A local characterization of affine holomorphic immersions with an anti-complex and ∇-parallel shape operator.

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Abstract

We study the complex hypersurfaces $f : M^{(n)} \to \mathbf{C}^{n+1}$ which together with their transversal bundles have the property that around any point of M there exists a local section of the transversal bundle inducing a ∇-parallel, anti-complex shape operator. We give a class of examples of such hypersurfaces with an arbitrary rank of S from 1 to $[\frac{n}{2}]$ and show that every such hypersurface with $tf > 0$ and $S \neq 0$ is locally of this kind, modulo an affine isomorphism of $\mathbf{C}^{n+1}$.

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1. Introduction. Among the connections induced on complex hypersurfaces $f : M^{(n)} \to \mathbf{C}^{n+1}$ by $\mathcal{C}^\infty$ complex transversal bundles there are two particular kinds of great interest - a holomorphic one and an affine Kähler one. The latter is meant to be a generalisation of the Kähler connection. In terms of the curvature tensor, the holomorphic affine connection is characterized by the condition

$$R(JX, Y) = JR(X, Y) \text{ for all vector fields } X, Y,$$

while for the affine Kähler connection we have, by definition,

$$R(JX, JY) = R(X, Y) \text{ for all } X, Y \text{ [NS].}$$

Since, if only the affine fundamental form h does not vanish on M, a holomorphic connection is induced by a holomorphic transversal bundle, it is possible to adapt the ideas from the real affine hypersurface geometry for this case. For instance, having a non-degenerate hypersurface one can construct a holomorphic analogue of affine normal vector field [DVV]. Then one can consider the condtion $S = \lambda I,\ \ \lambda =$ const, which describes the affine spheres [DVV].

On the contrary, the non-flat affine Kähler connections induced on hypersurfaces (in particular - the non-flat Kähler ones) can not be treated in this way. Instead of the holomorphic transversal bundles, with the complex shape operator, one has to consider the transversal bundles $\mathcal{N}$ having the property that the shape operator corresponding to sections of $\mathcal{N}$ is anti-complex. This property of $\mathcal{N}$ implies the desired condition for the curvature tensor R of ∇ and is necessary for ∇ to be affine Kähler if $tf > 1$ at some point of M [O].

Clearly, no section ξ of such a bundle can induce S proportional to the identity, except for the case $S = 0$. The ∇ - parallelity of S is a weaker condition than $S = \lambda I,\ \lambda =$ const. This condition is shown to have some non-trivial realisations even if we require S to be anti-complex. It is worth noting that we need to consider degenerate hypersurfaces, because the non-degeneracy implies $S = 0$ (Lemma 2).

2. Preliminaries. Let M be an n-dimensional connected complex manifold. We shall consider a holomorphic immersion $f : M \to \mathbf{C}^{n+1}$ together with a $\mathcal{C}^\infty$ complex transversal bundle $\mathcal{N}$. If $\xi : U \to \mathbf{C}^{n+1}$ is a local section of $\mathcal{N}$, then the induced connection ∇ on M, the second fundamental form h, the shape operator S and the transversal forms μ and ν are defined by the following Gauss and Weingarten formulas [NS]:

$$D_X f_* Y = f_* \nabla_X Y + h(X, Y)\xi - h(JX, Y)J\xi,$$

$$D_X \xi = -f_* SX + \mu(X)\xi + \nu(X)J\xi.$$

Here D denotes the standard connection on $\mathbf{C}^{n+1}$, and J the complex structure on M and on $\mathbf{C}^{n+1}$ as well.

Let $m \in M$. The complex rank of the $\mathbf{C}$-bilinear form $h^c_m(\cdot, \cdot) = h_m(\cdot, \cdot) - ih_m(J\cdot, \cdot)$ depends on f only. It is called the type number of f at m and denoted by tf_m [O]. We shall assume that it is positive everywhere on M.

Our first requirement on the transversal bundle is that the induced shape operator S is anti-complex, ie $SJ = -JS$ [O]. The fundamental equations satisfied by ∇, h, μ, ν and such an S are the following:

$$R(X,Y)Z = h(Y,Z)SX - h(X,Z)SY + h(JY,Z)SJX - h(JX,Z)SJY \qquad (Gauss),$$

$$(\nabla_X h)(Y,Z) + \mu(X)h(Y,Z) + \nu(X)h(JY,Z) = (\nabla_Y h)(X,Z) + \mu(Y)h(X,Z) + \nu(Y)h(JX,Z) \quad (Codazzi\ I),$$

$$(\nabla_X S)Y - \mu(X)SY + \nu(X)SJY = (\nabla_Y S)X - \mu(Y)SX + \nu(Y)SJX \qquad (Codazzi\ II),$$

$$h(X,SY) - h(Y,SX) = 2d\mu(X,Y) \quad (Ricci\ I),$$

$$h(SX,JY) - h(SY,JX) = 2d\nu(X,Y) \quad (Ricci\ II).$$

Furthermore, we shall assume that for every point $m \in M$ there exists a local section $\xi : U \to \mathbf{C}^{n+1}$ of $\mathcal{N}$ with $U \ni m$ such that $\nabla S^\xi = 0$, where S^ξ denotes the shape operator induced by ξ. If the points $m_1,\ m_2 \in U$ can be joined by a curve lying in U, and $\mathcal{B}_1$ is a basis of $T_{m_1}M$, then by parallel displacement we can obtain a basis $\mathcal{B}_2$ of $T_{m_2}M$ with respect to which $S^\xi_{m_2}$ has the same matrix as $S^\xi_{m_1}$ with respect to $\mathcal{B}_1$. Hence $\operatorname{rank} S^\xi_{m_1} = \operatorname{rank} S^\xi_{m_2}$, where $\operatorname{rank} S^\xi_m := \dim_{\mathbf{C}} \operatorname{im} S^\xi_m$. The assumed connectedness of M and the independence of $\operatorname{rank} S^\xi_m$ of ξ at a fixed m imply that $q := \operatorname{rank} S$ is well-defined for the whole M.

Studying the immersions with ∇-parallel shape operator we shall make use of the following remarks:

REMARK 1. *If $\nabla S = 0$, then for every X, Y, Z*

$$R(X,Y)SZ = S(R(X,Y)Z).$$

P r o o f. It is an obvious consequence of the commutativity of S and ∇_W for any W.

REMARK 2. *If $\nabla S = 0$ and $SJ = -JS$, then $\ker S \subset \ker \mu \cap \ker \nu$ or $S = 0$.*

P r o o f. Let $SX = 0$. By the second Codazzi equation we have $-\mu(X)SY + \nu(X)SJY = 0$ for any Y. If $S \neq 0$, then there exists Y such that $SY \neq 0$. Since SY and $SJY = -JSY$ are linearly independent over $\mathbf{R}$, it follows that $\mu(X) = 0$ and $\nu(X) = 0$.

REMARK 3. *If $\nabla S = 0$, $SJ = -JS$ and $S \neq 0$, then the section ξ inducing S is anti-holomorphic, ie $\nu(X) = \mu(JX)$ for any X.*

P r o o f. We may assume that $SX \neq 0$. The assertion follows easily from the second Codazzi equation, written for $Y = JX$.

For an anti-holomorphic ξ we can rewrite the first Codazzi equation as

$$\begin{aligned}&(\nabla_X h)(Y,Z) + \mu(X)h(Y,Z) + \mu(JX)h(JY,Z)\\ =&(\nabla_Y h)(X,Z) + \mu(Y)h(X,Z) + \mu(JY)h(JX,Z).\end{aligned}$$

3. Theorem. We can now formulate our main result.

THEOREM. *Let M be an n-dimensional connected complex manifold, $n > 1$, $f : M \to \mathbf{C}^{n+1}$ a holomorphic immersion and ∇ a linear connection induced on M by a transversal bundle $\mathcal{N}$. Let f and M satisfy the following assumptions:*

(1) *$tf > 0$ everywhere on M,*

(2) *for every $m \in M$ there exist its neighbourhood U and a local section $\xi : U \to \mathbf{C}^{n+1}$ of $\mathcal{N}$ inducing an anti-complex and ∇-parallel shape operator S,*

(3) *$q := \operatorname{rank} S > 0$.*

Under the conditions stated above,

(i) *$q \leq \frac{n}{2}$;*

(ii) *for every $m \in M$ there exist its neighbourhood V, a complex map $\tilde{\phi} : V \to \mathbf{C}^n \tilde{=} \mathbf{C}^q \times \mathbf{C}^{n-2q} \times \mathbf{C}^q$, a complex affine isomorphism $\tilde{A} : \mathbf{C}^{n+1} \to \mathbf{C}^{n+1}$, a holomorphic function $\tilde{\mathcal{F}}$ such that*

$$\begin{aligned}&\tilde{A} \circ f \circ \tilde{\phi}^{-1}(\tilde{x}^1, ..., \tilde{x}^q, \tilde{y}^1, ..., \tilde{y}^{n-2q}, \tilde{z}^1, ..., \tilde{z}^q)\\ &\qquad =(\tilde{x}^1, ..., \tilde{x}^q, \tilde{y}^1, ..., \tilde{y}^{n-2q}, \tilde{z}^1, ..., \tilde{z}^q, \tilde{\mathcal{F}}(\tilde{y}^1, ..., \tilde{y}^{n-2q}, \tilde{z}^1, ..., \tilde{z}^q));\end{aligned}$$

(iii) *if $q > 1$, then the local section $\vec{\tilde{A}} \circ \xi : V \to \mathbf{C}^{n+1}$ of $\vec{\tilde{A}}\mathcal{N}$ iducing the ∇-parallel shape operator in this map is described by the following formula:*

$$\vec{\tilde{A}} \circ \xi \circ \tilde{\phi}^{-1}(\tilde{x}^1, ..., \tilde{x}^q, \tilde{y}^1, ..., \tilde{y}^{n-2q}, \tilde{z}^1, ..., \tilde{z}^q) = (\overline{\tilde{z}^1}, ..., \overline{\tilde{z}^q}, \underbrace{0, ..., 0}_{n-q \text{ times}}, 1),$$

here $\vec{\tilde{A}}$ denotes the linear part of $\tilde{A}$;

(iv) *if $q = 1$, then*

$$\vec{\tilde{A}} \circ \xi \circ \tilde{\phi}^{-1}(\tilde{x}, \tilde{y}^1, ..., \tilde{y}^{n-2}, \tilde{z}) = (\overline{\tilde{\mathcal{G}}(\tilde{z})}, \underbrace{0, ..., 0}_{n-1 \text{ times}}, \overline{e^{\tilde{\mathcal{M}}(\tilde{z})}})$$

where $\tilde{\mathcal{G}}$ and $\tilde{\mathcal{M}}$ are holomorphic functions such that

$$\tilde{\mathcal{G}}'(\tilde{z}) - \tilde{\mathcal{M}}'(\tilde{z})\tilde{\mathcal{G}}(\tilde{z}) \equiv 1.$$

In the real representation, setting $\tilde{x}^k = x^{2k-1} + ix^{2k}$, $\tilde{y}^l = y^{2l-1} + iy^{2l}$, $\tilde{z}^j = z^{2j-1} + iz^{2j}$ *for* $j, k = 1, ..., q$; $l = 1, ..., n-2q$; $\tilde{\mathcal{F}} = \mathcal{F}^1 + i\mathcal{F}^2$, $\tilde{\mathcal{G}} = \mathcal{G}^1 + i\mathcal{G}^2$, $\tilde{\mathcal{M}} = \mathcal{M}^1 + i\mathcal{M}^2$, *we have:*

$$A \circ f \circ \phi^{-1}(x^1, ..., x^{2q}, y^1, ..., y^{2n-4q}, z^1, ..., z^{2q}) =$$
$$(x^1, ..., x^{2q}, y^1, ..., y^{2n-4q}, z^1, ..., z^{2q}, \mathcal{F}^1(y, z), \mathcal{F}^2(y, z)),$$

$$\vec{A} \circ \xi \circ \phi^{-1}(x^1, ..., x^{2q}, y^1, ..., y^{2n-4q}, z^1, ..., z^{2q}) =$$
$$(z^1, -z^2, ..., z^{2q-1}, -z^{2q}, \underbrace{0, ..., 0}_{2n-2q \text{ times}}, 1, 0)$$

if $q > 1$

and

$$\vec{A} \circ \xi \circ \phi^{-1}(x^1, x^2, y^1, ..., y^{2n-4}, z^1, z^2) =$$
$$(\mathcal{G}^1(z), -\mathcal{G}^2(z), \underbrace{0, ..., 0}_{2n-2 \text{ times}}, e^{\mathcal{M}^1(z)} \cos \mathcal{M}^2(z), -e^{\mathcal{M}^1(z)} \sin \mathcal{M}^2(z)),$$

if $q = 1$.

REMARK 4. An easy computation shows that the inverse is also true:

(a)*For any holomorphic function* $\tilde{\mathcal{F}}$ *of* $n - q$ *variables, where* $q \leq \frac{n}{2}$, *the shape operator* S *induced on the hypersurface*

$$f : \mathbf{C}^n \supset U \ni (\tilde{x}^1, \dots, \tilde{x}^q, \tilde{y}^1, \dots, \tilde{y}^{n-2q}, \tilde{z}^1, \dots, \tilde{z}^q) \mapsto$$
$$\mapsto (\tilde{x}^1, \dots, \tilde{x}^q, \tilde{y}^1, \dots, \tilde{y}^{n-2q}, \tilde{z}^1, \dots, \tilde{z}^q, \tilde{\mathcal{F}}(\tilde{y}, \tilde{z})) \in \mathbf{C}^{n+1}$$

endowed with the transversal field

$$\xi(\tilde{x}^1, \dots, \tilde{x}^q, \tilde{y}^1, \dots, \tilde{y}^{n-2q}, \tilde{z}^1, \dots, \tilde{z}^q) = (\overline{\tilde{z}^1}, \dots, \overline{\tilde{z}^q}, \underbrace{0, \dots, 0}_{\text{n-q times}}, 1)$$

is parallel with respect to the induced connection and $\operatorname{rank} S = q$.

(b)*For any holomorphic function* $\tilde{\mathcal{F}}$ *of* $n - 1$ *variables and for any holomorphic functions of one variable* $\tilde{\mathcal{G}}$ *and* $\tilde{\mathcal{M}}$ *satisfying the equation*

$$\tilde{\mathcal{G}}'(\tilde{z}) - \tilde{\mathcal{M}}'(\tilde{z})\tilde{\mathcal{G}}(\tilde{z}) \equiv 1.$$

the shape operator S *induced on the hypersurface*

$$f : \mathbf{C}^n \supset U \ni (\tilde{x}, \tilde{y}^1, \dots, \tilde{y}^{n-2}, \tilde{z}) \mapsto (\tilde{x}, \tilde{y}^1, \dots, \tilde{y}^{n-2}, \tilde{z}, \tilde{\mathcal{F}}(\tilde{y}, \tilde{z})) \in \mathbf{C}^{n+1}$$

endowed with the transversal field

$$\xi(\tilde{x}, \tilde{y}^1, \dots, \tilde{y}^{n-2}, \tilde{z}) = (\overline{\tilde{\mathcal{G}}(\tilde{z})}, \underbrace{0, \dots, 0}_{\text{n-1 times}}, \overline{e^{\tilde{\mathcal{M}}(\tilde{z})}})$$

is parallel with respect to the induced connection and $\operatorname{rank} S = 1$.

P r o o f o f t h e t h e o r e m. We begin by proving two lemmas in which we establish some inclusions between $\operatorname{im} S$, $\ker S$ and $\ker h$.

LEMMA 1. *If* $tf > 0$, $SJ = -JS$, *and* $\nabla S = 0$, *then the following conditions are equivalent:*

(1) $d\mu = 0$ *and* $d\nu = 0$,

(2) $\operatorname{im} S \subset \ker h$,

(3) $\operatorname{im} S \subset \ker S$.

P r o o f.
(1)⇒(2). Suppose that $d\mu = 0$ and $d\nu = 0$. Let $m' \in M$ and $X, Y \in T_{m'}M$. Applying the Ricci equations we have

$$\begin{aligned} 0 &= 2d\mu(X, Y) + 2d\nu(JX, Y) \\ &= h(X, SY) - h(Y, SX) - h(J^2X, SY) + h(JY, SJX) \\ &= h(X, SY) - h(Y, SX) + h(X, SY) + h(JY, -JSX) \\ &= h(X, SY) - h(Y, SX) + h(X, SY) + h(Y, SX) = 2h(X, SY), \end{aligned}$$

hence for any $X, Y \in T_{m'}M$ we have $h(X, SY) = 0$.
The Ricci equations make it obvious that (2) implies (1).
(2)⇒(3).
Suppose that $\operatorname{im} S \subset \ker h$, which yields $R(X, Y)SZ = 0$ by the Gauss equation. Since $tf_{m'} > 0$, there exist $X_0, Y_0 \in T_{m'}M$ such that $h(X_0, Y_0) \neq 0$. We first show that $S^2X_0 = 0$. Indeed, making use of Remark 1 we have

$$0 = S(R(X_0, JX_0)Y_0) = 2(h(JX_0, Y_0)S^2X_0 - h(X_0, Y_0)S^2JX_0)$$

with $h(X_0, Y_0) \neq 0$, which means that S^2X_0 and S^2JX_0 are linearly dependent over $\mathbf{R}$. It is possible only when $S^2X_0 = S^2JX_0 = 0$.
Now we take an arbitrary $Z \in T_{m'}M$. Then

$$0 = S(R(X_0, Z)Y_0) = -h(X_0, Y_0)S^2Z - h(JX_0, Y_0)S^2JZ,$$

and by a similar argument $S^2Z = 0$.
(3)⇒(2). If $S^2 = 0$, then the right-hand side of the equality $R(X, Y)SZ = S(R(X, Y)Z)$ vanishes for every X, Y, Z. For $S = 0$ (2) holds, therefore we can assume that $S \neq 0$. Let us take X_0 such that $SX_0 \neq 0$. Since

$$0 = R(X_0, JX_0)SZ = 2(h(JX_0, SZ)SX_0 - h(X_0, SZ)SJX_0)$$

and SX_0, SJX_0 are linearly independent over $\mathbf{R}$, we have

$$h(X_0, SZ) = h(JX_0, SZ) = 0 \quad \text{for any } Z.$$

Now we can write for any Y, Z

$$0 = R(X_0, Y)SZ = h(Y, SZ)SX_0 + h(JY, SZ)SJX_0.$$

Hence $h(Y, SZ) = 0$ for any Y, Z. ■

LEMMA 2. *Under the assumptions of Lemma 1, the equivalent conditions* (1), (2) *and* (3) *are satisfied.*

P r o o f. If $\nabla S = 0$, then $\operatorname{rank} S$ is constant on the domain of S. We have to consider three cases.

1°. $\operatorname{rank} S = 0$. Then, of course , (3) holds.

2°. $\operatorname{rank} S = 1$. We shall prove that in this case (3) holds, too.

Suppose, contrary to our claim, that $S^2 \neq 0$. We shall obtain a contradiction with the assumption $tf > 0$.

Let $m' \in M$. We fix $X_0 \in T_{m'}M$ such that $S^2 X_0 \neq 0$.

STEP 1. $\ker S \subset \ker h$.

Let $Z \in \ker S$. Then

$$0 = R(X_0, JX_0)SZ = S(R(X_0, JX_0)Z) = 2(h(JX_0, Z)S^2 X_0 - h(X_0, Z)S^2 JX_0),$$

hence $h(X_0, Z) = h(JX_0, Z) = 0$. For any Y we have now

$$0 = R(X_0, Y)SZ = S(R(X_0, Y)Z) = h(Y, Z)S^2 X_0 + h(JY, Z)S^2 JX_0,$$

which implies $h(Y, Z) = 0$ for any $Z \in \ker S$ and for any Y.

STEP 2.

(a) $T_{m'}M = \ker S \oplus \mathbf{C}X_0$

and

(b) $T_{m'}M = \ker S \oplus \mathbf{C}SX_0$.

Since $\dim_{\mathbf{C}} \ker S = n-1$, it is sufficient and easy to check that $\ker S \cap \mathbf{C}X_0 = \{0\}$ and $\ker S \cap \mathbf{C}SX_0 = \{0\}$.

STEP 3. $h(X_0, SX_0) = h(JX_0, SJX_0) = 0$.

If $h(X_0, X_0) = 0$ and $h(X_0, JX_0) = 0$ then $X_0 \in \ker h$ because of Steps 1 and 2(a). The claimed equality holds.

Assume now that $h(X_0, X_0) \neq 0$ or $h(X_0, JX_0) \neq 0$. We have

$$\begin{aligned} 0 &= R(X_0, JX_0)SX_0 - S(R(X_0, JX_0)X_0 \\ &= 2S(h(JX_0, SX_0)X_0 - h(X_0, SX_0)JX_0 - h(JX_0, X_0)SX_0 + h(X_0, X_0)SJX_0), \end{aligned}$$

therefore $Z_0 \in \ker S$, where

$$Z_0 := h(JX_0, SX_0)X_0 - h(X_0, SX_0)JX_0 - h(JX_0, X_0)SX_0 + h(X_0, X_0)SJX_0.$$

According to Step 1, we have $h(Z_0, X_0) = 0$ and $h(Z_0, JX_0) = 0$, hence

$$\begin{aligned} &h(JX_0, SX_0)h(X_0, X_0) - h(X_0, SX_0)h(JX_0, X_0) \\ &- h(JX_0, X_0)h(SX_0, X_0) + h(X_0, X_0)h(SJX_0, X_0) = 0 \end{aligned}$$

and

$$\begin{aligned} &h(JX_0, SX_0)h(X_0, JX_0) - h(X_0, SX_0)h(JX_0, JX_0) \\ &- h(JX_0, X_0)h(SX_0, JX_0) + h(X_0, X_0)h(SJX_0, JX_0) = 0. \end{aligned}$$

Thus we obtain

$$h(SX_0, X_0)h(JX_0, X_0) = 0 \quad \text{and} \quad h(SX_0, X_0)h(X_0, X_0) = 0,$$

which implies $h(SX_0, X_0) = 0$, and consequently, by the anti-complexity of S and the properties of $h(\cdot\,,\cdot)$, $h(SJX_0, JX_0) = 0$.

Step 4. *$h(Z, SW) + h(W, SZ) = 0$ for any $Z, W \in T_{m'}M$.*

By Step 2(a) we have $Z = Z_1 + \alpha X_0 + \beta JX_0$, $W = W_1 + \gamma X_0 + \delta JX_0$ with Z_1, $W_1 \in \ker S$ and $\alpha, \beta, \gamma, \delta \in \mathbf{R}$. An easy computation gives

$$\begin{aligned} &h(Z, SW) + h(W, SZ) \\ &= h(\alpha X_0 + \beta JX_0, S(\gamma X_0 + \delta JX_0)) + h(\gamma X_0 + \delta JX_0, S(\alpha X_0 + \beta JX_0)) \\ &= 2\alpha\gamma h(X_0, SX_0) + 2\beta\delta h(JX_0, SJX_0) + (\alpha\delta + \beta\gamma)h(X_0, (SJ + JS)X_0), \end{aligned}$$

which vanishes by the anti-complexity of S and by Step 3.

Step 5. *$(\nabla_W h)(X_0, SX_0) = (\nabla_W h)(JX_0, SJX_0) = 0$ for any W.*

We extend X_0 to the local vector field X_0 such that $S^2X_0 \neq 0$ at any point of the domain of X_0. We have

$$\begin{aligned} (\nabla_W h)(X_0, SX_0) &= W(h(X_0, SX_0)) - h(\nabla_W X_0, SX_0) - h(X_0, \nabla_W(SX_0)) \\ &= W(h(X_0, SX_0)) - (h(\nabla_W X_0, SX_0) + h(X_0, S(\nabla_W X_0))) = 0. \end{aligned}$$

The same is true for JX_0 in place of X_0.

Step 6. *$\nabla h = -2\mu \otimes h$.*

If $Y \in \ker S_{m'}$, then we can extend Y to a local section Y of $\ker S$. For any X, Z $\nabla_X Y \in \ker S$ and

$$\begin{aligned} &(\nabla_X h)(Y, Z) \\ &= X(h(Y, Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z) = 0 \\ &= -2\mu(X)h(Y, Z). \end{aligned}$$

Let $X \in \ker S$. Then for any Y, Z we have

$$
\begin{aligned}
&(\nabla_X h)(Y, Z) \underset{\text{Codazzi I}}{=} \\
&- \mu(X)h(Y, Z) - \mu(JX)h(JY, Z) \\
&+ (\nabla_Y h)(X, Z) + \mu(Y)h(X, Z) + \mu(JY)h(JX, Z) \\
&= 0 = -2\mu(X)h(Y, Z),
\end{aligned}
$$

since $\ker S \subset \ker \mu \cap \ker \nu$ and $\ker S \subset \ker h$.

It follows that

$$(\nabla_X h)(Y, \cdot) = -2\mu(X)h(Y, \cdot)$$

if $X \in \ker S$ or $Y \in \ker S$.

From the first Codazzi equation we obtain

$$
\begin{aligned}
&(\nabla_{X_0} h)(JX_0, SX_0) + \mu(X_0)h(JX_0, SX_0) + \mu(JX_0)h(J^2X_0, SX_0) \\
=&(\nabla_{JX_0} h)(X_0, SX_0) + \mu(JX_0)h(X_0, SX_0) + \mu(J^2X_0)h(JX_0, SX_0).
\end{aligned}
$$

Hence

$$(\nabla_{X_0} h)(JX_0, SX_0) = -2\mu(X_0)h(JX_0, SX_0),$$

by virtue of Steps 3 and 5.

Similarly,

$$
\begin{aligned}
&(\nabla_{X_0} h)(JX_0, SJX_0) + \mu(X_0)h(JX_0, SJX_0) + \mu(JX_0)h(J^2X_0, SJX_0) \\
=&(\nabla_{JX_0} h)(X_0, SJX_0) + \mu(JX_0)h(X_0, SJX_0) + \mu(J^2X_0)h(JX_0, SJX_0),
\end{aligned}
$$

which gives

$$(\nabla_{JX_0} h)(X_0, SJX_0) = -2\mu(JX_0)h(X_0, SJX_0).$$

We have now

$$
\begin{aligned}
&(\nabla_{X_0} h)(X_0, SX_0) = 0 = -2\mu(X_0)h(X_0, SX_0), \\
&(\nabla_{X_0} h)(X_0, SJX_0) = -(\nabla_{X_0} h)(JX_0, SX_0) \\
&= 2\mu(X_0)h(JX_0, SX_0) = -2\mu(X_0)h(X_0, SJX_0)
\end{aligned}
$$

and

$$(\nabla_{X_0} h)(X_0, Z) = -2\mu(X_0)h(X_0, Z)$$

for $Z \in \ker S$. Therefore

$$(\nabla_{X_0} h)(X_0, \cdot\ ) = -2\mu(X_0)h(X_0, \cdot\ )$$

by Step 2(b).

In the same manner we can see that

$$(\nabla_{X_0} h)(JX_0, \cdot\ ) = -2\mu(X_0)h(JX_0, \cdot\ ),$$

$$(\nabla_{JX_0} h)(X_0, \cdot\ ) = -2\mu(JX_0)h(X_0, \cdot\ ),$$

and

$$(\nabla_{JX_0}h)(JX_0,\cdot\,) = -2\mu(JX_0)h(JX_0,\cdot\,),$$

which completes the proof of Step 6.

As a consequence of Step 6 we obtain

STEP 7. *$R(X,Y)\cdot h = -4\; d\mu(X,Y)\; h$ for any X, Y.*

Applying the Ricci equation yields

STEP 8. *$R(X,Y)\cdot h = -2\;(h(X,SY) - h(Y,SX))\; h$.*
In particular, $R(X_0,JX_0)\cdot h = -4\; h(X_0,SJX_0)\; h$.

On the other hand, a direct computation gives

$$\begin{aligned}
&(R(X_0,JX_0)\cdot h)(X_0,X_0)\\
&= -2h(R(X_0,JX_0)X_0,X_0)\\
&= -2h(2(h(JX_0,X_0)SX_0 - h(X_0,X_0)SJX_0),X_0)\\
&= -4h(JX_0,X_0)h(SX_0,X_0) + 4h(X_0,X_0)h(SJX_0,X_0)\\
&= 4h(X_0,X_0)h(X_0,SJX_0),
\end{aligned}$$

and

$$\begin{aligned}
&(R(X_0,JX_0)\cdot h)(X_0,JX_0)\\
&= -h(R(X_0,JX_0)X_0,JX_0) - h(X_0,R(X_0,JX_0)JX_0)\\
&= -h(2(h(JX_0,X_0)SX_0 - h(X_0,X_0)SJX_0),JX_0)\\
&- h(X_0,2(h(JX_0,JX_0)SX_0 - h(X_0,JX_0)SJX_0))\\
&= -2h(JX_0,X_0)h(SX_0,JX_0) + 2h(X_0,X_0)h(SJX_0,JX_0)\\
&- 2h(JX_0,JX_0)h(X_0,SX_0) + 2h(X_0,JX_0)h(X_0,SJX_0)\\
&= 4h(JX_0,X_0)h(X_0,SJX_0).
\end{aligned}$$

A comparison with Step 8 gives

$$h(X_0,SJX_0) = 0 \quad \text{or} \quad h(X_0,X_0) = h(X_0,JX_0) = 0,$$

which both together with Steps 1, 2 and 3 lead to a contradiction with the assumption $h_{m'} \neq 0$.

3°. $\operatorname{rank} S > 1$.
If $SX = 0$, then, by Remark 2, $\mu(X) = \nu(X) = 0$. Let $SX \neq 0$. Then there exists Y such that SX and SY are linearly independent over $\mathbf{C}$. From the Codazzi II we have

$$(\mu(X) + i\nu(X))SY - (\mu(Y) + i\nu(Y))SX = 0,$$

which implies $\mu(X) = \nu(X) = 0$. Therefore (1) of the lemma 1. holds. ■

We now go back to the proof of the theorem.

Fix $m \in M$. Let $\xi : U \to \mathbf{C}^{n+1}$ be defined on a connected neighbourhood U of m and have the property described in the assumption (2) of the theorem.

LEMMA 3. *There exists a q-dimensional complex subspace $\mathcal{W}$ of $\mathbf{C}^{n+1}$ such that for every $m' \in U$ $f_*\mathrm{im}\, S_{m'} = \mathcal{W}$.*

P r o o f. Let $m' \in U$. It is sufficient to show that $f_*\mathrm{im}\, S_m = f_*\mathrm{im}\, S_{m'}$. Let $\gamma : [0,1] \to U$ be a $\mathcal{C}^1$ curve joining m and m'; $\gamma(0) = m$, $\gamma(1) = m'$. We choose $X_{1m}, ..., X_{qm} \in T_mM$ such that $SX_{1m}, ..., SX_{qm}$ form a basis over $\mathbf{C}$ of $f_*\mathrm{im}\, S_m$. Let $\tilde{X}_1, ..., \tilde{X}_q$ be the vector fields defined along the curve γ, parallel with respect to ∇, $\tilde{X}_i(0) = X_{im}$ for $i \in \{1, ..., q\}$. It is easy to check that the map

$$[0,1] \ni t \mapsto f_* S_{\gamma(t)}\tilde{X}_i(t) \in \mathbf{C}^{n+1}$$

is a constant one. Indeed,

$$\begin{aligned}
&\frac{d}{dt}(t \mapsto f_* S_{\gamma(t)}\tilde{X}_i(t)) \\
=&D_{\dot\gamma(t)} f_* S\tilde{X}_i \\
=&f_*\nabla_{\dot\gamma(t)} S\tilde{X}_i + h(\dot\gamma(t), S\tilde{X}_i(t))\xi_{\gamma(t)} - h(J\dot\gamma(t), S\tilde{X}_i(t))J\xi_{\gamma(t)}.
\end{aligned}$$

The last two terms vanish because $\mathrm{im}\, S \subset \ker h$, and

$$\nabla_{\dot\gamma(t)} S\tilde{X}_i = S\nabla_{\dot\gamma(t)}\tilde{X}_i = S0 = 0.$$

It follows that

$$\mathrm{span}_{\mathbf{C}}\{f_* S_m \tilde{X}_i(0) : i = 1, ..., q\} = \mathrm{span}_{\mathbf{C}}\{f_* S_{m'} \tilde{X}_i(1) : i = 1, ..., q\},$$

that means, $f_*\mathrm{im}\, S_m = f_*\mathrm{im}\, S_{m'} =: \mathcal{W}$. ■

Let $\tilde{A}_1 : \mathbf{C}^{n+1} \to \mathbf{C}^{n+1}$ be a linear isomorphism such that

$$\tilde{A}_1 \mathcal{W} = \mathrm{span}_{\mathbf{C}}\{\tilde{e}_1, ..., \tilde{e}_q\}.$$

Here and subsequently $\tilde{e}_1, \ldots, \tilde{e}_{n+1}$ denotes the standard basis of $\mathbf{C}^{n+1}$, whereas $e_1, \ldots, e_{2n+2}$ the standard basis of $\mathbf{R}^{2n+2}$.

LEMMA 4. *There exists $i_0 \in \{q+1, ..., n+1\}$ such that $\tilde{e}_{i_0} \notin (\tilde{A}_1 \circ f)_* T_mM$ and the i_0-th coordinate of $\tilde{A}_1\xi_m$ does not vanish.*

P r o o f. Suppose the assertion of the lemma is false. Then $\tilde{e}_j \in (\tilde{A}_1 \circ f)_* T_mM$ for every $j \in \{q+1, ..., n+1\}$ such that the j-th coordinate of $\tilde{A}_1\xi$ does not vanish at m. Then obviously $\tilde{A}_1\xi_m \in (\tilde{A}_1 \circ f)_* T_mM$, which contradicts the transversality of ξ. ■

Let $\tilde{A}_2^0$ be the linear isomorphism of $\mathbf{C}^{n+1}$, defined on its basis in the following way:

$$\tilde{A}_2^0 \tilde{e}_k := \begin{cases} \tilde{e}_k & \text{if } k \notin \{i_0, n+1\} \\ \tilde{e}_{n+1} & \text{if } k = i_0 \\ \tilde{e}_{i_0} & \text{if } k = n+1. \end{cases}$$

Let $\tilde{A}_2 := \tilde{A}_2^0 \circ \tilde{A}_1$. Now $\tilde{e}_{n+1}$ is transversal to $(\tilde{A}_2 \circ f)_* T_m M$ and $\tilde{A}_2 \xi$ has the non-vanishing $n+1$-th coordinate at m. Moreover, the isomorphism $\tilde{A}_2^0$ does not change the subspace $(\tilde{A}_1 \circ f)_* \mathrm{im}\, S = \mathrm{span}_{\mathbf{C}}\{\tilde{e}_1, ..., \tilde{e}_q\}$.

We denote by π the projection $\pi : \mathbf{C}^{n+1} \ni (\zeta^1, ..., \zeta^{n+1}) \mapsto (\zeta^1, ..., \zeta^n) \in \mathbf{C}^n$. It is easy to check that

$$d_m(\pi \circ \tilde{A}_2 \circ f) : T_m M \to \mathbf{C}^n$$

is a monomorphism. Indeed, if $d_m(\pi \circ \tilde{A}_2 \circ f).V = 0$, then $(\tilde{A}_2 \circ f)_* V \in \mathbf{C}\tilde{e}_{n+1}$. But $(\tilde{A}_2 \circ f)_* T_m M \cap \mathbf{C}\tilde{e}_{n+1} = \{0\}$ and $(\tilde{A}_2 \circ f)_*$ is a monomorphism; therefore $V = 0$.

We can now take $\tilde{\phi}_1 := \pi \circ \tilde{A}_2 \circ f$ as a complex map on some neighbourhood $U_1 \subset U$ of m. In this map

$$\tilde{A}_2 \circ f \circ \tilde{\phi}_1^{-1}(\zeta^1, ..., \zeta^n) = (\zeta^1, ..., \zeta^n, \tilde{\varphi}(\zeta)),$$

with a holomorphic function $\tilde{\varphi}$.

In the real representation, identifying $\mathbf{C}^k$ with $\mathbf{R}^{2k}$

$$\iota_k : \mathbf{R}^{2k} \ni (w^1, ..., w^{2k}) \mapsto (w^1 + iw^2, ..., w^{2k-1} + iw^{2k}) \in \mathbf{C}^k,$$

we can write

$$A_2 \circ f \circ \phi_1^{-1}(w^1, ..., w^{2n}) = (w^1, ..., w^{2n}, \varphi^1(w), \varphi^2(w)).$$

Here $A_2 := {\iota_{n+1}}^{-1} \circ \tilde{A}_2$ and $\phi_1 := {\iota_n}^{-1} \circ \tilde{\phi}_1$.

LEMMA 5.

(a) $$\frac{\partial \varphi^k}{\partial w^s} = 0 \quad \mathit{for} \quad k = 1, 2 \quad \mathit{and} \quad s = 1, ..., 2q.$$

(b) $$\mathrm{im}\, S = \mathrm{span}_{\mathbf{R}}\{\frac{\partial}{\partial w^s} : s = 1, ..., 2q\}.$$

P r o o f. At any point $m' \in U_1$ we have

$$(A_2 \circ f)_*(\frac{\partial}{\partial w^s}) = e_s + \frac{\partial \varphi^1}{\partial w^s} e_{2n+1} + \frac{\partial \varphi^2}{\partial w^s} e_{2n+2}.$$

If $s \in \{1, ..., 2q\}$, then $e_s = (A_2 \circ f)_* S W_s$ with some $W_s \in T_{m'} M$, because

$$e_s \in \mathrm{span}_{\mathbf{R}}\{e_1, ..., e_{2q}\} = (A_2 \circ f)_* \mathrm{im}\, S_{m'}.$$

Therefore we have

$$(A_2 \circ f)_*(\frac{\partial}{\partial w^s} - S W_s) = \frac{\partial \varphi^1}{\partial w^s} e_{2n+1} + \frac{\partial \varphi^2}{\partial w^s} e_{2n+2}.$$

From the transversality of e_{2n+1} and e_{2n+2} to $(A_2 \circ f)_* TM$ and from the injectivness of $(A_2 \circ f)_*$ it follows that (a) holds, and $\frac{\partial}{\partial w^s} - SW_s = 0$ for $s = 1, ..., 2q$. According to the last equality,

$$\operatorname{span}_{\mathbf{R}}\{\frac{\partial}{\partial w^s} : s = 1, ..., 2q\} \subset \operatorname{im} S,$$

which implies (b), because the dimensions are equal. ■

LEMMA 6. *The transversal field $A_2\xi$ does not depend on the variables $w^1, ..., w^{2q}$.*

P r o o f. We use the Weingarten formula

$$D_{\frac{\partial}{\partial w^s}} A_2\xi = -(A_2 \circ f)_* S\frac{\partial}{\partial w^s} + \mu(\frac{\partial}{\partial w^s})A_2\xi + \nu(\frac{\partial}{\partial w^s})JA_2\xi.$$

According to Lemmas 5(b), 2 and Remark 2, for $s = 1, ..., 2q$

$$\frac{\partial}{\partial w^s} \in \operatorname{im} S \subset \ker S \subset \ker \mu \cap \ker \nu,$$

hence

$$D_{\frac{\partial}{\partial w^s}} A_2\xi = 0. ■$$

We introduce now the functions $\Xi^1, ..., \Xi^{2n+2}$ by the following formula

$$(A_2\xi \circ {\phi_1}^{-1})(w) = \sum_{k=1}^{n+1}[\Xi^{2k-1}(w)e_{2k-1} - \Xi^{2k}(w)e_{2k}].$$

LEMMA 7.

$$\operatorname{rank}_{\mathbf{R}} \left[\frac{\partial \Xi^k}{\partial w^j}(w)\right]_{k=1,...,2n+2;j=2q+1,...,2n} = 2q$$

for $w \in \phi_1(U_1)$.

P r o o f.

$$\begin{aligned}
&\operatorname{rank}_{\mathbf{R}} \left[\frac{\partial \Xi^k}{\partial w^j}(w)\right]_{k=1,\dots,2n+2;j=2q+1,\dots,2n} \\
&= \operatorname{rank}_{\mathbf{R}} \left[(-1)^{k-1}\frac{\partial \Xi^k}{\partial w^j}(w)\right]_{k=1,\dots,2n+2;j=2q+1,\dots,2n} \\
&= \dim_{\mathbf{R}} \operatorname{span}\{D_{\frac{\partial}{\partial w^j}} A_2\xi : j = 2q+1, \dots, 2n\} \\
&= \dim_{\mathbf{R}} \operatorname{span}\{D_{\frac{\partial}{\partial w^j}} A_2\xi : j = 1, \dots, 2n\} \\
&= \dim_{\mathbf{R}} \operatorname{im} S_{{\phi_1}^{-1}(w)}.
\end{aligned}$$

The last equality is due to the isomorphism $\operatorname{im} S$ and $\operatorname{im}\{\, X \mapsto D_X A_2\xi\}$.

A consequence of Lemma 7 is the following

COROLLARY.
$$n \geq 2q.$$

LEMMA 8.
$$\tilde{\Xi}^k := \Xi^{2k-1} + i\Xi^{2k}$$
is a holomorphic function for $k = 1, \ldots, n+1$.

P r o o f. It is sufficient to show that Ξ^{2k-1} and Ξ^{2k} satisfy the Cauchy-Riemann equations. From Remark 3 it follows that

$$D_{JX}(A_2\xi) = -JD_X(A_2\xi)$$

for any $X \in T_{m'}M$. Let $X = \frac{\partial}{\partial w^{2s-1}}$, then $JX = \frac{\partial}{\partial w^{2s}}$ and

$$D_{JX}(A_2\xi) = \sum_{l=1}^{n+1}\left[\frac{\partial \Xi^{2l-1}}{\partial w^{2s}}e_{2l-1} - \frac{\partial \Xi^{2l}}{\partial w^{2s}}e_{2l}\right],$$

$$-JD_X(A_2\xi) = -J\sum_{l=1}^{n+1}\left[\frac{\partial \Xi^{2l-1}}{\partial w^{2s-1}}e_{2l-1} - \frac{\partial \Xi^{2l}}{\partial w^{2s-1}}e_{2l}\right]$$
$$= \sum_{l=1}^{n+1}\left[-\frac{\partial \Xi^{2l-1}}{\partial w^{2s-1}}e_{2l} - \frac{\partial \Xi^{2l}}{\partial w^{2s-1}}e_{2l-1}\right].$$

Therefore
$$\frac{\partial \Xi^{2l-1}}{\partial w^{2s}} = -\frac{\partial \Xi^{2l}}{\partial \Xi^{2s-1}} \quad \text{and} \quad \frac{\partial \Xi^{2l-1}}{\partial w^{2s-1}} = \frac{\partial \Xi^{2l}}{\partial w^{2s}}. \blacksquare$$

Now $\tilde{\Xi}^k$, $k = 1, \ldots, n+1$ are holomorphic functions of complex variables $\zeta^s = w^{2s-1} + iw^{2s}$, $s = q+1, \ldots, n$. According to Lemma 7, we have

$$\operatorname{rank}_{\mathbf{C}}\left[\frac{\partial \tilde{\Xi}^k}{\partial \zeta^l}\right]_{k=1,\ldots,n+1;\ l=q+1,\ldots,n} = q.$$

LEMMA 9. *Let $r \leq N \leq M$. Let $\mathcal{U}$ be an open set in $\mathbf{K}^N$, where $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$. Let $F : \mathcal{U} \to \mathbf{K}^M$ be a $\mathcal{C}^1$ mapping such that for every $x \in \mathcal{U}$ $\operatorname{rank} F'(x) = r$.*

Let $x_0 \in \mathcal{U}$ and let $i_1 < i_2 < \cdots < i_r$ and $j_1 < j_2 < \cdots < j_r$ be chosen so that

$$\det\left[\frac{\partial F^{i_k}}{\partial x^{j_l}}(x_0)\right]_{k=1,\ldots,r;\quad l=1,\ldots,r} \neq 0.$$

Then there exist a neighbourhood $\mathcal{U}' \subset \mathcal{U}$ of x_0 and a diffeomorphism $\Phi : \mathcal{U}' \to \Phi(\mathcal{U}') \subset \mathbf{K}^N$ such that

$$(F \circ \Phi^{-1})^{i_k}(y^1, \ldots, y^N) \equiv y^{j_k} \quad \textit{for} \quad k = 1, \ldots, r$$

and

$$\frac{\partial (F\circ\Phi^{-1})^k}{\partial y^l} \equiv 0 \quad if \quad l \notin \{j_1,\dots,j_r\}, \quad k\in\{1,\dots,M\}.$$

P r o o f. We define

$$\hat{\Phi}(x^1,\dots,x^N)^k = \begin{cases} x^k & \text{if } \; k\notin\{j_1,\dots,j_r\} \\ F^{i_s}(x^1,\dots,x^N) & \text{if } \; k=j_s. \end{cases}$$

Since

$$\det\left[\frac{\partial\hat{\Phi}^k}{\partial x^l}(x_0)\right]_{k=1,\dots,N;\; l=1,\dots,N} = \det\left[\frac{\partial F^{i_s}}{\partial x^{j_s}}(x_0)\right]_{s=1,\dots,r} \neq 0,$$

there exists a neighbourhood $\mathcal{U}'\subset\mathcal{U}$ of x_0 such that $\Phi := \hat{\Phi}\Big|_{\mathcal{U}'}$ is a diffeomorphism.

It remains to prove that $F\circ\Phi^{-1}$ depends only on the variables $y^{j_1},\dots,y^{j_r}$.

Let $y\in\Phi(\mathcal{U}')$. Then

$$(F\circ\Phi^{-1})'(y) = F'(\Phi^{-1}(y))\circ(\Phi^{-1})'(y)$$

and

$$\operatorname{rank}(F\circ\Phi^{-1})'(y) = \operatorname{rank} F'(\Phi^{-1}(y)) = r,$$

because $(\Phi^{-1})'(y)$ is an isomorphism.

Suppose that for some $k_0\in\{1,\dots,M\}$, $\;l_0\in\{1,\dots,N\}\setminus\{j_1,\dots,j_r\}$

$$\frac{\partial(F\circ\Phi^{-1})^{k_0}}{\partial y^{l_0}}(y)\neq 0.$$

Then $k_0\notin\{i_1,\dots,i_r\}$ and

$$\det\left[\frac{\partial(F\circ\Phi^{-1})^k}{\partial y^l}(y)\right]_{k\in\{i_1,\dots,i_r,\;k_0\};\; l\in\{j_1,\dots,j_r,\;l_0\}} \neq 0,$$

which contradicts the rank assumption. ■

We now restrict our attention to the case $q>1$.

Lemma 10. *If $q>1$, then*

(a)

$$\operatorname{rank}_{\mathbf{C}}\left[\frac{\partial\tilde{\Xi}^k}{\partial\zeta^l}\right]_{k=1,\dots,q;\; l=q+1,\dots,n} = q,$$

(b)

$$\tilde{\Xi}^k(\cdot) = b^k = \text{const} \quad for \quad k=q+1,\dots,n+1,$$

(c)

$$b^{n+1}\neq 0.$$

P r o o f. Since, for $q > 1$, $\mu = 0$ and $\nu = 0$ (see proof of Lemma 2,3°), we have

$$D_{\frac{\partial}{\partial w^l}} A_2 \xi \in (A_2 \circ f)_* \mathrm{im}\, S = \mathrm{span}_{\mathbf{R}}\{e_1, \ldots, e_{2q}\},$$

therefore

$$\frac{\partial \Xi^k}{\partial w^l} = 0$$

for $k = 2q+1, \ldots, 2n+2$; $\quad l = 2q+1, \ldots, 2n$; which implies

$$\frac{\partial \tilde{\Xi}^k}{\partial \zeta^l} = 0 \quad \text{for} \quad k = q+1, \ldots, n+1; \; l = q+1, \ldots, n$$

and

$$\mathrm{rank}_{\mathbf{C}} \left[\frac{\partial \tilde{\Xi}^k}{\partial \zeta^l} \right]_{k=1,\ldots,n+1;\; l=q+1,\ldots,n} = \mathrm{rank}_{\mathbf{C}} \left[\frac{\partial \tilde{\Xi}^k}{\partial \zeta^l} \right]_{k=1,\ldots,q;\; l=q+1,\ldots,n}.$$

The point (c) is a consequence of Lemma 4 and the definition of $\tilde{A}_2$. ■

We may now apply Lemma 9, taking $r = q$, $N = n$, $M = n+1$, $\mathcal{U} = \tilde{\phi}_1(U_1)$, $\mathbf{K} = \mathbf{C}$, $F = (\tilde{\Xi}^1, \ldots, \tilde{\Xi}^{n+1})$, $(i_1, \ldots, i_q) = (1, \ldots, q)$ and $j_1 < \cdots < j_q$ from $\{q+1, \ldots, n\}$ chosen so that

$$\det \left(\left[\frac{\partial \tilde{\Xi}^k}{\partial \zeta^{j_s}} (\tilde{\phi}_1(m)) \right]_{k=1,\ldots,q;\; s=1,\ldots,q} \right) \neq 0.$$

In this way we obtain a new map $\tilde{\phi}_2 := \Phi \circ \tilde{\phi}_1$ on the neighbourhood $U_2 := \tilde{\phi}_1^{-1}(\mathcal{U}')$ of m.

Since

$$(\tilde{\Xi}^k \circ \Phi^{-1})(\eta^1, \ldots, \eta^n) = \eta^{j_k}$$

for $k = 1, \ldots, q$ and

$$(\tilde{\Xi}^k \circ \Phi^{-1})(\eta^1, \ldots, \eta^n) = b^k$$

for $k = q+1, \ldots, n+1$, the transversal field is now described by the formula

$$(\tilde{A}_2 \xi \circ \tilde{\phi}_2^{-1})(\eta^1, \ldots, \eta^n) = \sum_{k=1}^{q} \overline{\eta^{j_k}} \tilde{e}_k + \sum_{k=q+1}^{n+1} \overline{b^k} \tilde{e}_k.$$

Let $\tilde{A}_3^0$ be the linear isomorphism of $\mathbf{C}^{n+1}$ which transforms the basis

$$(\tilde{e}_1, \ldots, \tilde{e}_{n+1})$$

onto the basis

$$(\tilde{e}_1, \ldots, \tilde{e}_q, \tilde{e}_{q+1}, \ldots, \widehat{\tilde{e}_{j_1}}, \ldots, \widehat{\tilde{e}_{j_q}}, \ldots, \tilde{e}_n, \tilde{e}_{j_1}, \ldots, \tilde{e}_{j_q}, \tilde{e}_{n+1}),$$

and let

$$\tilde{\phi}_3^0(\eta^1,\dots,\eta^n) := (\eta^1,\dots,\eta^q,\eta^{q+1},\dots,\widehat{\eta^{j_1}},\dots,\widehat{\eta^{j_q}},\dots,\eta^n,\eta^{j_1},\dots,\eta^{j_q}).$$

Taking $\tilde{A}_3 := \tilde{A}_3^0 \circ \tilde{A}_2$, $\tilde{\phi}_3 := \tilde{\phi}_3^0 \circ \tilde{\phi}_2$, we may write:

$$\begin{aligned}
&\tilde{A}_3 \circ f \circ \tilde{\phi}_3^{-1}(\tilde{s}^1,\dots,\tilde{s}^q,\tilde{t}^1,\dots,\tilde{t}^{n-2q},\tilde{u}^1,\dots,\tilde{u}^q)\\
&= (\tilde{s}^1,\dots,\tilde{s}^q,\tilde{t}^1,\dots,\tilde{t}^{n-2q},\tilde{\chi}^1(\tilde{t},\tilde{u}),\dots,\tilde{\chi}^q(\tilde{t},\tilde{u}),\tilde{\rho}(\tilde{t},\tilde{u}))
\end{aligned}$$

and

$$\tilde{A}_3\xi \circ \tilde{\phi}_3^{-1}(\tilde{s}^1,\dots,\tilde{s}^q,\tilde{t}^1,\dots,\tilde{t}^{n-2q},\tilde{u}^1,\dots,\tilde{u}^q) = \sum_{k=1}^{q}\overline{\tilde{u}^k}\tilde{e}_k + \sum_{k=q+1}^{n+1}\overline{a^k}\tilde{e}_k.$$

Applying now the isomorphism $\tilde{A}_4^0$, where

$$\tilde{A}_4^0\tilde{e}_k := \begin{cases} \tilde{e}_k & \text{for } k=1,\dots,n\\ \frac{1}{\overline{a^{n+1}}}(-\sum_{k=q+1}^{n}\overline{a^k}\tilde{e}_k + \tilde{e}_{n+1}) & \text{for } k=n+1,\end{cases}$$

$\tilde{A}_4 := \tilde{A}_4^0 \circ \tilde{A}_3$, we obtain

$$\begin{aligned}
&\tilde{A}_4 \circ f \circ \tilde{\phi}_3^{-1}(\tilde{s}^1,\dots,\tilde{s}^q,\tilde{t}^1,\dots,\tilde{t}^{n-2q},\tilde{u}^1,\dots,\tilde{u}^q)\\
&= (\tilde{s}^1,\dots,\tilde{s}^q,\tilde{\sigma}^1(\tilde{t},\tilde{u}),\dots,\tilde{\sigma}^{n-q}(\tilde{t},\tilde{u}),\tilde{\mathcal{Q}}(\tilde{t},\tilde{u}))
\end{aligned}$$

and

$$\tilde{A}_4\xi \circ \tilde{\phi}_3^{-1}(\tilde{s}^1,\dots,\tilde{s}^q,\tilde{t}^1,\dots,\tilde{t}^{n-2q},\tilde{u}^1,\dots,\tilde{u}^q) = \sum_{k=1}^{q}\overline{\tilde{u}^k}\tilde{e}_k + \tilde{e}_{n+1}.$$

Here

$$\tilde{\sigma}^k(\tilde{t},\tilde{u}) = \begin{cases} \tilde{t}^k - \frac{\overline{a^{q+k}}}{\overline{a^{n+1}}}\tilde{\rho}(\tilde{t},\tilde{u}) & \text{for } \ k=1,\dots,n-2q\\ \tilde{\chi}^{k-(n-2q)}(\tilde{t},\tilde{u}) - \frac{\overline{a^{q+k}}}{\overline{a^{n+1}}}\tilde{\rho}(\tilde{t},\tilde{u}) & \text{for } \ k=n-2q+1,\dots,n-q;\end{cases}$$

and $\tilde{\mathcal{Q}} = \frac{1}{\overline{a^{n+1}}}\tilde{\rho}$.

We now turn to the case $q=1$.

By virtue of Lemmas 7, 8 and 9 we may define on a connected neighbourhood $\tilde{\tilde{U}}_2$ of m a complex map $\tilde{\tilde{\phi}}_2$ in which the coordinates of the transversal field are the functions of one variable η^{i_0} only, with $i_0 > 1$.

Next we apply the isomorphisms

$$\tilde{\tilde{\phi}}_3^0(\eta^1,\dots,\eta^n) := (\eta^1,\eta^2,\dots,\widehat{\eta^{i_0}},\dots,\eta^n,\eta^{i_0})$$

and

$$\tilde{\tilde{A}}_3^0(\theta^1,\dots,\theta^{n+1}) := (\theta^1,\theta^2,\dots,\widehat{\theta^{i_0}},\dots,\theta^n,\theta^{i_0},\theta^{n+1}),$$

to obtain

$$\begin{aligned}&\tilde{\tilde{A}}_3 \circ f \circ \tilde{\tilde{\phi}}_3^{-1}(\tilde{s}, \tilde{t}^1, \dots, \tilde{t}^{n-2}, \tilde{u})\\&= (\tilde{s}, \tilde{t}^1, \dots, \tilde{t}^{n-2}, \tilde{\tilde{\chi}}(\tilde{t}, \tilde{u}), \tilde{\tilde{\rho}}(\tilde{t}, \tilde{u}))\end{aligned}$$

and

$$\tilde{\tilde{A}}_3 \xi \circ \tilde{\tilde{\phi}}_3^{-1}(\tilde{s}, \tilde{t}^1, \dots, \tilde{t}^{n-2}, \tilde{u}) = \sum_{k=1}^{n+1} \overline{\Theta^k(\tilde{u})} \tilde{e}_k$$

with $\Theta^{n+1}(\tilde{\tilde{\phi}}_3(m)) \neq 0$; $\tilde{\tilde{\phi}}_3 := \tilde{\tilde{\phi}}_3^0 \circ \tilde{\tilde{\phi}}_2$ and $\tilde{\tilde{A}}_3 := \tilde{\tilde{A}}_3^0 \circ \tilde{A}_2$.

In the real representation $\tilde{\tilde{\chi}} = \chi^1 + i\chi^2$, $\tilde{\tilde{\rho}} = \rho^1 + i\rho^2$, $\Theta^k = \vartheta^{2k-1} + i\vartheta^{2k}$, we have

$$\begin{aligned}&A_3 \circ f \circ \phi_3^{-1}(s^1, s^2, t^1, \dots, t^{2n-4}, u^1, u^2)\\&= (s^1, s^2, t^1, \dots, t^{2n-4}, \chi^1(t, u), \chi^2(t, u), \rho^1(t, u), \rho^2(t, u))\end{aligned}$$

and

$$A_3 \xi \circ \phi_3^{-1}(s^1, s^2, t^1, \dots, t^{2n-4}, u^1, u^2) = \sum_{k=1}^{n+1} \left[\vartheta^{2k-1}(u) e_{2k-1} - \vartheta^{2k}(u) e_{2k}\right].$$

Let $\hat{\pi}$ denote the projection

$$\mathbf{C}^n \ni (\tilde{s}, \tilde{t}^1, \dots, \tilde{t}^{n-2}, \tilde{u}) \mapsto \tilde{u} \in \mathbf{C}.$$

LEMMA 11. *There exist $c_2, \dots, c_{n+1} \in \mathbf{C}$, $c_{n+1} \neq 0$, a neighbourhood $\tilde{\tilde{U}}_3$ of m and a holomorphic function $\tilde{\mathcal{H}}$ such that*

$$\Theta^k(\tilde{u}) = c_k e^{\tilde{\mathcal{H}}(\tilde{u})}$$

for $k = 1, \dots, n+1$; $\tilde{u} \in \hat{\pi}(\tilde{\tilde{\phi}}_3(\tilde{\tilde{U}}_3))$.

P r o o f. We fix $j \in \{2, \dots, n+1\}$. Since Θ^j is a holomorphic function, and since $\tilde{\tilde{U}}_2$ is assumed to be connected, there are two possibilities - either $\Theta^j \equiv 0$ on $\hat{\pi} \circ \tilde{\tilde{\phi}}_3(\tilde{\tilde{U}}_2)$ or there exist a neighbourhood $\tilde{W}_j$ of $\tilde{u}_0 := \hat{\pi}(\tilde{\tilde{\phi}}_3(m))$ such that $\Theta^j(\tilde{u}) \neq 0$ for any $\tilde{u} \in \tilde{W}_j \setminus \{\tilde{u}_0\}$.

In the former case we take $c_j = 0$.

Suppose that $\Theta^j \not\equiv 0$. We can find $r > 0$ such that $K(\tilde{u}_0, r) \subset \tilde{W}_j$. Then

$$\tilde{W}_j' := K(\tilde{u}_0, r) \setminus \{\tilde{u} : \operatorname{Re} \tilde{u} = \operatorname{Re} \tilde{u}_0, \operatorname{Im} \tilde{u} \leq \operatorname{Im} \tilde{u}_0\}$$

is a simply connected domain and $\Theta^j(\tilde{u}) \neq 0$ for any $\tilde{u} \in \tilde{W}_j'$. If this is the case, there exists a holomorphic function λ^j on $\tilde{W}_j'$ such that

$$e^{\lambda^j} = \Theta^j\big|_{\tilde{W}_j'}$$

and

$$(\lambda^j)' = \frac{\Theta^{j\,\prime}}{\Theta^j}$$

on $\tilde{W}'_j$.

On the other hand, using the real representation we may write

$$(\Theta^j)' = \frac{\partial \vartheta^{2j-1}}{\partial u^1} + i\frac{\partial \vartheta^{2j}}{\partial u^1}.$$

From the Weingarten formula

$$D_{\frac{\partial}{\partial u^1}} A_3\xi = -(A_3 \circ f)_* S\frac{\partial}{\partial u^1} + \mu(\frac{\partial}{\partial u^1})A_3\xi + \nu(\frac{\partial}{\partial u^1})JA_3\xi$$

it follows that for any $j \in \{2, \dots, n+1\}$

$$\frac{\partial \vartheta^{2j-1}}{\partial u^1} = \mu(\frac{\partial}{\partial u^1})\vartheta^{2j-1} + \nu(\frac{\partial}{\partial u^1})\vartheta^{2j}$$

and

$$\frac{\partial \vartheta^{2j}}{\partial u^1} = \mu(\frac{\partial}{\partial u^1})\vartheta^{2j} - \nu(\frac{\partial}{\partial u^1})\vartheta^{2j-1}.$$

Hence

$$(\Theta^j)' = (\mu(\frac{\partial}{\partial u^1}) - i\nu(\frac{\partial}{\partial u^1}))(\vartheta^{2j-1} + i\vartheta^{2j}).$$

We may also assume that $\tilde{\tilde{U}}_2$ is simply connected, and from Lemma 2 we know that the 1-forms μ, ν are closed; therefore there exist functions $\mathcal{K}$ and $\mathcal{L}$ on $\tilde{\tilde{U}}_2$ such that $\mu = d\mathcal{K}$, $\nu = -d\mathcal{L}$. The functions $\mathcal{K} \circ \phi_3^{-1}$ and $\mathcal{L} \circ \phi_3^{-1}$ do not depend on the variables s and t, because $\frac{\partial}{\partial s^i}, \frac{\partial}{\partial s^i} \in \ker S \subset \ker \mu \cap \ker \nu$,

$$\frac{\partial \mathcal{K} \circ \phi_3^{-1}}{\partial s^i} = d\mathcal{K}(\frac{\partial}{\partial s^i}) = \mu(\frac{\partial}{\partial s^i}) = 0,$$

and similarly for $\mathcal{L}$ in place of $\mathcal{K}$ or t in place of s. It follows that there exist functions $\mathcal{H}^1$ and $\mathcal{H}^2$ defined on some open subset of $\mathbf{R}^2$ such that $\mathcal{K} \circ \phi_3^{-1} = \mathcal{H}^1 \circ \hat{\pi}$ and $\mathcal{L} \circ \phi_3^{-1} = \mathcal{H}^2 \circ \hat{\pi}$. We have now

$$\mu(\frac{\partial}{\partial u^s}) = d\mathcal{K}((\phi_3^{-1})_* e_{2n-2+s}) = d\mathcal{H}^1(\hat{\pi}_* e_{2n-2+s}) = d\mathcal{H}^1(e_s) = \frac{\partial \mathcal{H}^1}{\partial u^s}$$

and

$$\nu(\frac{\partial}{\partial u^s}) = -d\mathcal{L}((\phi_3^{-1})_* e_{2n-2+s}) = -d\mathcal{H}^2(\hat{\pi}_* e_{2n-2+s}) = -d\mathcal{H}^2(e_s) = -\frac{\partial \mathcal{H}^2}{\partial u^s}.$$

According to Remark 3, $\mathcal{H}^1$ and $\mathcal{H}^2$ satisfy the Cauchy-Riemann equations, therefore $\tilde{\mathcal{H}} := \mathcal{H}^1 + i\mathcal{H}^2$ is a holomorphic function.

Since

$$(\tilde{\mathcal{H}})' = \frac{\partial \mathcal{H}^1}{\partial u^1} + i\frac{\partial \mathcal{H}^2}{\partial u^1} = \mu(\frac{\partial}{\partial u^1}) - i\nu(\frac{\partial}{\partial u^1}),$$

we may go back to the $(\Theta^j)'$ and write

$$(\Theta^j)' = \tilde{\mathcal{H}}'\Theta^j$$

on some neighbourhood $\tilde{\pi}(\tilde{\tilde{\phi}}_3(\tilde{\tilde{U}}_3'))$ of $\tilde{u}_0$ (including $\tilde{u}_0$). Comparing this with $(\lambda^j)'$ we obtain

$$\tilde{\mathcal{H}}' = (\lambda^j)'$$

on $\tilde{W}_j'' := \tilde{W}_j' \cap \tilde{\pi}(\tilde{\tilde{\phi}}_3(\tilde{\tilde{U}}_3'))$. Hence there exists $d_j \in \mathbf{C}$ such that $\lambda^j = \tilde{\mathcal{H}} + d_j$ and

$$\Theta^j = e^{\tilde{\mathcal{H}}+d_j} = c_j e^{\tilde{\mathcal{H}}}$$

with a non-zero constant $c_j := e^{d_j}$. We may expand this equality from $\tilde{W}_j''$ to some neighbourhood $\tilde{\pi}(\tilde{\tilde{\phi}}_3(\tilde{\tilde{U}}_3))$ of $\tilde{u}_0$, because both sides are continuous and well-defined in the neighbourhood of $\tilde{u}_0$. Since $\Theta^{n+1} \not\equiv 0$, we have in particular $c_{n+1} \neq 0$. ■

Next we use the following isomorphism of $\mathbf{C}^{n+1}$:

$$\tilde{\tilde{A}}_4^0 \tilde{e}_k := \begin{cases} \tilde{e}_k & \text{for } \; k = 1, \ldots, n \\ \frac{1}{\overline{c_{n+1}}}(\tilde{e}_{n+1} - \sum_{s=2}^n \overline{c_s}\tilde{e}_s) & \text{for } \; k = n+1. \end{cases}$$

In this way we obtain

$$\begin{aligned} &\tilde{\tilde{A}}_4 \circ f \circ \tilde{\tilde{\phi}}_3^{-1}(\tilde{s}, \tilde{t}^1, \ldots, \tilde{t}^{n-2}, \tilde{u}) \\ &= (\tilde{s}, \tilde{\tilde{\sigma}}^1(\tilde{t}, \tilde{u}), \ldots, \tilde{\tilde{\sigma}}^{n-1}(\tilde{t}, \tilde{u}), \tilde{\tilde{\mathcal{Q}}}(\tilde{t}, \tilde{u})) \end{aligned}$$

and

$$\tilde{\tilde{A}}_4 \xi \circ \tilde{\tilde{\phi}}_3^{-1}(\tilde{s}, \tilde{t}^1, \ldots, \tilde{t}^{n-2}, \tilde{u}) = \overline{\Theta^1(\tilde{u})}\tilde{e}_1 + \overline{e^{\tilde{\mathcal{H}}(\tilde{u})}}\tilde{e}_{n+1},$$

with $\tilde{\tilde{A}}_4 := \tilde{\tilde{A}}_4^0 \circ \tilde{\tilde{A}}_3$, $\tilde{\tilde{\sigma}}^k(\tilde{t}, \tilde{u}) := \tilde{t}^k - \overline{(\frac{c_{k+1}}{c_{n+1}})}\tilde{\rho}(\tilde{t}, \tilde{u})$ for $k = 1, \ldots, n-2$,
$\tilde{\tilde{\sigma}}^{n-1} := \tilde{\chi}(\tilde{t}, \tilde{u}) - \overline{(\frac{c_{k+1}}{c_{n+1}})}\tilde{\rho}(\tilde{t}, \tilde{u})$ and $\tilde{\tilde{\mathcal{Q}}}(\tilde{t}, \tilde{u}) := \overline{(\frac{1}{c_{n+1}})}\tilde{\rho}(\tilde{t}, \tilde{u})$.

Thus for any $q > 0$ it is possible to find a map $\hat{\phi}_3$ and an isomorphism $\hat{A}_4$ of $\mathbf{C}^{n+1}$ such that the immersion and the transversal field have the shape

$$\begin{aligned} &\hat{A}_4 \circ f \circ \hat{\phi}_3^{-1}(\tilde{s}^1, \ldots, \tilde{s}^q, \tilde{t}^1, \ldots, \tilde{t}^{n-2q}, \tilde{u}^1, \ldots, \tilde{u}^q) \\ &= (\tilde{s}^1, \ldots, \tilde{s}^q, \hat{\sigma}^1(\tilde{t}, \tilde{u}), \ldots, \hat{\sigma}^{n-q}(\tilde{t}, \tilde{u}), \hat{\mathcal{Q}}(\tilde{t}, \tilde{u})) \end{aligned}$$

and

$$\hat{A}_4 \xi \circ \hat{\phi}_3^{-1}(\tilde{s}^1, \ldots, \tilde{s}^q, \tilde{t}^1, \ldots, \tilde{t}^{n-2q}, \tilde{u}^1, \ldots, \tilde{u}^q) = \sum_{k=1}^q \overline{\hat{\Theta}^k(\tilde{u})}\tilde{e}_k + \overline{\hat{\Theta}^{n+1}(\tilde{u})}\tilde{e}_{n+1},$$

here $\hat{\sigma}^i$, $\hat{\mathcal{Q}}$, $\hat{\Theta}^j$ are holomorphic functions and $\hat{\Theta}^{n+1} \neq 0$.

Since $\hat{A}_4 \circ f$ is an immersion, and $\hat{A}_4 \circ \xi$ is transversal to $(\hat{A}_4 \circ f)_* TM$,

$$\left|\begin{array}{ccccc} I_q & \begin{matrix}0 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & 0\end{matrix} & \begin{matrix}0 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & 0\end{matrix} & \begin{matrix}\overline{\hat{\Theta}^1}\\ \vdots\\ \overline{\hat{\Theta}^q}\end{matrix} \\ \begin{matrix}0 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & 0\end{matrix} & \left[\frac{\partial\hat{\sigma}^k}{\partial\tilde{t}^l}\right]_{1\le k\le n-q;\ 1\le l\le n-2q} & \left[\frac{\partial\hat{\sigma}^k}{\partial\tilde{u}^l}\right]_{1\le k\le n-q;\ 1\le l\le q} & \begin{matrix}0\\ \vdots\\ 0\end{matrix} \\ 0 \ \cdots \ 0 & \frac{\partial\hat{\mathcal{Q}}}{\partial\tilde{t}^1} \ \cdots \ \frac{\partial\hat{\mathcal{Q}}}{\partial\tilde{t}^{n-2q}} & \frac{\partial\hat{\mathcal{Q}}}{\partial\tilde{u}^1} \ \cdots \ \frac{\partial\hat{\mathcal{Q}}}{\partial\tilde{u}^q} & \overline{\hat{\Theta}^{n+1}} \end{array}\right| \neq 0.$$

Therefore

$$\det\left(\left[\frac{\partial\hat{\sigma}^k}{\partial\tilde{t}^l}\right]_{k=1,\dots,n-q;\ l=1,\dots,n-2q}\left[\frac{\partial\hat{\sigma}^k}{\partial\tilde{u}^l}\right]_{k=1,\dots,n-q;\ l=1,\dots,q}\right)\neq 0$$

and there exist $i_1 < \cdots < i_{n-2q}$ such that

$$\det\left(\left[\frac{\partial\hat{\sigma}^{i_k}}{\partial\tilde{t}^l}\right]_{k=1,\dots,n-2q;\ l=1,\dots,n-2q}\right)\neq 0.$$

By the appropiate isomorphism $\hat{A}_5^0$ we may vary the order of basis vectors in $\mathbf{C}^{n+1}$, putting $\hat{\sigma}^{i_1},\dots,\hat{\sigma}^{i_{n-2q}}$ at the positions $q+1,\dots,n-q$. This permutation does not affect the field $\hat{A}_4\xi$, because its coordinates from the $q+1$-th to the n-th vanish.

Applying now the local diffeomorphism

$$\begin{aligned}&\hat{\phi}_4^0(\tilde{s}^1,\dots,\tilde{s}^q,\tilde{t}^1,\dots,\tilde{t}^{n-2q},\tilde{u}^1,\dots,\tilde{u}^q)\\ &:=(\tilde{s}^1,\dots,\tilde{s}^q,\hat{\sigma}^{i_1}(\tilde{t},\tilde{u}),\dots,\hat{\sigma}^{i_{n-2q}}(\tilde{t},\tilde{u}),\tilde{u}^1,\dots,\tilde{u}^q)\end{aligned}$$

gives a new map $\hat{\phi}_4 := \hat{\phi}_4^0\circ\hat{\phi}_3$ such that $\hat{A}_5\circ f$ and $\hat{A}_5\circ\xi$ are described by the formulas

$$\begin{aligned}&\hat{A}_5\circ f\circ\hat{\phi}_4^{-1}(\tilde{x}^1,\dots,\tilde{x}^q,\tilde{y}^1,\dots,\tilde{y}^{n-2q},\tilde{z}^1,\dots,\tilde{z}^q)\\ &=(\tilde{x}^1,\dots,\tilde{x}^q,\tilde{y}^1,\dots,\tilde{y}^{n-2q},\hat{\psi}^1(\tilde{y},\tilde{z}),\dots,\hat{\psi}^q(\tilde{y},\tilde{z}),\hat{\mathcal{F}}(\tilde{y},\tilde{z}))\end{aligned}$$

and

$$\hat{A}_5\circ\xi\circ\hat{\phi}_4^{-1}(\tilde{x}^1,\dots,\tilde{x}^q,\tilde{y}^1,\dots,\tilde{y}^{n-2q},\tilde{z}^1,\dots,\tilde{z}^q)=\sum_{k=1}^{q}\overline{\hat{\Theta}^k(\tilde{z})}\tilde{e}_k+\overline{\hat{\Theta}^{n+1}(\tilde{z})}\tilde{e}_{n+1}.$$

LEMMA 12. *For $k=1,\dots,q$*

$$\hat{\psi}^k(\tilde{y},\tilde{z})=\sum_{s=1}^{q}C^k{}_s\tilde{y}^s\ +\ \hat{\Pi}^k(\tilde{z}),$$

where $C^k{}_s$, $s=1,\dots,n-2q$ are the complex numbers, and $\hat{\Pi}^k$ is a holomorphic function.

P r o o f. We shall now use the real representation of $\hat{A}_5 \circ f$ and $\hat{A}_5 \circ \xi$, setting $\hat{\psi}^l = \check{\psi}^{2l-1} + i\check{\psi}^{2l}$ for $l = 1, \dots, q$, $\hat{\mathcal{F}} = \check{\mathcal{F}}^1 + i\check{\mathcal{F}}^2$, and $\hat{\Theta}^s = \check{\Theta}^{2s-1} + i\check{\Theta}^{2s}$:

$$\begin{aligned}
&\check{A}_5 \circ f \circ \check{\phi}_4^{-1}(x^1, \dots, x^{2q}, y^1, \dots, y^{2n-4q}, z^1, \dots, z^{2q}) \\
&= (x^1, \dots, x^{2q}, y^1, \dots, y^{2n-4q}, \check{\psi}^1(y,z), \dots, \check{\psi}^{2q}(y,z), \check{\mathcal{F}}^1(y,z), \check{\mathcal{F}}^2(y,z)), \\
&\check{A}_5 \circ \xi \circ \check{\phi}_4^{-1}(x^1, \dots, x^{2q}, y^1, \dots, y^{2n-4q}, z^1, \dots, z^{2q}) \\
&= \sum_{k=1}^{q} [\check{\Theta}^{2k-1}(z) e_{2k-1} - \check{\Theta}^{2k}(z) e_{2k}] + \check{\Theta}^{2n+1}(z) e_{2n+1} - \check{\Theta}^{2n+2}(z) e_{2n+2}.
\end{aligned}$$

At any point m' of the domain $\check{U}$ of $\check{\phi}_4$ $\ker S$ is spanned by

$$\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^{2q}}, \frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^{2n-4q}}.$$

For any $W \in T_{m'}M$, for any $j = 1, \dots, 2n-4q$

$$S(\nabla_W \frac{\partial}{\partial y^j}) = \nabla_W (S \frac{\partial}{\partial y^j}) = 0,$$

therefore

$$\nabla_{\frac{\partial}{\partial y^s}} \frac{\partial}{\partial y^j} = \sum_{k=1}^{2q} \alpha^k_{sj} \frac{\partial}{\partial x^k} + \sum_{l=1}^{2n-4q} \beta^l_{sj} \frac{\partial}{\partial y^l}$$

and

$$\nabla_{\frac{\partial}{\partial z^s}} \frac{\partial}{\partial y^j} = \sum_{k=1}^{2q} \gamma^k_{sj} \frac{\partial}{\partial x^k} + \sum_{l=1}^{2n-4q} \delta^l_{sj} \frac{\partial}{\partial y^l}.$$

We have

$$(\check{A}_5 \circ f)_* (\frac{\partial}{\partial x^k}) = e_k,$$

$$(\check{A}_5 \circ f)_* (\frac{\partial}{\partial y^l}) = e_{2q+l} + \sum_{r=1}^{2q} \frac{\partial \check{\psi}^r}{\partial y^l} e_{2n-2q+r} + \frac{\partial \check{\mathcal{F}}^1}{\partial y^l} e_{2n+1} + \frac{\partial \check{\mathcal{F}}^2}{\partial y^l} e_{2n+2};$$

hence

$$\begin{aligned}
&(\check{A}_5 \circ f)_* (\nabla_{\frac{\partial}{\partial y^s}} \frac{\partial}{\partial y^j}) \\
&= \sum_{k=1}^{2q} \alpha^k_{sj} e_k + \sum_{l=1}^{2n-4q} \beta^l_{sj} e_{2q+l} + \sum_{r=1}^{2q} (\sum_{l=1}^{2n-4q} \beta^l_{sj} \frac{\partial \check{\psi}^r}{\partial y^l}) e_{2n-2q+r} \\
&+ (\sum_{l=1}^{2n-4q} \beta^l_{sj} \frac{\partial \check{\mathcal{F}}^1}{\partial y^l}) e_{2n+1} + (\sum_{l=1}^{2n-4q} \beta^l_{sj} \frac{\partial \check{\mathcal{F}}^2}{\partial y^l}) e_{2n+2}
\end{aligned}$$

and

$$\begin{aligned}
&(\check{A}_5 \circ f)_* (\nabla_{\frac{\partial}{\partial z^s}} \frac{\partial}{\partial y^j}) \\
&= \sum_{k=1}^{2q} \gamma^k_{sj} e_k + \sum_{l=1}^{2n-4q} \delta^l_{sj} e_{2q+l} + \sum_{r=1}^{2q} (\sum_{l=1}^{2n-4q} \delta^l_{sj} \frac{\partial \check{\psi}^r}{\partial y^l}) e_{2n-2q+r} \\
&+ (\sum_{l=1}^{2n-4q} \delta^l_{sj} \frac{\partial \check{\mathcal{F}}^1}{\partial y^l}) e_{2n+1} + (\sum_{l=1}^{2n-4q} \delta^l_{sj} \frac{\partial \check{\mathcal{F}}^2}{\partial y^l}) e_{2n+2}.
\end{aligned}$$

On the other hand, using the Gauss formula we may write

$$
\begin{aligned}
&(\breve{A}_5\circ f)_*(\nabla_{\frac{\partial}{\partial y^s}}\frac{\partial}{\partial y^j})\\
&= D_{\frac{\partial}{\partial y^s}}(\breve{A}_5\circ f)_*\frac{\partial}{\partial y^j} - h(\frac{\partial}{\partial y^s},\frac{\partial}{\partial y^j})\breve{A}_5\xi + h(J\frac{\partial}{\partial y^s},\frac{\partial}{\partial y^j})J\breve{A}_5\xi\\
&= \sum_{k=1}^{q}[(-h(\frac{\partial}{\partial y^s},\frac{\partial}{\partial y^j})\breve{\Theta}^{2k-1}(z) + h(J\frac{\partial}{\partial y^s},\frac{\partial}{\partial y^j})\breve{\Theta}^{2k}(z))e_{2k-1}\\
&+ (h(\frac{\partial}{\partial y^s},\frac{\partial}{\partial y^j})\breve{\Theta}^{2k}(z) + h(J\frac{\partial}{\partial y^s},\frac{\partial}{\partial y^j})\breve{\Theta}^{2k-1}(z))e_{2k}]\\
&+ \sum_{r=1}^{2q}\frac{\partial^2\breve{\psi}^r}{\partial y^s\partial y^j}e_{2n-2q+r}\\
&+ (\frac{\partial^2\breve{\mathcal{F}}^1}{\partial y^s\partial y^j} - h(\frac{\partial}{\partial y^s},\frac{\partial}{\partial y^j})\breve{\Theta}^{2n+1}(z) + h(J\frac{\partial}{\partial y^s},\frac{\partial}{\partial y^j})\breve{\Theta}^{2n+2}(z))e_{2n+1}\\
&+ (\frac{\partial^2\breve{\mathcal{F}}^2}{\partial y^s\partial y^j} + h(\frac{\partial}{\partial y^s},\frac{\partial}{\partial y^j})\breve{\Theta}^{2n+2}(z) + h(J\frac{\partial}{\partial y^s},\frac{\partial}{\partial y^j})\breve{\Theta}^{2n+1}(z))e_{2n+2}
\end{aligned}
$$

and similarly

$$
\begin{aligned}
&(\breve{A}_5\circ f)_*(\nabla_{\frac{\partial}{\partial z^s}}\frac{\partial}{\partial y^j})\\
&= D_{\frac{\partial}{\partial z^s}}(\breve{A}_5\circ f)_*\frac{\partial}{\partial y^j} - h(\frac{\partial}{\partial z^s},\frac{\partial}{\partial y^j})\breve{A}_5\xi + h(J\frac{\partial}{\partial z^s},\frac{\partial}{\partial y^j})J\breve{A}_5\xi\\
&= \sum_{k=1}^{q}[(-h(\frac{\partial}{\partial z^s},\frac{\partial}{\partial y^j})\breve{\Theta}^{2k-1}(z) + h(J\frac{\partial}{\partial z^s},\frac{\partial}{\partial y^j})\breve{\Theta}^{2k}(z))e_{2k-1}\\
&+ (h(\frac{\partial}{\partial z^s},\frac{\partial}{\partial y^j})\breve{\Theta}^{2k}(z) + h(J\frac{\partial}{\partial z^s},\frac{\partial}{\partial y^j})\breve{\Theta}^{2k-1}(z))e_{2k}]\\
&+ \sum_{r=1}^{2q}\frac{\partial^2\breve{\psi}^r}{\partial z^s\partial y^j}e_{2n-2q+r}\\
&+ (\frac{\partial^2\breve{\mathcal{F}}^1}{\partial z^s\partial y^j} - h(\frac{\partial}{\partial z^s},\frac{\partial}{\partial y^j})\breve{\Theta}^{2n+1}(z) + h(J\frac{\partial}{\partial z^s},\frac{\partial}{\partial y^j})\breve{\Theta}^{2n+2}(z))e_{2n+1}\\
&+ (\frac{\partial^2\breve{\mathcal{F}}^2}{\partial z^s\partial y^j} + h(\frac{\partial}{\partial z^s},\frac{\partial}{\partial y^j})\breve{\Theta}^{2n+2}(z) + h(J\frac{\partial}{\partial z^s},\frac{\partial}{\partial y^j})\breve{\Theta}^{2n+1}(z))e_{2n+2}
\end{aligned}
$$

Since in the second pair of expressions there are no terms containing the basis vectors e_t with $t\in\{2q+1,\ldots,2n-2q\}$, we conclude that $\beta^l_{sj}=0$ and $\delta^l_{sj}=0$ for any l,s,j. Comparing now the coefficients of $e_{t'}$ with $t'\in\{2n-2q+1,\ldots,2n\}$ we obtain

$$\frac{\partial^2\breve{\psi}^r}{\partial y^s\partial y^j}=0$$

and

$$\frac{\partial^2\breve{\psi}^r}{\partial z^s\partial y^j}=0$$

for any r, s, j.

It follows that

$$\frac{\partial \check{\psi}^r}{\partial y^j} = E^r_j = \text{const}$$

and

$$\check{\psi}^r(y, z) = \sum_{j=1}^{2n-4q} E^r_j y^j + \check{\Pi}^r(z).$$

The Cauchy-Riemann equations for $\hat{\psi}^r$ imply that $\hat{\Pi}^r := \check{\Pi}^{2r-1} + i\check{\Pi}^{2r}$ is a holomorphic function and $E^{2r-1}_{2j-1} = E^{2r}_{2j}$, $E^{2r-1}_{2j} = -E^{2r}_{2j-1}$ for $r = 1, \ldots, q$, $j = 1, \ldots, n-2q$. We set $C^k{}_s := E^{2k-1}_{2j-1} + iE^{2k}_{2j-1}$ and the lemma follows. ■

We define the isomorphism $\hat{A}^0_6$ of $\mathbf{C}^{n+1}$,

$$\hat{A}^0_6 \tilde{e}_k := \begin{cases} \tilde{e}_k & \text{if } \ k \notin \{q+1, \ldots, n-q\} \\ \tilde{e}_k - \sum_{j=1}^q C^j{}_k \tilde{e}_{n-q+j} & \text{if } \ k \in \{q+1, \ldots, n-q\}. \end{cases}$$

Let $\hat{A}_6 := \hat{A}^0_6 \circ \hat{A}_5$, then

$$\begin{aligned} &\hat{A}_6 \circ f \circ \hat{\phi}_4^{-1}(\tilde{x}^1, \ldots, \tilde{x}^q, \tilde{y}^1, \ldots, \tilde{y}^{n-2q}, \tilde{z}^1, \ldots, \tilde{z}^q) \\ &= (\tilde{x}^1, \ldots, \tilde{x}^q, \tilde{y}^1, \ldots, \tilde{y}^{n-2q}, \hat{\Pi}^1(\tilde{z}), \ldots, \hat{\Pi}^q(\tilde{z}), \hat{\mathcal{F}}(\tilde{y}, \tilde{z})) \end{aligned}$$

and $\hat{A}_6 \xi \circ \hat{\phi}_4^{-1}$ is given by the same formula as $\hat{A}_5 \xi \circ \hat{\phi}_4^{-1}$.

Lemma 13. *If $q > 1$ then*

$$\hat{\Pi}^r(\tilde{z}) = \sum_{s=1}^q C_1{}^r{}_s \tilde{z}^s \ + \ C_2{}^r$$

with $C_1{}^r{}_s, \ C_2{}^r \in \mathbf{C}$.

P r o o f. Recall that for $q > 1$

$$\hat{\Theta}^k(\tilde{z}^1, \ldots, \tilde{z}^q) = \tilde{z}^k \ \text{ for } \ k = 1, \ldots, q$$

and

$$\hat{\Theta}^{n+1}(\tilde{z}^1, \ldots, \tilde{z}^q) \equiv 1.$$

Therefore we have

$$\begin{aligned} &A_6 \xi \circ \phi_4^{-1}(x^1, \ldots, x^{2q}, y^1, \ldots, y^{2n-4q}, z^1, \ldots, z^{2q}) \\ &= \sum_{k=1}^q [z^{2k-1} e_{2k-1} - z^{2k} e_{2k}] \ + \ e_{2n+1}, \end{aligned}$$

$$D_{\frac{\partial}{\partial z^s}}(A_6 \xi) = (-1)^{s-1} e_s = -(A_6 \circ f)_*((-1)^s \frac{\partial}{\partial x^s})$$

which implies

$$S\frac{\partial}{\partial z^s} = (-1)^s\frac{\partial}{\partial x^s}.$$

Applying the covariant derivative ∇_W to the right-hand side of the above formula we obtain zero, therefore

$$S(\nabla_W\frac{\partial}{\partial z^s}) = \nabla_W(S\frac{\partial}{\partial z^s}) = 0$$

and

$$\nabla_W\frac{\partial}{\partial z^s} \in \ker S$$

for any tangent vector W. We can now proceed analogously to the proof of Lemma 12. ∎

The matrix $({C_1}^r{}_s)_{r,s=1,\ldots,q}$ is invertible, since $\hat{A}_6 \circ f$ is an immersion and

$$\hat{A}_6\xi \circ \hat{\phi}_4^{-1}(\tilde{x}^1,\ldots,\tilde{x}^q,\tilde{y}^1,\ldots,\tilde{y}^{n-2q},\tilde{z}^1,\ldots,\tilde{z}^q) = \sum_{k=1}^{q}\overline{\tilde{z}^k}\tilde{e}_k + \tilde{e}_{n+1}$$

is a transversal field. Let

$$({C_3}^i{}_j)_{i,j=1,\ldots,q} = [({C_1}^r{}_s)_{r,s=1,\ldots,q}]^{-1}.$$

To complete the proof of the theorem in the case $q > 1$ it remains to apply the affine isomorphism $\tilde{A}_7^0$ of $\mathbf{C}^{n+1}$,

$$\tilde{A}_7^0(\theta^1,\ldots,\theta^{n+1}) :=$$
$$(\theta^1,\ldots,\theta^{n-q},\sum_{j_1=1}^{q}{C_3}^1{}_{j_1}(\theta^{n-q+j_1} - {C_2}^{j_1}),\ldots,\sum_{j_q=1}^{q}{C_3}^q{}_{j_q}(\theta^{n-q+j_q} - {C_2}^{j_q}),\theta^{n+1}).$$

The linear part of $\tilde{A}_7^0$ does not change the transversal field $\hat{A}_6\xi$, therefore we have $\tilde{A} \circ f$ and $\tilde{A}^{\rightarrow}\xi$ as claimed, with $\tilde{A} := \tilde{A}_7^0 \circ \hat{A}_6$, $\tilde{\mathcal{F}} = \hat{\mathcal{F}}$, $\tilde{\phi} = \hat{\phi}_4$.

For $q = 1$

$$\hat{A}_6 \circ f \circ \hat{\phi}_4^{-1}(\tilde{x},\tilde{y}^1,\ldots,\tilde{y}^{n-2},\tilde{z}) = (\tilde{x},\tilde{y}^1,\ldots,\tilde{y}^{n-2},\hat{\Pi}(\tilde{z}),\hat{\mathcal{F}}(\tilde{y},\tilde{z}))$$

with $\hat{\Pi}' \neq 0$ in the neighbourhood of $\tilde{\tilde{\phi}}_4(m)$. We define a local diffeomorphism

$$\tilde{\tilde{\phi}}_5^0(\tilde{x},\tilde{y}^1,\ldots,\tilde{y}^{n-2},\tilde{z}) := (\tilde{x},\tilde{y}^1,\ldots,\tilde{y}^{n-2},\hat{\Pi}(\tilde{z}))$$

and obtain

$$\hat{A}_6 \circ f \circ \tilde{\tilde{\phi}}_5^{-1}(\tilde{x},\tilde{y}^1,\ldots,\tilde{y}^{n-2},\tilde{z}) = (\tilde{x},\tilde{y}^1,\ldots,\tilde{y}^{n-2},\tilde{z},\hat{\hat{\mathcal{F}}}(\tilde{y},\tilde{z})),$$

$$\hat{A}_6\xi \circ \tilde{\tilde{\phi}}_5^{-1}(\tilde{x}, \tilde{y}^1, \ldots, \tilde{y}^{n-2}, \tilde{z}) = \overline{\tilde{\tilde{\mathcal{G}}}(\tilde{z})}\tilde{e}_1 \ + \ \overline{e^{\tilde{\tilde{\mathcal{M}}}(\tilde{z})}}\tilde{e}_{n+1}$$

with $\hat{\hat{\mathcal{F}}} = \hat{\mathcal{F}} \circ \hat{\Pi}^{-1}$, $\tilde{\tilde{\mathcal{M}}} = \tilde{\mathcal{H}} \circ \hat{\Pi}^{-1}$, $\tilde{\tilde{\mathcal{G}}} = \Theta^1 \circ \hat{\Pi}^{-1}$, $\tilde{\tilde{\phi}}_5 = \tilde{\tilde{\phi}}_5^0 \circ \hat{\phi}_4$.

LEMMA 14.

$$\tilde{\tilde{\mathcal{G}}}' - \tilde{\tilde{\mathcal{M}}}'\tilde{\tilde{\mathcal{G}}} = \text{const} \ \neq 0.$$

P r o o f. Let $\tilde{\tilde{\mathcal{G}}} = G^1 + iG^2$, $\tilde{\tilde{\mathcal{M}}} = M^1 + iM^2$. Then

$$\begin{aligned}
&\tilde{\tilde{\mathcal{G}}}' - \tilde{\tilde{\mathcal{M}}}'\tilde{\tilde{\mathcal{G}}} \\
&= \left(\frac{\partial G^1}{\partial z^1} + i\frac{\partial G^2}{\partial z^1}\right) - \left(\frac{\partial M^1}{\partial z^1} + i\frac{\partial M^2}{\partial z^1}\right)(G^1 + iG^2) \\
&= \left(\frac{\partial G^1}{\partial z^1} - \frac{\partial M^1}{\partial z^1}G^1 + \frac{\partial M^2}{\partial z^1}G^2\right) + i\left(\frac{\partial G^2}{\partial z^1} - \frac{\partial M^2}{\partial z^1}G^1 - \frac{\partial M^1}{\partial z^1}G^2\right).
\end{aligned}$$

It is easily seen that

$$S\frac{\partial}{\partial z^1} = \left(-\frac{\partial G^1}{\partial z^1} + \frac{\partial M^1}{\partial z^1}G^1 - \frac{\partial M^2}{\partial z^1}G^2\right)\frac{\partial}{\partial x^1} + \left(\frac{\partial G^2}{\partial z^1} - \frac{\partial M^2}{\partial z^1}G^1 - \frac{\partial M^1}{\partial z^1}G^2\right)\frac{\partial}{\partial x^2},$$

because

$$\begin{aligned}
D_{\frac{\partial}{\partial z^1}}A_6\xi &= \frac{\partial G^1}{\partial z^1}e_1 - \frac{\partial G^2}{\partial z^1}e_2 + \left(\frac{\partial M^1}{\partial z^1}e^{M^1}\cos M^2 - \frac{\partial M^2}{\partial z^1}e^{M^1}\sin M^2\right)e_{2n+1} \\
&\quad + \left(-\frac{\partial M^1}{\partial z^1}e^{M^1}\sin M^2 - \frac{\partial M^2}{\partial z^1}e^{M^1}\cos M^2\right)e_{2n+2} \\
&= \left(\frac{\partial G^1}{\partial z^1} - \frac{\partial M^1}{\partial z^1}G^1 + \frac{\partial M^2}{\partial z^1}G^2\right)e_1 \\
&\quad + \left(-\frac{\partial G^2}{\partial z^1} + \frac{\partial M^2}{\partial z^1}G^1 + \frac{\partial M^1}{\partial z^1}G^2\right)e_2 + \frac{\partial M^1}{\partial z^1}A_6\xi - \frac{\partial M^2}{\partial z^1}JA_6\xi.
\end{aligned}$$

For any vector W

$$\nabla_W\frac{\partial}{\partial z^1} \in \operatorname{span}_{\mathbf{R}}\left\{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}\right\} \subset \ker S,$$

since

$$D_W(A_6 \circ f)_*\frac{\partial}{\partial z^1} = W(\frac{\partial \check{\mathcal{F}}^1}{\partial z^1})e_{2n+1} + W(\frac{\partial \check{\mathcal{F}}^2}{\partial z^1})e_{2n+2} \in \operatorname{span}_{\mathbf{R}}\{e_1, e_2, A_6\xi, JA_6\xi\}.$$

It follows that $\nabla_W(S\frac{\partial}{\partial z^1}) = 0$ for any W. Therefore

$$-\frac{\partial G^1}{\partial z^1} + \frac{\partial M^1}{\partial z^1}G^1 - \frac{\partial M^2}{\partial z^1}G^2 =: -B_1 = \text{const}$$

and

$$\frac{\partial G^2}{\partial z^1} - \frac{\partial M^2}{\partial z^1} G^1 - \frac{\partial M^1}{\partial z^1} G^2 =: B_2 = \text{const.}$$

Moreover, $(B_1)^2 + (B_2)^2 \neq 0$, because $S \neq 0$. ■

Let

$$\tilde{\tilde{\phi}}{}^0_6(\tilde{x}, \tilde{y}^1, \dots, \tilde{y}^{n-2}, \tilde{z}) := (\tilde{x}, \tilde{y}^1, \dots, \tilde{y}^{n-2}, (B_1 + iB_2)\tilde{z}),$$

$$\tilde{\tilde{A}}{}^0_7(\zeta^1, \dots, \zeta^{n+1}) := (\zeta^1, \dots, \zeta^{n-1}, (B_1 + iB_2)\zeta^n, \zeta^{n+1}).$$

Now $\tilde{\tilde{A}}_7 \circ f \circ \tilde{\tilde{\phi}}{}_6^{-1}$ and $\tilde{\tilde{A}}_7 \xi \circ \tilde{\tilde{\phi}}{}_6^{-1}$ have the same shape as $\hat{A}_6 \circ f \circ \tilde{\tilde{\phi}}{}_5^{-1}$ and $\hat{A}_6 \xi \circ \tilde{\tilde{\phi}}{}_5^{-1}$ with $\hat{\tilde{\mathcal{F}}}$, $\tilde{\tilde{\mathcal{G}}}$, $\tilde{\tilde{\mathcal{M}}}$ replaced by $\tilde{\mathcal{F}}$, $\tilde{\mathcal{G}}$ $\tilde{\mathcal{M}}$;

$$\tilde{\mathcal{F}}(\tilde{z}) := \hat{\tilde{\mathcal{F}}}(\frac{\tilde{z}}{B_1 + iB_2}), \quad \tilde{\mathcal{G}}(\tilde{z}) := \tilde{\tilde{\mathcal{G}}}(\frac{\tilde{z}}{B_1 + iB_2}), \quad \tilde{\mathcal{M}}(\tilde{z}) := \tilde{\tilde{\mathcal{M}}}(\frac{\tilde{z}}{B_1 + iB_2}).$$

It is easy to check that

$$\tilde{\mathcal{G}}' - \tilde{\mathcal{M}}'\tilde{\mathcal{G}} = 1.$$

This finishes the proof of the theorem. ■

There are many examples of functions $\tilde{\mathcal{G}}$ and $\tilde{\mathcal{M}}$ satisfying the above equation. In fact, for any holomorphic $\tilde{\mathcal{M}}$ (and in this way for any μ, ν, because $\mu(\frac{\partial}{\partial z^k}) = \frac{\partial \mathcal{M}^1}{\partial z^k}$ and $\nu(\frac{\partial}{\partial z^k}) = -\frac{\partial \mathcal{M}^1}{\partial z^k}$) there exists $\tilde{\mathcal{G}}$ such that $\tilde{\mathcal{G}}' - \tilde{\mathcal{M}}'\tilde{\mathcal{G}} \equiv 1$. For example

$$\tilde{\mathcal{G}}(\tilde{z}) = \tilde{z}, \quad \tilde{\mathcal{M}}(\tilde{z}) = 0$$

(that is what we obtain also for $q > 1$, with $\mu = \nu = 0$);

$$\tilde{\mathcal{G}}(\tilde{z}) = 1, \quad \tilde{\mathcal{M}}(\tilde{z}) = -\tilde{z};$$

$$\tilde{\mathcal{G}}(\tilde{z}) = e^{\tilde{z}}, \quad \tilde{\mathcal{M}}(\tilde{z}) = \tilde{z} + e^{-\tilde{z}}.$$

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