [CKNS] one can find $u_j, v_j \in PSH(B) \cap C^\infty(\overline{B})$ solving the Dirichlet problems for the Monge-Ampère equation

$$\begin{aligned} (dd^c u_j)^n &= g_j dV, \quad u_j = f_j \ \text{ on } \partial B \\ (dd^c v_j)^n &= g_j dV, \quad v_j = h_j \ \text{ on } \partial B. \end{aligned}$$

By L^2 estimates from [K1] u_j and v_j tend uniformly to u and v respectively. Hence one can apply the convergence theorem [BT2] and the statement for smooth functions to the effect

$$\begin{aligned} (dd^c u)^k \wedge (dd^c v)^{n-k} &= \lim_{j\to\infty} (dd^c u_j)^k \wedge (dd^c v_j)^{n-k} \\ &\geq \lim_{j\to\infty} g_j\, dV \geq g\, dV, \end{aligned}$$

For the general case we take an increasing sequence $g_j \uparrow g$ with $g_j \in L^2(B)$ and repeat the above argument using [K1] to solve the suitable Dirichlet problems. Now the convergence of the approximating sequences is not uniform, but the sequences are decreasing due to the comparison principle [BT2]. So the convergence theorem still applies in this case.

2. Comparison principle

Now we shall prove the comparison principle for the Monge-Ampère operator on compact Kähler manifolds.

Theorem 2.1. *If φ and ψ are ω-psh on M then for $\Omega = \{\varphi < \psi\}$ we have*

$$\int_\Omega \omega_\psi^n \leq \int_\Omega \omega_\varphi^n.$$

Proof. Suppose first that φ, ψ and the boundary of Ω are smooth. Set $\varphi_t = \max(\varphi + t, \psi),\ t > 0$. Then close to $\partial\Omega$ we have $\varphi_t = \varphi + t$. Define the (closed) current

$$T_t = \sum_{k=1}^{n} \binom{n}{k} (dd^c\varphi_t)^{k-1} \wedge \omega^{n-k}$$

and set $T = \lim_{t\to 0} T_t$. By Stokes' theorem

$$\begin{aligned}\int_\Omega \omega_{\varphi_t}^n &= \int_\Omega dd^c\varphi_t \wedge T_t + \omega^n = \int_{\partial\Omega} d^c\varphi_t \wedge T_t + \int_\Omega \omega^n \\ &= \int_{\partial\Omega} d^c\varphi \wedge T + \int_\Omega \omega^n = \int_\Omega \omega_\varphi^n\end{aligned}$$

Since $\varphi_t \downarrow \psi$ in Ω as $t \to 0$ we get applying the convergence theorem [BT2] that

$$\omega_{\varphi_t}^n \to \omega_\psi^n \quad \text{in } \Omega$$

as measures. Hence for a test function χ in Ω with $0 \leq \chi \leq 1$ we get

$$\int_\Omega \chi\omega_\psi^n = \lim_{t\to 0} \int_\Omega \chi\omega_{\varphi_t}^n \leq \liminf_{t\to 0} \int_\Omega \omega_{\varphi_t}^n.$$

So

$$\int_\Omega \omega_\psi^n \leq \liminf_{t\to 0} \int_\Omega \omega_{\varphi_t}^n = \int_\Omega \omega_\varphi^n,$$

which completes the proof for smooth functions. Suppose now that φ and ψ are continuous and that they satisfy the extra assumption

$$dd^c\varphi \geq (\delta - 1)\omega, \quad dd^c\psi \geq (\delta - 1)\omega \tag{2.1}$$

for some $\delta > 0$. Then, applying Lemma 1.1, one can find two sequences of ω-psh functions φ_j and ψ_j converging uniformly to φ and ψ respectively. Given a compact set $K \subset \Omega$ we find $t > 0$ and a positive integer j_0 such that $K \subset \Omega(t,j) = \{\varphi_j < \psi_j - t\} \subset \Omega$ for $j > j_0$ and the boundary of $\Omega(t,j)$ is smooth (using Sards' theorem). Applying the first part of the proof and the convergence theorem we obtain

$$\int_K \omega_\psi^n \leq \liminf_{j\to\infty} \int_{\Omega(t,j)} \omega_{\psi_j}^n \leq \liminf_{j\to\infty} \int_{\Omega(t,j)} \omega_{\varphi_j}^n \leq \int_\Omega \omega_\varphi^n.$$

Exhausting Ω by compact sets we get the desired inequality in this case. It remains to get rid of the extra assumption. Note that for fixed $t \in (0,1)$ and ω-psh functions φ, ψ the functions $t\varphi$ and $t\psi$ satisfy (2.1) for some $\delta > 0$. Fix a compact set $K \subset \Omega = \{\varphi < \psi\}$ and consider $\delta > 0$ and $t \in (0,1)$ such that $K \subset \Omega(\delta, t) = \{\varphi < \psi - \delta/t\}$. By the above and the convergence theorem we have

$$\int_K \omega_\psi^n \leq \liminf_{t\to 1} \int_{\Omega(\delta,t)} \omega_{t\psi}^n \leq \liminf_{t\to 1} \int_{\Omega(\delta,t)} \omega_{t\varphi}^n \leq \int_\Omega \omega_\varphi^n.$$

Again to complete the proof it is enough to consider an exhaustion sequence of compact subsets of Ω.

Remark. *In the definition of ω-psh function we assumed that it is continuous (since only with such functions we deal in this paper). In the comparison principle this assumption is redundant. This can be shown using quasicontinuity of plurisubharmonic functions as in [BT2] (see also [K3]).*

The following lemma is a variation of Lemma 2.3.1 in [K2]. The reason the proof must be modified is that, unlike for plurisubharmonic functions, a sum of ω-psh is not necessarily ω-psh.

Lemma 2.2. *Let φ and ψ be ω-psh functions on M with $0 \le \psi \le C$. Assume that $U(s) := \{\varphi - s < \psi\}$ are nonempty for $s \in [S, S+D]$. Suppose that for some positive number A and an increasing function $h : \mathbb{R}_+ \to (1, \infty)$ satisfying*

$$\int_1^\infty (yh^{1/n}(y))^{-1}\, dy < \infty,$$

the following inequality holds

$$\int_K \omega_\varphi^n \le F(cap_\omega(K)), \quad \text{with} \quad F(t) = \frac{At}{h(t^{-1/n})}, \quad A > 0, \tag{2.2}$$

for any compact set K. Then

$$D \le \kappa(a(S+D)), \quad a(s) := cap_\omega(U(s))$$

where

$$\kappa(s) = c(n)A^{1/n}(1+C)[\int_{s^{-1/n}}^\infty y^{-1}h^{-1/n}(y)\, dy + h^{-1/n}(s^{-1/n})].$$

Proof. Set

$$b(s) = \int_{U(s)} \omega_\varphi^n.$$

First we shall prove the inequality

$$t^n a(s) \le b(s + t + Ct) \ \text{ for } 0 < t < \frac{S+D-s}{C+1}. \tag{2.3}$$

Indeed, take $\rho \in PSH(\omega)$ with $-1 \le \rho \le 0$ and put $V(s) := \{\varphi - s - t - Ct < t\rho + (1-t)\psi\}$. One easily verifies that $U(s) \subset V(s) \subset U(s+t+Ct)$. We can now apply the comparison principle (Theorem 2.1) to obtain

$$\begin{aligned} t^n \int_{U(s)} \omega_\rho^n &\le \int_{V(s)} (t\omega_\rho + (1-t)\omega_\psi)^n \\ &\le \int_{V(s)} \omega_\varphi^n \le \int_{U(s+t+Ct)} \omega_\varphi^n = b(s+t+Ct). \end{aligned}$$

Taking supremum over ρ we get (2.3).

Next we define an increasing sequence $s_0, s_1, ..., s_N$, setting $s_0 := S$ and

$$s_j := \sup\{s : a(s) \leq \lim_{t \to s_{j-1}+} ea(t)\}$$

for $j = 1, 2, ..., N$. Then this sequence is increasing and

$$a(s_j) \geq ea(s_{j-2}).$$

The integer N is chosen to be the greatest one satisfying $s_N \leq S + D$. Then

$$a(S + D) \leq \lim_{t \to s_N+} ea(t)$$

(otherwise we would have $s_{N+1} \leq S+D$.) From the last inequality, the assumptions and (2.3) it follows that for any $s \in (s_N, S + D)$ we have

$$(\frac{S + D - t}{1 + C})^n a(s) \leq b(S + D) \leq F(a(S + D)) \leq Aea(s)h^{-1}([a(S + D)]^{-1/n}).$$

Hence

$$S + D - s_N \leq (Ae)^{1/n}(1 + C)h^{-1/n}([a(S + D)]^{-1/n}) \tag{2.4}$$

Consider two numbers $S < s' < s < S + D$ such that $a(s) \leq ea(s')$ and take $t := \frac{s-s'}{1+C}$. Therefore, by the assumptions and (2.2), we have

$$\begin{aligned} a(s') &\leq t^{-n}b(s) \leq At^{-n}a(s)h^{-1}([a(s)]^{-1/n}) \\ &\leq Aet^{-n}a(s')h^{-1}([a(s)]^{-1/n}). \end{aligned}$$

Then proceeding like in [K2] we obtain (letting s to s_{j+1} and s' to s_j)
(2.5)

$$\begin{aligned} s_N - S &= \sum_{j=0}^{N-1} t_j \\ &\leq 2(Ae)^{1/n}[n \int_{[a(S+D)]^{-1/n}}^{\infty} [yh^{1/n}(y)]^{-1}\, dy + (h(a(S + D)^{-1/n}))^{-1/n}]. \end{aligned}$$

It follows from (2.4) and (2.5) that

$$D \leq 2n(Ae)^{1/n}(1 + C)[\int_{[a(S+D)]^{-1/n}}^{\infty} [yh^{1/n}(y)]^{-1}\, dy + (h(a(S + D)^{-1/n}))^{-1/n}].$$

The proof is complete.

3. Convergence with respect to capacity

By [X] and (1.1) the convergence with respect to capacity of a sequence of ω-psh functions entails the convergence of Monge-Ampère measures associated to those functions. Below we prove that a weakly convergent sequence of solutions of (0.1) for the right hand side uniformly bounded in L^p norm, $p > 1$, is convergent with respect to capacity. Later, using this we shall show that it is in fact uniformly convergent.

Lemma 3.1. *Let $\varphi_j \in PSH(\omega)$ and $\varphi_j \to \varphi \in PSH(\omega)$ in the sense of distributions. Suppose that*

$$\omega_{\varphi_j}^n = f_j \omega^n$$

where the norms $||f_j||_{L^p(\omega^n)}$ are uniformly bounded by C for some $p > 1$. Then

$$cap_\omega(\{\varphi_j + 2s < \varphi\}) \le C(||\varphi|| + 1)s^{-n-1}||\varphi - \varphi_j||_{L^q(\omega^n)},$$

where q is such that $p^{-1} + q^{-1} = 1$.

In particular $\varphi_j \to \varphi$ with respect to capacity.

Proof. Denote

$$E_j(s) = \{\varphi_j + s < \varphi\} \quad \text{and} \quad a_j := cap_\omega(E_j(2s)).$$

Suppose $0 < \varphi < N - 1$. Take $\rho \in PSH(\omega)$ with $-1 \le \rho \le 0$ and set $V := \{\varphi_j < \frac{s}{N}\rho + (1 - \frac{s}{N})\varphi - s\}$. Then

$$E_j(2s) \subset V \subset E_j(s).$$

By comparison principle

$$\begin{aligned} \frac{s^n}{N^n} \int_{E_j(2s)} \omega_\rho^n &\le \int_V (\frac{s}{N}\omega_\rho + (1 - \frac{s}{N})\omega_\varphi)^n \\ &\le \int_V \omega_{\varphi_j}^n \le \int_{E_j(s)} f_j \omega^n. \end{aligned}$$

Taking supremum over ρ one gets

$$\frac{s^n}{N^n} a_j \le \int_{E_j(s)} f_j \omega^n. \tag{3.1}$$

Hence, applying Hölder's inequality we infer

$$\begin{aligned} s^{n+1} N^{-n} a_j &\le \int_M |\varphi - \varphi_j| f_j \omega^n \le (\int_M |\varphi - \varphi_j|^q \omega^n)^{1/q} (\int_M f_j^p \omega^n)^{1/p} \\ &\le C(\int_M |\varphi - \varphi_j|^q \omega^n)^{1/q}. \end{aligned}$$

Thus we have obtained the estimate from the statement. It is known (see e.g. [H]) that a sequence of plurisubharmonic functions which is weakly convergent converges also in L^q_{loc} for any $1 \le q < \infty$. The same is true for ω-psh functions if we apply the above statement for

$$v + \varphi_j \to v + \varphi$$

where v is a local potential for ω. Therefore

$$\lim_{j\to\infty} a_j \le \lim_{j\to\infty} C N^n s^{-n-1} ||\varphi - \varphi_j||_{L^q(\omega^n)} = 0.$$

This proves our statement.

4. Stability

In this paper we call a function $F : \mathbb{R}_+ \to \mathbb{R}_+$ *admissible* if F has the form

$$F(x) = \frac{Ax}{h(x^{-1/n})},$$

with $A > 0$ and $h : \mathbb{R}_+ \to [1, \infty)$ is nondecreasing and such that

$$\int_1^\infty y^{-1} h^{-1/n}(y)\, dy < \infty.$$

We shall study the stability of the solutions of the complex Monge-Ampère equation when the function on the right hand side belongs to the set

$$\mathcal{F}(F) = \{f \in L^1(M) : f \geq 0, \int_M f\omega^n = 1,$$
$$\int_E f\omega^n \leq F(cap_\omega(E)) \text{ for any Borel set } E \subset M\},$$

where F is an admissible function.

It easily follows from the results of [K2] and Lemma 2.2 that the Monge-Ampère equation has a continuous solution for any $f \in \mathcal{F}(F)$. Indeed, let us construct $f_j = t_j g_j$, where $g_j = \min(f, j)$ and $t_j > 0$ is chosen so that $\int_M f_j \omega^n = 1$. Since $f \in L^1(M)$ we have $\lim_{j\to\infty} t_j = 1$ and so for j big enough $f_j \in \mathcal{F}(2F)$. Therefore the ω-psh solutions of

$$\omega_{\varphi_j}^n = f_j \omega^n, \quad \max_M \varphi_j = 0$$

are uniformly bounded (see Lemma 2.2). Those solutions exist by [K2] since $f_j \in L^2(M)$. Finally, Lemma 2.2.1 from [K2] shows that $\varphi = (\limsup \varphi_j)^*$ solves

$$\omega_\varphi^n = f\omega^n.$$

By Lemma 2.2, given admissible F, there exists a positive constant which shall be denoted by $a(F)$ such that for any $f \in \mathcal{F}(F)$ the ω-psh solution φ of

$$\omega_\varphi^n = f\omega^n, \quad \max_M \varphi = 0,$$

satisfies $\varphi \geq -a(F)$. We shall denote by κ_F the function from Lemma 2.2 with $C = a(2F)$. Thus, for $F(x) = \frac{Ax}{h(x^{-1/n})}$ we set

$$\kappa_F(s) = c(n) A^{1/n} (1 + a(2F)) [\int_{s^{-1/n}}^\infty y^{-1} h^{-1/n}(y)\, dy + h^{-1/n}(s^{-1/n})].$$

Theorem 4.1. *Let us consider two functions f and g belonging to $\mathcal{F}(F)$ for some admissible F and the corresponding solutions of*

$$\omega_\varphi^n = f\omega^n, \quad \omega_\psi^n = g\omega^n$$

normalized by

$$\max_M(\varphi - \psi) = \max_M(\psi - \varphi).$$

Set $q = q(n) = (\frac{2}{3})^{1/n}$ and $\delta = \delta(n) = \frac{1}{2}(q-1)(1-2^{-n})$. Define an incraesing function on $\mathbb{R}_+$ by

$$\gamma(t) = \frac{(2a(2F))^n}{(a(2F)+1)^n}\frac{\delta}{\delta+1}\kappa_F^{-1}(t)$$

(where $\kappa_F^{-1}(t)$ denotes the inverse of $\kappa_F(t)$). Then the inequality

$$||f-g||_1 \leq \gamma(t)t^{n+1}$$

implies

$$||\varphi - \psi||_\infty \leq (4a(2F)+2)t$$

for $t < t_0$ with $t_0 > 0$ depending on γ.

Proof. Put $a = a(2F)$. One can assume that

$$\int_{\{\psi<\varphi\}} (f+g)\omega^n \leq 1 \tag{4.1}$$

since otherwise we may interchange the roles of φ and ψ. Since $\lim_{t\to 0}\gamma(t) = 0$ one can fix $t_0 < \delta/(2\delta+1)$ and such that $\gamma(t_0)t_0^{n+1} < 1/3$. From now on we shall work with fixed $t < t_0$. Denote by E_k the set $\{\psi < \varphi - kat\}$ and put

$$C_0 = \int_{E_2} g\omega^n.$$

Then by (4.1) and the assumptions

$$\begin{aligned}\int_{E_0} g\omega^n &= \frac{1}{2}\int_{E_0}[(f+g)+(g-f)]\omega^n \\ &\leq \frac{1}{2}(1+\gamma(t_0)t_0^{n+1}) \leq \frac{2}{3}.\end{aligned} \tag{4.2}$$

Define ω-psh function ρ as the solution of

$$\omega_\rho^n = g_1\omega^n, \quad \max_M \rho = 0.$$

where $g_1 = (3/2)g$ on E_0 and h is equal to a constant $c_0 \geq 0$ elsewhere, with c_0 chosen so that $\int_M g_1\omega^n = 1$. (Observe that (4.2) implies $c_0 \geq 0$.) The solution belongs to $\mathcal{F}(2F)$ and so

$$\rho \geq -a.$$

Adding the same constant to φ and ψ (which does not influence neither the hypothesis nor the assertion) one can assume that

$$-a \le \varphi \le 0.$$

The last two inequalities entail

$$E_2 \subset E := \{\psi < (1-t)\varphi + t\rho - at\} \subset E_0.$$

Let us denote by G the set $\{f < (1-t)g\}$. From Lemma 1.2 we know that for $k < n$ the following inequalities hold on $E_0 \setminus G$:

$$\omega_\varphi^k \wedge \omega_\rho^{n-k} \ge q^{n-k}(1-t)^{k/n} g\omega^n \ge q(1-t)g\omega^n.$$

Therefore on $E_0 \setminus G$

$$\omega^n_{t\rho+(1-t)\varphi} \ge (q - \frac{q-1}{2^n})(1-t)g\omega^n. \tag{4.3}$$

By the assumptions we have

$$t\int_G g\omega^n \le \int_G (g-f)\omega^n \le \gamma(t)t^{n+1}.$$

Hence

$$\int_G g\omega^n \le \gamma(t)t^n. \tag{4.4}$$

The following chain of inequalities is obtained by applying, in turn, formula (4.3), the comparison principle, and formula (4.4):

$$\begin{aligned}(q - \frac{q-1}{2^n})(1-t)\int_{E\setminus G} g\omega^n &\le \int_E \omega^n_{t\rho+(1-t)\varphi} \le \int_E g\omega^n \\ &\le \int_{E\setminus G} g\omega^n + \gamma(t)t^n.\end{aligned} \tag{4.5}$$

Since, by the choice of of δ and t_0

$$(q - \frac{q-1}{2^n})(1-t) - 1 = (2\delta+1)(1 - \frac{\delta}{2\delta+1}) - 1 = \delta,$$

we infer from (4.5) that

$$\delta \int_{E\setminus G} g\omega^n \le \gamma(t)t^n.$$

Therefore

$$\delta(C_0 - \gamma(t)t^n) \le \delta(\int_{E_2} g\omega^n - \int_G g\omega^n) \le \gamma(t)t^n,$$

and further

$$C_0 \le (1+\frac{1}{\delta})\gamma(t)t^n.$$

We also have an estimate which is analogous to (3.1) with the suitable change of notation

$$cap_\omega(E_4) \le \frac{(||(1-t)\varphi + t\rho|| + 1)^n}{(2at)^n}\int_{E_2} g\omega^n.$$

Observe that $-a \le (1-t)\varphi + t\rho \le 0$. So coupling the last two estimates one obtains

$$\begin{aligned} cap_\omega(E_4) &\le (2ta)^{-n}(a+1)^n C_0 \le (a+1)^n(2a)^{-n}(1+\delta^{-1})\gamma(t) \\ &\le (a+1)^n(2a)^{-n}(1+\delta^{-1})\gamma(t). \end{aligned}$$

Suppose $E' = \{\psi < \varphi - (4a+2)t\}$ were nonempty. Then by Lemma 2.2, the above estimate and the definition of γ we would have

$$2t \le \kappa_F(cap_\omega(E_4)) \le \kappa_F((a+1)^n(2a)^{-n}(1+\delta^{-1})\gamma(t)) = t$$

which is a contradiction. Therefore E' is empty which translates into the desired estimate

$$\max(\psi - \varphi) = \max(\varphi - \psi) \le (4a+2)t.$$

The proof is completed.

It follows from Theorem 4.1 that for $f \in \mathcal{F}(F)$ with admissible F the solution of (0.1), normalized by $\max_M \varphi = 0$, is unique. This ia a generalization of the result of Calabi [C].

Corollary 4.2. *If φ_1 and φ_2 solve*

$$\omega_{\varphi_1}^n = f\omega^n = \omega_{\varphi_2}^n$$

with $f \in \mathcal{F}(F)$ for some admissible F then $\varphi_1 - \varphi_2 = const.$

Next we shall make the estimate from Theorem 4.1 more explicit in the case when f, g belong to a bounded set in $L^p(\omega^n)$. For $p > 1$ we denote by $\mathcal{F}_p(c_0)$ the set of functions

$$\{f \in L^p(M) : f \ge 0, \int_M f\omega^n = 1, \int_M f^p\omega^n \le c_0\}.$$

An analogue of the following estimate was proved in [K4] for subsets of strictly pseudoconvex domains in $\mathbb{C}^n$.

Lemma 4.3. *Let $p > 1$, and $f \in L^p(M)$. Then for some $c_M > 0$ and any compact K in M*

$$\int_K f\,dV \le c(p,M)||f||_p a(K)\exp(-c_M a^{-1/n}(K))$$

where we use the notation $a(K) = cap_\omega(K)$ and the constant $c(p, M)$ depends on p, and M.

Proof. Consider the covering V_s, the sets K_s and numbers δ, N as in Section 1. Then, by Lemma 4 in [K4] we have for $a_s = cap(K_s, V_s)$ and $m < \min(4\pi - 1, p-1)$:

$$\int_K f\, dV \le \sum_s \int_{K_s} f\, dV \le c'(p, M)||f||_p \sum_s a_s \exp(-m a_s^{-1/n}).$$

Using the equivalence of cap_ω and cap'_ω from Section 1 we therefore get

$$\int_K f\, dV \le c'(p, M)\delta N ||f||_p a(K) \exp(-m\delta^{-1/n} a^{-1/n}(K)).$$

So we get the statement with $c(p, M) = c'(p, M)\delta N$ and $c_M = m\delta^{-1/n}$.

It follows from Lemma 4.3 that

$$\mathcal{F}_p(c_0) \subset \mathcal{F}(F)$$

with $F(x) = Ac_0 \exp(-c_M x^{-1/n})$ for some $A > 0$ depending on and M. We estimate κ_F for F given above:

$$\kappa_F(s) \le c(n) A^{1/n}(1 + a(F))(1 + \frac{n}{c_M}) \exp(-\frac{c_M}{n} s^{-1/n}) \le c_F \exp(-\frac{c_M}{n} s^{-1/n})$$

where the constant c_F depends on F. Hence

$$\begin{aligned}\gamma(t) &= (2a(2F))^n (a(2F) + 1)^{-n} \frac{\delta}{\delta + 1} \kappa_F^{-1}(t) \\ &\ge (2a(2F))^n (a(2F) + 1)^{-n} \frac{\delta}{\delta + 1} (-\frac{n}{c_M} \log \frac{t}{c_F})^{-n} = c_0 \log^{-n} \frac{c_F}{t}.\end{aligned}$$

Thus we have obtained the following corollary to Theorem 4.2.

Corollary 4.4. *Given $p > 1$ and $c_0 > 0$ there exist positive constants α, β and t_0 such that for $f, g \in \mathcal{F}_p(c_0)$ satisfying*

$$||f - g||_1 \le \alpha(-\log t)^{-n} t^{n+1}, \quad t < t_0,$$

one can find ω-psh solutions of the equations

$$\omega_\varphi^n = f\omega^n, \quad \omega_\psi^n = g\omega^n$$

subject to

$$||\varphi - \psi||_\infty \le \beta t, \quad t < t_0.$$

In particular for any $\epsilon > 0$ there exists $c(\epsilon)$ with

$$||\varphi - \psi||_\infty \le c(\epsilon) ||f - g||_1^{1/(n+1+\epsilon)}$$

(for suitably normalized φ and ψ).

Let us denote by $\mathcal{P}_p(c)$ the inverse image of $\mathcal{F}_p(c)$ via the Monge-Ampère operator

$$\mathcal{P}_p(c) = \{\varphi \in PSH(\omega) : \omega_\varphi^n = f\omega^n, ||f||_p \le c\}.$$

In this family the weak convergence is equivalent to the uniform convergence.

Theorem 4.5. *Suppose $\varphi_j \in \mathcal{P}_p(c)$ and $\varphi_j \to \varphi$ weakly. Then φ_j tends to φ uniformly.*

Proof. As it was mentioned in Section 1, under our assumptions, $\varphi_j \to \varphi$ in $L^q(M)$, where q is chosen to be conjugate to p. Applying Lemma 3.1 (and its notation) we have for given $s > 0$

$$cap_\omega(E_j(2s)) \to 0$$

as $j \to \infty$. Then from Lemma 2.2 we deduce that $E_j(2s)$ is empty for j large enough. Since, by Hartogs' lemma, $\varphi = (\limsup \varphi_j)^*$ the statement follows.

Corollary 4.6. *$\mathcal{P}_p(c)$ is equicontinuous.*

Proof. Arguing by contradiction, suppose that for some $\epsilon > 0, \varphi_j \in \mathcal{P}_p(c)$ and for some two sequences z_j, w_j of points in M we had $dist(z_j - w_j) < j^{-1}$ and $\varphi_j(z_j) - \varphi_j(w_j) > \epsilon$. One can pick subsequences z_{j_k}, w_{j_k} converging to z and φ_{j_k} converging in L^1 norm to $\varphi \in PSH(\omega)$. By Theorem 4.5 φ_{j_k} converges uniformly. From this fact and the triangle inequality we easily deduce a contradiction for k large enough:

$$|\varphi_{j_k}(z_{j_k}) - \varphi_{j_k}(w_{j_k})| < \epsilon.$$

Theorem 4.5 and the corollary following it hold in wider families of solutions. For $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\psi(x)/x$ increases to ∞ as $x \to \infty$ set

$$L_c^\psi(M) = \{g \in L^1(M) : \int_M \psi(g)\omega^n \leq c\}.$$

Then for any admissible F the inverse image through the Monge-Ampère operator of $\mathcal{F}(F) \cap L_c^\psi(M)$ is equicontinuous. As an example: the solutions corresponding to $f \in L_c^\psi(M)$ with $\psi(x) = x \log^m(1+x),\ m > n$ are equicontinuous. To show this one has to strengthen Lemma 3.1 along the same lines as in [K4, proof of Theorem 1].

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