ON SPECIAL VALUES FOR PENCILS OF PLANE CURVE SINGULARITIES

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Abstract

Let $(F_t : t \in \mathbf{P}^1)$ be a pencil of plane curve singularities and let μ_0^t be the Milnor number of the fiber F_t. We prove a formula for the jumps $\mu_0^t - \inf\{\mu_0^t : t \in \mathbf{P}^1\}$. As application we give a description of the special values of the pencil $(F_t : t \in \mathbf{P}^1)$.

Introduction

Let $(F_t : t \in \mathbf{P}^1)$, $\mathbf{P}^1 = \mathbf{C} \cup \{\infty\}$ be a pencil of plane curve singularities defined by two coprime power series $f, g \in \mathbf{C}\{X, Y\}$ without constant term. That is $F_t = f - tg$ for $t \in \mathbf{C}$ and $F_\infty = g$. Let μ_0^t be the Milnor number of the fiber F_t and let

$$\mu_0^{\min} = \inf\{\mu_0^t : t \in \mathbf{P}^1\} .$$

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Our aim is to give a formula for the jumps $\mu_0^t - \mu_0^{\min}$ by means of the meromorphic fraction f/g considered on the branches of the jacobian curve

$$j(F) = \frac{\partial f}{\partial X}\frac{\partial g}{\partial Y} - \frac{\partial f}{\partial Y}\frac{\partial g}{\partial X} = 0 \; .$$

Roughly speaking we will show that $\mu_0^t - \mu_0^{\min}$ = the number of zeros of $f/g - t$ if $t \in \mathbf{C}$ and $\mu_0^{\infty} - \mu_0^{\min}$ = the number of poles of f/g on the branches of the jacobian curve $j(F) = 0$ provided that $\mu_0^t \neq +\infty$ (resp. $\mu_0^{\infty} \neq +\infty$). Then we prove a known result on the special values of the pencil $(F_t : t \in \mathbf{P}^1)$.

1 Preliminaries

Let $f \in \mathbf{C}\{X, Y\}$ be a non-zero power series without constant term. We say that the curve $f = 0$ is singular if $\partial f/\partial X(0,0) = \partial f/\partial Y(0,0) = 0$. A branch P is a prime ideal of $\mathbf{C}\{X, Y\}$ generated by an irreducible power series p. Let $\mathcal{B}$ be the set of all branches. For any curve $f = 0$ we put

$$\mathcal{B}(f) = \{P \in \mathcal{B} : \; f \equiv 0 (\operatorname{mod} P)\} \; .$$

We put by definition $(f, g)_0 = \dim_{\mathbf{C}} \mathbf{C}\{X, Y\}/(f, g)$ the intersection multiplicity of f and g. Note that $(f, g)_0 = +\infty$ if and only if f and g have a common factor. The Milnor number $\mu_0(f)$ is defined to be $\mu_0(f) = (\partial f/\partial X, \partial f/\partial Y)_0$. Then $\mu_0(f) = +\infty$ if and only if the curve $f = 0$ is not reduced (i. e. the power series f has a multiple factor).

The following lemma is well-known (see [GLM], [Ca] and [P]).

Lemma 1.1 *Let $f = 0$ and $g = 0$ be two curves without common branch. Let $j(f, g) = (\partial f/\partial X)(\partial g/\partial Y) - (\partial f/\partial Y)(\partial g/\partial X)$. Then*

$$(f, j(f, g))_0 = \mu_0(f) + (f, g)_0 - 1 \; .$$

In particular the curve $f = 0$ is not reduced if and only if the curves $f = 0$ and $j(f, g) = 0$ share a common branch.

Proof. We may assume that $f(0, Y) \neq 0$. Using Delgado's formula ([Ca], Prop. 7.4.1) we get

$$(f, j(f, g))_0 = (f, \partial f/\partial Y)_0 + (f, g)_0 - (f, X)_0 \; . \tag{1}$$

On the other hand by Teissier's formula ([T], Chap. II, Prop. 1.2) one can write

$$(f, \partial f/\partial Y)_0 = \mu_0(f) + (f, X)_0 - 1 \tag{2}$$

and the lemma follows.

For every branch P we denote by $\mathcal{M}_P$ the field of fractions of the ring $\mathbf{C}\{X,Y\}/P$. Let $f, g \in \mathbf{C}\{X,Y\}$ be coprime power series. Put

$$\mathcal{D}(f/g) = \{P \in \mathcal{B} : \; g \not\equiv 0 (\operatorname{mod} P)\} \,.$$

Then for every $P \in \mathcal{D}(f/g)$ the fraction f/g defines an element of $\mathcal{M}_P$ which we denote also f/g. We put for $P \in \mathcal{D}(f/g)$:

$$\operatorname{ord}_P(f/g) = (f,p)_0 - (g,p)_0 \,,$$

where p is a generator of P. Clearly ord_P is a valuation of the field $\mathcal{M}_P$.

Let $\mathbf{P}^1 = \mathbf{C} \cup \{\infty\}$ and let us denote by $f/g \mapsto (f/g)(P) \in \mathbf{P}^1$ the place associated with ord_P.

Recall that $(f/g)(P) = \infty$ if $\operatorname{ord}_P f < \operatorname{ord}_P g < +\infty$ and $(f/g)(P) = 0$ if $\operatorname{ord}_P g < \operatorname{ord}_P f$.

Lemma 1.2 *Suppose that* $\operatorname{ord}_P f/g \geq 0$ *for a* $P \in \mathcal{D}(f/g)$ *and let* $t_0 = (f/g)(P)$. *Then* $\operatorname{ord}_P(f - tg) = \operatorname{ord}_P g$ *if* $t \neq t_0$ *and* $\operatorname{ord}_P(f - t_0 g) > \operatorname{ord}_P g$.

Proof. Obvious.

2 Result

Let $f = 0$ and $g = 0$ be two curves without common branch. We put $F = (f,g)$ and $j(F) = j(f,g)$. Let us consider the pencil $(F_t : \, t \in \mathbf{P}^1)$ where $F_t = f - tg$ for $t \in \mathbf{C}$ and $F_\infty = g$. Let $\mu_0^t = \mu_0(F_t)$. If $j(F)(0,0) \neq 0$ then $\mu_0^t = 0$ for all $t \in \mathbf{P}^1$. In the sequel we assume that $j(F)(0,0) = 0$.

Proposition 2.1 *Let* $t \in \mathbf{P}^1$. *Then the following two conditions are equivalent:*

(i) $\mu_0^t = +\infty$,

(ii) *the curves* $j(F) = 0$ *and* $F_t = 0$ *share a common branch.*

Proof. We use Lemma 1.1 to power series F_t, g if $t \in \mathbf{C}$ and to power series $F_\infty = g$ and f if $t = \infty$.

Proposition 2.1 implies Bertini's theorem "the set $\{t \in \mathbf{P}^1 : F_t$ is not reduced$\}$ is finite". Indeed, it is easy to check that

$$\#\{t \in \mathbf{P}^1 : \mu_0^t = +\infty\} \leq \mathcal{B}(j(F)) .$$

Let $\mu_0^{\min} = \inf\{\mu_o^t : t \in \mathbf{P}^1\}$. By Bertini's theorem $\mu_0^{\min}$ is an integer.

Let us put

$$\mathcal{U}(F) = \{P \in \mathcal{B}(j(F)) : \operatorname{ord}_P f \geq \operatorname{ord}_P g\}$$

and

$$\mathcal{U}(F)^c = \{P \in \mathcal{B}(j(F)) : \operatorname{ord}_P f < \operatorname{ord}_P g\} .$$

Thus $\mathcal{U}(F) \subset \mathcal{D}(f/g)$ and $\mathcal{U}(F)^c \subset \mathcal{D}(f/g)$ provided that $\mu_0^\infty < +\infty$.

For every branch P of the jacobian curve $j(F) = 0$ we denote by $m(P)$ the multiplicity of P i.e. the greatest integer $m > 0$ such that $j(F) \equiv 0(\bmod P^m)$. By convention a sum extended over an empty set equals zero.

Our main result is the following

Theorem 2.2 *With the notation introduced above*

(i) *if $\mu_0^t \neq +\infty$ for a $t \in \mathbf{C}$, then*

$$\mu_0^t - \mu_0^{\min} = \sum_{P \in \mathcal{U}(F)} m(P) \operatorname{ord}_P(f/g - t) .$$

(ii) *If $\mu_0^\infty \neq +\infty$, then*

$$\mu_0^\infty - \mu_0^{\min} = - \sum_{P \in \mathcal{U}(F)^c} m(P) \operatorname{ord}_P(f/g) .$$

Proof. Let us fix a $t \in \mathbf{C}$ such that $\mu_0^t \neq +\infty$. We have $j(F_t, g) = j(f, g)$ and $(F_t, g)_0 = (f, g)_0$. Applying Lemma 1.1 to F_t and g we get

$$\mu_0^t = (F_t, j(f,g))_0 - (f,g)_0 + 1 . \tag{3}$$

Let us write

$$j(f,g)=\prod_{i=1}^{k} p_i$$

with irreducible $p_i \in \mathbf{C}\{X,Y\}$ and let $P_i=(p_i)\mathbf{C}\{X,Y\}$. Therefore $(P_i)_{i=1,\dots,k}$ is a sequence of branches of $j(f,g)=0$ counted with multiplicities. Let

$$I=\{i\in[1,k]:\mathrm{ord}_{P_i}f \geq \mathrm{ord}_{P_i}g\}$$

and observe that $\mathrm{ord}_{P_i}F_t=\mathrm{ord}_{P_i}f$ for $i\notin I$. Then

$$(F_t,j(f,g))_0=\sum_{i=1}^{k}\mathrm{ord}_{P_i}F_t=\sum_{i\in I}\mathrm{ord}_{P_i}F_t+\sum_{i\notin I}\mathrm{ord}_{P_i}f$$

and by (3) we get

$$\mu_0^t=\sum_{i\in I}\mathrm{ord}_{P_i}F_t+\sum_{i\notin I}\mathrm{ord}_{P_i}f-(f,g)_0+1\ . \tag{4}$$

If $i\in I$ then by Lemma 1.2 we have $\mathrm{ord}_{P_i}F_t\geq\mathrm{ord}_{P_i}g$ with equality for $t\neq(f/g)(P_i)$. Using (4) we get

$$\mu_0^{\min}=\sum_{i\in I}\mathrm{ord}_{P_i}g+\sum_{i\notin I}\mathrm{ord}_{P_i}f-(f,g)_0+1\ . \tag{5}$$

and consequently

$$\begin{aligned}\mu_0^t-\mu_0^{\min}&=\sum_{i\in I}\mathrm{ord}_{P_i}F_t-\mathrm{ord}_{P_i}g=\sum_{i\in I}\mathrm{ord}_{P_i}(F_t/g)\\&=\sum_{P\in\mathcal{U}(F)}m(P)\mathrm{ord}_P(f/g-t)\end{aligned}$$

for $F_t/g=f/g-t$. We have thus proved (i).

Let us suppose that $\mu_0^\infty\neq+\infty$. By Lemma 1.1 applied to g and f we get

$$\mu_0^\infty=(g,j(f,g))_0-(f,g)_0+1=\sum_{i\in I}\mathrm{ord}_{P_i}g+\sum_{i\notin I}\mathrm{ord}_{P_i}g-(f,g)_0+1\ . \tag{6}$$

Now, by (5) and (6) we get

$$\begin{aligned}\mu_0^\infty-\mu_0^{\min}&=\sum_{i\notin I}\mathrm{ord}_{P_i}g-\mathrm{ord}_{P_i}f=-\sum_{i\notin I}\mathrm{ord}_{P_i}(f/g)\\&=-\sum_{P\in\mathcal{U}(F)^c}m(P)\mathrm{ord}_P(f/g)\end{aligned}$$

which proves (ii).

3 Description of special values

Let

$$\Lambda(F) = \{t \in \mathbf{P}^1 : \ \mu_0^t > \mu_0^{\min}\}$$

be the set of special values of the pencil $(F_t : t \in \mathbf{P}^1)$ (see [LW]). We put by convention $(f/g)(P) = \infty$ if $g \equiv 0(\operatorname{mod} P)$.

The following description of the special values is due to different authors:

Theorem 3.1 (see [MM], Théorème 1, [LO], pp 410-411, [Ca], 7.4).
One has

$$\Lambda(F) = \{(f/g)(P) : \ P \in \mathcal{B}(j(F))\} .$$

Proof. First we prove the following

$$\{t \in \mathbf{C} : \ \mu_0^t > \mu_0^{\min}\} = \{(f/g)(P) : \ P \in \mathcal{U}(F)\} . \tag{7}$$

Fix $t \in \mathbf{C}$. We will check that $\mu_o^t > \mu_0^{\min}$ if and only if there exists a $P \in \mathcal{U}(F)$ such that $(f/g)(P) = t$. If $\mu_0^t = +\infty$ then F_t has multiple factors. Thus there exists a branch P such that $F_t \equiv 0(\operatorname{mod} P^2)$. It is easy to check that $P \in \mathcal{U}(F)$ and $(f/g)(P) = t$.

Now suppose that $\mu_0^t < +\infty$. According to Theorem 2.2(i) the inequality $\mu_0^t > \mu_0^{\min}$ holds if and only if there exists $P \in \mathcal{U}(F)$ such that $\operatorname{ord}_P(f - tg) > \operatorname{ord}_P g$. The last inequality is equivalent to the condition $(f/g)(P) = t$. This proves (7).

Let us check the following property

$$\mu_0^\infty > \mu_0^{\min} \text{ if and only if } \mathcal{U}(F)^c \neq \emptyset . \tag{8}$$

Indeed, if $\mu_0^\infty = +\infty$ then there is a branch P such that $g \equiv 0(\operatorname{mod} P)$ and $P \in \mathcal{B}(j(F))$ by Proposition 2.1. Obviously $f \not\equiv 0(\operatorname{mod} P)$ and we get $\operatorname{ord}_P g = +\infty > \operatorname{ord}_P f$. Thus $P \in \mathcal{U}(F)^c$. If $\mu_0^\infty < +\infty$ then (8) follows from Theorem 2.2(ii). Theorem 3.1 follows from (7) and (8) for $(f/g)(P) = \infty$ if $P \in \mathcal{U}(F)^c$.

When studying the singularities at infinity of a polynomial in two complex variables of degree $N > 1$ one considers the pencil defined by $F = (f, l^N)$ where $l = 0$ is a smooth curve which is not a component of the curve $f = 0$. Clearly $\mu_0^\infty = +\infty$. Using Theorem 3.1 we get

Corollary 3.2 ((see [GBP], Prop. 2.2))

$$\Lambda(F) \cap \mathbf{C} = \{(f/l^N)(P) : \ P \in \mathcal{B}(j(f,l)) \ \textit{and} \ \operatorname{ord}_P f / \operatorname{ord}_P l \geq N\} .$$

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