Łojasiewicz exponents and singularities at infinity of polynomials in two complex variables

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Abstract

For every polynomial $F : \mathbb{C}^2 \to \mathbb{C}$ we calculate the Łojasiewicz exponents $\mathcal{L}_{p,t}(F)$ measuring the growth of the gradient ∇F on the branches centered at points p at infinity.

Introduction

The notion of the Łojasiewicz exponent was introduced and studied by Lejeune–Jalabert and Teissier [LJ-T]. In the case of isolated singularities of hypersurfaces Teissier [T] showed that the Łojasiewicz exponent of the gradient can be calculated by means of the polar invariants. Then Ha [H] defined the Łojasiewicz exponents at infinity $\mathcal{L}_{\infty,t}(F)$ and $\mathcal{L}_\infty(F)$ for the polynomial F of two complex variables and applied these notions to the singularities at infinity. His results were completed by Chądzyński and Krasiński [ChK1], [ChK2]. Moreover Cassou-Noguès and Ha [CN-H] gave a formula for $\mathcal{L}_\infty(F)$ using the Eisenbud and Neumann diagrams. In this note we consider the Łojasiewicz exponens $\mathcal{L}_{p,t}(F)$ at points (p,t) at infinity. To be more specific let us consider a polynomial $F : \mathbb{C}^2 \to \mathbb{C}$ of degree $d > 0$. To study the growth of

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the gradient $\nabla F = \left(\frac{\partial F}{\partial X}, \frac{\partial F}{\partial Y}\right)$ at infinity we extend $\mathbb{C}^2$ to the projective plane $\mathbb{P}^2(\mathbb{C}) = \mathbb{C}^2 \cup \mathbb{L}_\infty$ and consider the Łojasiewicz exponents $\mathcal{L}_{p,t}(F)$ for every pair $(p,t) \in \mathbb{L}_\infty \times \mathbb{C}$. Roughly speaking (see Section 1 of this paper for the precise definition) the Łojasiewicz exponent $\mathcal{L}_{p,t}(F)$ measures the growth of the gradient ∇F on the branches γ of $\mathbb{P}^2(\mathbb{C})$ centered at p such that F approaches $t \in \mathbb{C}$ along γ. Let $F^*(X,Y,Z) = Z^d F(\frac{X}{Z}, \frac{Y}{Z})$ be the homogeneous form corresponding to $F = F(X,Y)$. We will show how to calculate $\mathcal{L}_{p,t}(F)$ in terms of the local invariants of singularities at infinity of the pencil of projective curves $F^*(X,Y,Z) - tZ^d$, $t \in \mathbb{C}$. It turns out that if (p,t) is a critical point at infinity for the polynomial F (see [D] and Preliminaries 0.3 of this note) then the exponent $\mathcal{L}_{p,t}(F)$ can be calculated by means of the polar invariants, like in the local case (see [T]). If (p,t) is a regular point at infinity then we need another invariant of singularity to calculate $\mathcal{L}_{p,t}(F)$ (see Section 2 and Theorem 4.2). Putting $\mathcal{L}_{\infty,t}(F) = \inf\{\mathcal{L}_{p,t}(F) : p \in \mathbb{L}_\infty\}$ we get the Łojasiewicz exponent of the gradient ∇F along the fiber $F = t$. This notion was studied by Ha [H] and recently by Chądzyński and Krasiński in [ChK2] in a global affine context. Our main theorem (Theorem 1.2) generalizes the results obtained in [H] and [ChK2] and shows that the Łojasiewicz exponent at infinity is a purely local notion.

0 Preliminaries

In this section we fix our notation and recall some useful notions and results.

0.1 Branches at infinity

We use the notions of the classical theory of plane algebraic curves. Let $\mathbb{P}^2(\mathbb{C}) = \mathbb{C}^2 \cup \mathbb{L}_\infty$ where $\mathbb{L}_\infty$ is the line at infinity. A plane branch γ will be called a branch at infinity if it is centered at a point $p \in \mathbb{L}_\infty$ and it is not a branch of $\mathbb{L}_\infty$. We denote by $\mathcal{B}_{\infty,p}$ the set of all branches at infinity centered at p and put $\mathcal{B}_\infty = \bigcup_p \mathcal{B}_{\infty,p}$.

Consider a projective coordinate system $(X : Y : Z)$ such that $Z = 0$ is the equation of $\mathbb{L}_\infty$. If $F = F(X,Y) \in \mathbb{C}[X,Y]$ is a polynomial of degree $d > 0$ and $F^* = F^*(X,Y,Z) \in \mathbb{C}[X,Y,Z]$ is the homogeneous form corresponding to F, then we put

$$\deg_\gamma F = -\operatorname{ord}_\gamma \frac{F^*}{Z^d} = d \operatorname{ord}_\gamma Z - \operatorname{ord}_\gamma F^*, \quad \deg \gamma = \operatorname{ord}_\gamma Z$$

for every branch $\gamma \in \mathcal{B}_\infty$.

Here $\operatorname{ord}_\gamma F^*$ stands for the order of vanishing of a homogeneous form $F^* = F^*(X,Y,Z)$ at the branch γ. We admit usual conventions on the symbol ∞. Note that $\deg_\gamma F \in \mathbb{Z} \cup \{-\infty\}$ with $\deg_\gamma F = -\infty$ if and only if γ is a branch of the projective curve $F^* = 0$. For every branch $\gamma \in \mathcal{B}_\infty$ we define the value $F(\gamma) \in \mathbb{C} \cup \{\infty\}$ as follows: if $\deg_\gamma F \leqslant 0$ then $F(\gamma)$ is the unique $t \in \mathbb{C}$ such that $\deg_\gamma(F - t) < 0$; if $\deg_\gamma F > 0$ then $F(\gamma) = \infty$. Let $\gamma \in \mathcal{B}_\infty$ be a branch at infinity. We say that a pair $p(T) = (x(T), y(T))$ of Laurent series $x(T)$, $y(T) \in \mathbb{C}((T))$ is a meromorphic parametrization of γ if $\operatorname{ord} p(T)$ equal by definition $\min\{\operatorname{ord} x(T), \operatorname{ord} y(T)\}$ is negative and γ is given in projective coordinates by $(T^k x(T) : T^k y(T) : T^k)$ where $k = -\operatorname{ord} p(T)$. It is easy to check that $\deg \gamma = -\operatorname{ord} p(T)$, $\deg_\gamma F = -\operatorname{ord} F(p(T))$ and $F(\gamma) = F(p(T))|_{T=0}$. For every nonconstant polynomial $F \in \mathbb{C}[X, Y]$ we consider its gradient $\nabla F = \left(\frac{\partial F}{\partial X}, \frac{\partial F}{\partial Y}\right)$. For every $\gamma \in \mathcal{B}_\infty$ we put $\deg_\gamma(\nabla F) = \sup\left\{\deg_\gamma \frac{\partial F}{\partial X}, \deg_\gamma \frac{\partial F}{\partial Y}\right\}$. Using meromorphic parametrizations we check that $\deg_\gamma \nabla F = -\infty$ if and only if γ is a branch at infinity of a multiple component of the curve $F^* = 0$.

Moreover for every $\gamma \in \mathcal{B}_\infty$ if $\deg_\gamma F \neq 0$ then $\deg_\gamma F \leqslant \deg_\gamma \nabla F + \deg \gamma$.

0.2 Germs of curves

Our main reference is [Ca]. We will consider the germs γ, γ', …of analytic curves at a given point p of a projective plane. We denote by $(\gamma \cdot \gamma')_p$ the intersection multiplicity of γ and γ' at p and by $\mu_p(\gamma)$ the Milnor number of γ. If γ is a branch then $S(\gamma)$ denotes the semigroup of γ generated by all intersection numbers $(\gamma \cdot \gamma')_p$. Let λ be a smooth branch at p. We say that two reduced germs γ, γ' are λ-equisingular if $\lambda \not\subset \gamma \cup \gamma'$ and there are decompositions $\gamma = \bigcup_{i=1}^r \gamma_i$ and $\gamma' = \bigcup_{i=1}^r \gamma'_i$ as unions of the same number $r > 0$ of branches such that

- • $S(\gamma_i) = S(\gamma'_i)$,
- •• $(\gamma_i \cdot \gamma_j)_p = (\gamma'_i \cdot \gamma'_j)_p$,
- ••• $(\gamma_i \cdot \lambda)_p = (\gamma'_i \cdot \lambda)_p$

for all $i, j = 1, \ldots, r$.
Let U be an open and connected subset of $\mathbb{C}$. Using the Zariski discriminant criterion [Z] we get

Equisingularity Criterion

Let $(\gamma^t : t \in U)$ be an analytic family of germs such that:

(i) there is an integer $n > 0$ such that $(\gamma^t \cdot \lambda)_p = n$ for all $t \in U$,

(ii) there is an integer $\mu \geqslant 0$ such that $\mu_0(\gamma^t) = \mu$ for all $t \in U$.

Then any two germs of the family $(\gamma^t : t \in U)$ are λ-equisingular.

0.3 Critical points at infinity

For any projective plane curve C we denote by $|C|$ the support of C. We identify C and $|C|$ if C has no multiple components. For any two projective plane curves C, C' we denote by $(C \cdot C')_p$ the intersection multiplicity of the germs (C, p) and (C', p) and by $\mu_p(C)$ the Milnor number of the germ (C, p). Note that $\mu_p(C) < +\infty$ if and only if there is no multiple component of C passing through p. The following construction is due to Broughton [B]. Let $F = F(X, Y)$ be a polynomial of degree $d > 0$ and let $F^* = F^*(X, Y, Z)$ be the homogeneous form corresponding to F. Let us consider the pencil $C^t : F^*(X, Y, Z) - tZ^d = 0$, $t \in \mathbb{C}$ of projective curves. The set C_∞ given by equations $F^*(X, Y, Z) = Z = 0$ is the set of base points of the pencil $(C^t : t \in \mathbb{C})$. Fix a point $p \in C_\infty$ and let $\mu_p^{min} = \inf\{\mu_p(C^t) : t \in \mathbb{C}\}$ and $\Lambda_p(F) = \{t \in \mathbb{C} : \mu_p(C^t) > \mu_p^{min}\}$. Thus μ_p^{min} is an integer and $\Lambda_p(F)$ is a finite subset of $\mathbb{C}$. Clearly the integer $d_p = (C^t \cdot \mathbb{L}_\infty)_p$ does not depend on t. Applying the Equisingularity Criterion to the family (C^t, p), $t \in \mathbb{C} \setminus \Lambda_p(F)$ and to $\lambda = (\mathbb{L}_\infty, p)$ we get

Equisingularity at Infinity Property

For every $p \in C_\infty$ any two germs of the family (C^t, p), $t \in \mathbb{C} \setminus \Lambda_p(F)$ are $(\mathbb{L}_\infty, p)$-equisingular.

The pairs $(p, t) \in \mathbb{L}_\infty \times \mathbb{C}$ where $p \in C_\infty$ and $t \in \Lambda_p(F)$ are called critical points at infinity of the polynomial F (see [D] for others definitions and examples).

1 Result

We keep the notation introduced in Preliminaries. Let $F : \mathbb{C}^2 \to \mathbb{C}$ be a polynomial of degree $d > 1$ and let $\nabla F = \left(\frac{\partial F}{\partial X}, \frac{\partial F}{\partial Y}\right)$ be its gradient. For every pair $(p, t) \in \mathbb{L}_\infty \times \mathbb{C}$ we put

$$\pounds_{p,t}(F) = \inf\left\{\frac{\deg_\gamma \nabla F}{\deg \gamma} : \gamma \in \mathcal{B}_{\infty,p} \text{ and } F(\gamma) = t\right\}$$

and call $\pounds_{p,t}(F)$ the Łojasiewicz exponent of the polynomial F at point (p, t).

Let C be the projective closure of the affine curve $F(X, Y) = 0$ and let $C_\infty = |C| \cap \mathbb{L}_\infty$.

Property 1.1 *Let $(p, t) \in \mathbb{L}_\infty \times \mathbb{C}$. If $p \notin C_\infty$ then $\pounds_{p,t}(F) = +\infty$. If $p \in C_\infty$ and a multiple component of C^t passes through p then $\pounds_{p,t}(F) = -\infty$.*

Proof. If $p \notin C_\infty$ then the set $\{\gamma \in \mathcal{B}_{\infty,p} : F(\gamma) = t\}$ is empty and consequently $\pounds_{p,t}(F) = \inf \emptyset = +\infty$. If $p \in C_\infty$ and a multiple component of C^t passes through p then $\deg_\gamma(\nabla F) = -\infty$ for every branch γ of C centered at p and $\pounds_{p,t}(F) = -\infty$.

In the sequel we assume that $p \in C_\infty$. We say that the Łojasiewicz exponent $\pounds_{p,t}(F)$ is attained on an affine curve Γ if there is a branch γ of Γ centered at p such that $F(\gamma) = t$ and $\frac{\deg_\gamma \nabla F}{\deg \gamma} = \pounds_{p,t}(F)$.

Recall that C^t is the projective closure of the fiber $F(X, Y) - t = 0$ (C^t may have multiple components). Our main result is the following

Theorem 1.2 *Let $F : \mathbb{C}^2 \to \mathbb{C}$ be a polynomial of degree $d > 1$ and let $(C^t : t \in \mathbb{C})$ be the pencil of projective curves associated with F. Assume that $(p, t) \in C_\infty \times \mathbb{C}$ and there is no multiple component of C^t passing through p.*

Then the Łojasiewicz exponent $\pounds_{p,t}(F)$ is determined by the class of $(\mathbb{L}_\infty, p)$-equisingularity of the germ (C^t, p) and the following holds

(i) *there exists a rational number* $l_p(F) \geqslant 0$ *such that* $\mathcal{L}_{p,t}(F) = l_p(F)$ *for all* $t \in \mathbb{C} \setminus \Lambda_p(F)$.
For every $t \in \mathbb{C} \setminus \Lambda_p(F)$ *the exponent* $\mathcal{L}_{p,t}(F)$ *is attained on the fiber* $F = t$.

(ii) $l_p(F) = 0$ *if and only if* C *is a pencil of lines passing through* p.

(iii) *If* $t \in \Lambda_p(F)$ *then* $\mathcal{L}_{p,t}(F) < -1$. *Let* $q = (a : b : 0) \notin C_\infty$. *Then the exponent* $\mathcal{L}_{p,t}(F)$ *is attained on the polar curve* $\nabla_q F = a\frac{\partial F}{\partial X} + b\frac{\partial F}{\partial Y} = 0$.

The proof of the above theorem is given in Section 4. We will give also formulae for $\mathcal{L}_{p,t}(F)$ in terms of invariants of the germ (C^t, p) with respect to $(\mathbb{L}_\infty, p)$. Now let us present some applications.

Corollary 1.3 (cf. [H], [D]) *The following conditions are equivalent*

(M) *the pair* $(p, t) \in \mathbb{L}_\infty \times \mathbb{C}$ *is a critical point at infinity for polynomial* F,

(L) $\mathcal{L}_{p,t}(F) < -1$,

(G) *there exists a branch* $\gamma \in \mathcal{B}_{\infty,p}$ *such that* $\frac{\partial F}{\partial X}(\gamma) = \frac{\partial F}{\partial Y}(\gamma) = 0$ *and* $F(\gamma) = t$.

Proof. Conditions *(M)* and *(L)* are equivalent by Theorem 1.2 *(i)* and *(iii)*. To check that *(G)* implies *(L)* take a branch $\gamma \in \mathcal{B}_{\infty,p}$ satisfying *(G)*. Then $\deg_\gamma \nabla F < 0$ and by definition of the Łojasiewicz exponent $\mathcal{L}_{p,t}(F) < 0$. Therefore $\mathcal{L}_{p,t}(F) < -1$ by Theorem 1.2. The implication *(L)* $\Rightarrow$ *(G)* is obvious.

Following [ChK2] we put

$$\mathcal{L}_{\infty,t}(F) = \inf\left\{\frac{\deg_\gamma \nabla F}{\deg \gamma} : \gamma \in \mathcal{B}_\infty \text{ and } F(\gamma) = t\right\}$$

and call $\mathcal{L}_{\infty,t}(F)$ the Łojasiewicz exponent of F along the fiber $F = t$.

It is easy to see that $\mathcal{L}_{\infty,t}(F) = \inf\{\mathcal{L}_{\infty,p}(F) : p \in C_\infty\}$. We say that the exponent $\mathcal{L}_{\infty,t}(F)$ is attained on an affine curve Γ if there is a branch at infinity γ of Γ such that $\frac{\deg_\gamma \nabla F}{\deg \gamma} = \mathcal{L}_{\infty,t}(F)$. Let $\Lambda(F) = \bigcup_{p \in C_\infty} \Lambda_p(F)$.

Corollary 1.4 (cf. [ChK2] and [H]) *There exists a constant* $l_\infty(F) \geqslant 0$ *such that for all* $t \in \mathbb{C} \setminus \Lambda(F)$ $\mathcal{L}_{\infty,t}(F) = l_\infty(F)$. *For such* t *the exponent* $\mathcal{L}_{\infty,t}(F)$ *is attained on the fiber* $F = t$. *If* $t \in \Lambda(F)$ *then* $\mathcal{L}_{\infty,t}(F) < -1$ *and the exponent* $\mathcal{L}_{\infty,t}(F)$ *is attained on every polar curve* $\nabla_q F = 0$, $q \notin C_\infty$.

Proof. We put $l_\infty(F) = \inf\{l_p(F) : p \in C_\infty\}$ and use Theorem 1.2.

Finally, let us consider the total Łojasiewicz exponent $\mathcal{L}_\infty(F)$ (see [H] and [ChK1]):

$$\mathcal{L}_\infty(F) = \inf\left\{\frac{\deg_\gamma \nabla F}{\deg \gamma} : \gamma \in \mathcal{B}_\infty\right\}.$$

Using Corollary 1.4 we get easily

Corollary 1.5 (cf. [H]) *If $\Lambda(F) \neq \emptyset$ then $\pounds_\infty(F) = \inf\limits_{t\in\Lambda(F)} \pounds_{\infty,t}(F)$.*

Remark 1.6 *If $\Lambda(F) = \emptyset$ and the projective closure of the affine curve $F(X,Y) = 0$ crosses the line at infinity $\mathbb{L}_\infty$ at $c \neq \deg F$ distinct points, then $\pounds_\infty(F) < ł_\infty(F)$.*

2 Local invariants of singularities

We keep the notation introduced in Preliminaries. Let λ be a smooth branch at a point p of a complex surface. For any germ γ of an analytic curve we consider the maximal polar quotient $\eta_p(\gamma,\lambda)$ (cf. [T] and [Pł2]). To recall the definition of $\eta_p(\gamma,\lambda)$ choose a system of local coordinates (X,Y) such that $X(p) = Y(p) = 0$ and identify the local ring at 0 with the ring of convergent power series $\mathbb{C}\{X,Y\}$. Let $f(X,Y) = 0$ and $l(X,Y) = 0$ be the local reduced equations od γ and λ, respectively. Let $j(f,l) = \frac{\partial f}{\partial X}\frac{\partial l}{\partial Y} - \frac{\partial f}{\partial Y}\frac{\partial l}{\partial X}$ and put

$$\eta_p(\gamma,\lambda) = \sup\left\{\frac{(f,g)_0}{(l,g)_0} : g \text{ is an irreducible factor of } j(f,l)\right\}.$$

In [Pł2] we gave an explicit formula for $\eta_p(\gamma,\lambda)$ which shows that $\eta_p(\gamma,\lambda)$ is a λ-invariant of equisingularity.

Now, let us put by definition

$$\Theta_p(\gamma,\lambda) = \sup\left\{\frac{(h,j(f,l))_0}{(l,h)_0} : h \text{ is an irreducible factor of } f\right\}.$$

Lemma 2.1 *Let $\gamma = \bigcup_{i=1}^r \gamma_i$ with branches γ_i pairwise different. Let μ_i be the Milnor numbers of γ_i. Then*

$$\Theta_p(\gamma,\lambda) = \sup_{i=1}^{r}\left\{\frac{\mu_i - 1}{(\gamma_i\cdot\lambda)_p} + 1 + \frac{1}{(\gamma_i\cdot\lambda)_p}\sum_{j\neq i}(\gamma_i\cdot\gamma_j)_p\right\}.$$

Proof. By the well-known properties of intersection numbers

$$(f_i, j(f,l))_0 = (f_i, j(f_i,l))_0 + \sum_{j\neq i}(f_i,f_j)_0 = \mu_0(f_i) + (f_i,l)_0 - 1 + \sum_{j\neq i}(f_i,f_j)_0$$

and the lemma follows.

Remark 2.2

1. *If $(\gamma\cdot\lambda)_p = 2$ (γ may be irreducible or not) then $\Theta_p(\gamma,\lambda) = \frac{\mu_p(\gamma)+1}{2}$.*
2. *$\Theta_p(\gamma,\lambda) \geqslant 0$ with equality if and only if γ and λ are smooth and transversal.*

3. $\Theta_p(\gamma,\lambda) \geqslant \frac{\mu_p(\gamma)-1}{(\gamma\cdot\lambda)_p}+1$ *with equality if* γ *is a branch.*

Both invariants $\eta_p(\gamma,\lambda)$ and $\Theta_p(\gamma,\lambda)$ can be calculated by means of Puiseux series. Let $\mathbb{C}\{X\}^* = \bigcup_{n\geqslant 1}\mathbb{C}\{X^{\frac{1}{n}}\}$.

Proposition 2.3 *Let* (X,Y) *be a system of coordinates such that* λ *has the equation* $X=0$. *Suppose that* $f(X,Y) = U(X,Y)\prod_{i=1}^n (Y-y_i(X))$ *with* $y_i = y_i(X) \in \mathbb{C}\{X\}^*$ *without constant terms and* $U(X,Y) \in \mathbb{C}\{X,Y\}$ *such that* $U(0,0)\neq 0$. *Then*

(i) $(\gamma\cdot\lambda)_p = n$,

(ii) $\Theta_p(\gamma,\lambda) = \sup_{i=1}^{n}\left\{\sum_{j\neq i}\operatorname{ord}(y_i-y_j)\right\}$,

(iii) $\eta_p(\gamma,\lambda) = \sup_{i=1}^{n}\left\{\sum_{j\neq i}\operatorname{ord}(y_i-y_j) + \max_{j\neq i}\{\operatorname{ord}(y_i-y_j)\}\right\}$.

Proof. Properties *(i)* and *(ii)* follow easily from the definitions. The proof of *(iii)* is given in [Pł2] (Proposition 2.2).

Proposition 2.4 *If* $n=(\gamma\cdot\lambda)_p > 1$ *then* $\Theta_p(\gamma,\lambda) \leqslant \frac{n-1}{n}\eta_p(\gamma,\lambda)$.

Proof. With the notation introduced above we get $\Theta_p(\gamma,\lambda) = \sum_{j\neq i_0}\operatorname{ord}(y_{i_0}-y_j)$ for an $i_0\in\{1,\dots,n\}$. Therefore $\Theta_p(\gamma,\lambda)\leqslant (n-1)\max_{j\neq i_0}\{\operatorname{ord}(y_{i_0}-y_j)\}$ and

$$\Theta_p(\gamma,\lambda) + \frac{1}{n-1}\Theta_p(\gamma,\lambda) \leqslant \sum_{j\neq i_0}\operatorname{ord}(y_{i_0}-y_j) + \max_{j\neq i_0}\{\operatorname{ord}(y_{i_0}-y_j)\} \leqslant \eta_p(\gamma,\lambda)$$

by Proposition 2.3 *(iii)*.

Proposition 2.5 *Let* $n=(\gamma\cdot\lambda)_p>1$. *If* $\Theta_p(\gamma,\lambda) = n-1$ *and* $\eta_p(\gamma,\lambda)=n$ *then* $n = \operatorname{ord}\gamma$ *that is* γ *and* λ *are transversal.*

Proof. With the notations of Proposition 2.3 we get

$$\sum_{j\neq i_0}\operatorname{ord}(y_{i_0}-y_j) = n-1 \text{ for a } i_0\in\{1,\dots,n\}, \tag{1}$$

$$\sum_{j\neq i_0}\operatorname{ord}(y_{i_0}-y_j) + \max_{j\neq i_0}\{\operatorname{ord}(y_{i_0}-y_j)\} \leqslant n. \tag{2}$$

From (1) and (2) we get

$$\operatorname{ord}(y_{i_0}-y_j) = 1 \text{ for all } j\neq i_0. \tag{3}$$

Now we can check that $y_{i_0} = y_{i_0}(X) \in \mathbb{C}\{X\}$. In fact, let $f_1(X,Y) \in \mathbb{C}\{X,Y\}$ be the irreducible power series such that $f_1(X, y_{i_0}(X)) = 0$. If we had $(f_1, X)_0 > 1$ then there would exists a solution $y_{i_1}(X) \neq y_{i_0}(X)$ of the equation $f_1(X,Y) = 0$ such that $\operatorname{ord}(y_{i_0}(X) - y_{i_1}(X)) \neq 1$ (see, for example [Pł2], Proposition 3.1) and we would get contradiction with (3). Therefore $y_{i_0}(X) \in \mathbb{C}\{X\}$ and consequently $\operatorname{ord} y_{i_0}(X) \geqslant 1$. Now, by (3) we get $\operatorname{ord} y_j(X) \geqslant 1$ for all $j = 1, \dots, n$ and $\operatorname{ord} f(X,Y) = \operatorname{ord} \prod_{j=1}^{n}(Y - y_j(X)) = \sum_{j=1}^{n} \operatorname{ord}(Y - y_j(X)) = n = (f, X)_0$.

Let us keep the notation introduced in the beginning of this section. The germ given by equation $j(f,l) = 0$ will be called a local polar of γ with respect to λ.

Proposition 2.6 *Let γ' be the local polar of γ with respect to λ. Then for every branch ξ*

$$\frac{(\gamma \cdot \xi)_p}{(\lambda \cdot \xi)_p} \leqslant \eta_p(\gamma, \lambda) \ \text{or}\ \frac{(\gamma' \cdot \xi)_p}{(\lambda \cdot \xi)_p} \leqslant \Theta_p(\gamma, \lambda).$$

Proof. Let $f = f_1 \dots f_r$ and $j(f,l) = g_1 \dots g_s$ be the decompositions into irreducible factors and let $h = 0$ be the reduced equation of ξ. Then we have

$$\frac{(f,h)_0}{(l,h)_0} \leqslant \max_{j=1}^{s} \left\{ \frac{(f, g_j)_0}{(l, g_j)_0} \right\} \ \text{or}\ \frac{(j(f,l),h)_0}{(l,h)_0} \leqslant \max_{i=1}^{r} \left\{ \frac{(f_i, j(f,l)_0}{(l, f_i)_0} \right\}$$

by [ChP] (Theorem 1.1 and Concluding Remarks). Now we use the definitions of η_p and Θ_p.

3 Polar curves

Let $F : \mathbb{C}^2 \to \mathbb{C}$ be a polynomial of degree $d > 1$ and let C be the projective curve $F^*(X,Y,Z) = 0$. The polar curve $\nabla_q C$ with equation $a\frac{\partial F^*}{\partial X} + b\frac{\partial F^*}{\partial Y} = 0$ will be called generic at infinity if $q = (a:b:0) \in \mathbb{L}\infty \setminus |C|$ that is if $q = (a:b:0)$ and $F^*(a,b,0) \neq 0$. The following is well–known

Lemma 3.1 *Let D be a polar of C, generic at infinity. Suppose that the germ (C,p) is reduced and $(C \cdot \mathbb{L}_\infty)_p > 1$. Then (D,p) is a local polar curve of (C,p) with respect to $(\mathbb{L}_\infty, p)$.*

Let C'_∞ be the set of all $p \in C_\infty = |C| \cap \mathbb{L}_\infty$ such that $(C \cdot \mathbb{L}_\infty)_p > 1$.

Lemma 3.2 *Let D be a polar of C generic at infinity. Then*

(i) $|C| \cap |D| \cap \mathbb{L}_\infty = C'_\infty$,

(ii) *if $\gamma \in \mathcal{B}_\infty$ is a branch of D such that* $\deg_\gamma F \neq 0$ *then* $\deg_\gamma F = \deg_\gamma \nabla F + \deg \gamma$.

Proof.

(i) It suffices to observe that $p = (x_0 : y_0 : 0) \in |C| \cap |D|$ if and only if the linear form $y_0X - x_0Y$ is a multiple factor of $F^*(X, Y, 0)$.

(ii) Let $p(T) = (x(T), y(T))$ be a meromorphic parametrization of γ and let $l(X, Y) = bX - aY$. We get $\operatorname{ord} l(p(T)) = -\deg_\gamma l = \deg \gamma$ for the line $l = 0$ does not intersect the polar D at infinity. From $a\frac{\partial F}{\partial X}(p(T)) + b\frac{\partial F}{\partial Y}(p(T)) = 0$ and $\dot{x}(T)a\frac{\partial F}{\partial X}(p(T)) + \dot{y}(T)b\frac{\partial F}{\partial Y}(p(T)) = \frac{d}{dT}F(p(T))$ we get $\frac{d}{dt}l(T)\frac{\partial F}{\partial X}(p(T)) = -b\frac{d}{dT}F(p(T))$. Taking orders (if $\operatorname{ord}\phi(T) \neq 0$ then $\operatorname{ord}\dot{\phi}(T) = \operatorname{ord}\phi(T) - 1$) we get $\deg_\gamma \frac{\partial F}{\partial X} = \deg_\gamma F - \deg\gamma$ if $b \neq 0$ and $\deg_\gamma \frac{\partial F}{\partial X} = -\infty$ if $b = 0$ and similarly for $\deg_\gamma \frac{\partial F}{\partial Y}$. Hence *(ii)* follows.

Lemma 3.3 *Let D be a polar of C generic at infinity. Let $p \in C_\infty$ and suppose that the germ (C, p) is reduced. Then we have*

(i) If $p \in C'_\infty$ then $\eta_p(C, \mathbb{L}_\infty) = \sup\left\{\frac{\operatorname{ord}_\gamma C}{\operatorname{ord}_\gamma \mathbb{L}_\infty} : \gamma \in \mathcal{B}_{\infty,p} \text{ is a branch of } D\right\}$,

(ii) $\Theta_p(C, \mathbb{L}_\infty) = \sup\left\{\frac{\operatorname{ord}_\gamma D}{\operatorname{ord}_\gamma \mathbb{L}_\infty} : \gamma \in \mathcal{B}_{\infty,p} \text{ is a branch of } C\right\}$.

Proof. Use Property 3.1 and definitions of η_p and Θ_p.

Proposition 3.4 *Let D be a generic polar of C. Let $p \in C_\infty$ be such that (C, p) is reduced. Then for every branch $\gamma \in \mathcal{B}_{\infty,p}$:*

$$\frac{\operatorname{ord}_\gamma C}{\operatorname{ord}_\gamma \mathbb{L}_\infty} \leqslant \eta_p(C, \mathbb{L}_\infty) \text{ or } \frac{\operatorname{ord}_\gamma D}{\operatorname{ord}_\gamma \mathbb{L}_\infty} \leqslant \Theta_p(C, \mathbb{L}_\infty).$$

Proof. Use Proposition 2.6 and Property 3.1.

4 Łojasiewicz exponents and invariants of singularities

We keep the notation introduced above. The following proposition is well–known (see [CaP], pp. 35–37, [Pł1], Corollary 1.3 and [GaP], Proposition 1.4).

Proposition 4.1 *Let $F : \mathbb{C}^2 \to \mathbb{C}$ be a polynomial mapping of degree $d > 1$.*

(i) If $\Lambda_p(F) = \emptyset$ then $\eta_p(C^t, \mathbb{L}_\infty) = \eta_p(C, \mathbb{L}_\infty) < d$,

(ii) if $\Lambda_p(F) \neq \emptyset$ then $\eta(C^t, \mathbb{L}_\infty) = d$ for $t \in \mathbb{C} \setminus \Lambda_p(F)$ and $\eta_p(C^t, \mathbb{L}_\infty) > d$ for $t \in \Lambda_p(F)$ (if C^t is not reduced at p that $\eta_p(C^t, \mathbb{L}_\infty) = +\infty > d$ by convention).

The following theorem is basic for us.

Theorem 4.2 *Let $F : \mathbb{C}^2 \to \mathbb{C}$ be a polynomial mapping of degree $d > 1$ and let $(C^t : t \in \mathbb{C})$ be the pencil of projective curves associated with F. Then*

(i) $\pounds_{p,t}(F) = d - 1 - \Theta_p(C^t, \mathbb{L}_\infty)$ *if* $t \in \mathbb{C} \setminus \Lambda_p(F)$. *Moreover* $\pounds_{p,t}(F)$ *is attained on the fiber* $F = t$,

(ii) $\pounds_{p,t}(F) = d - 1 - \eta_p(C^t, \mathbb{L}_\infty)$ *if* $t \in \Lambda_p(F)$ *and* $\pounds_{p,t}(F)$ *is attained on every polar* $\nabla_q F = 0$, $q \notin C_\infty$.

Proof. We may assume that $\deg_X F = \deg_Y F = d$. Let D_1 be the polar $\frac{\partial F^*}{\partial X} = 0$ and D_2 the polar $\frac{\partial F^*}{\partial Y} = 0$ (note that $\left(\frac{\partial F}{\partial X}\right)^* = \frac{\partial F^*}{\partial X}$ provided that $\deg_X F = d$). Then, by definitions, we get

$$\frac{\deg_\gamma \nabla F}{\deg \gamma} = d - 1 - \inf_i \left\{ \frac{\operatorname{ord}_\gamma D_i}{\operatorname{ord}_\gamma \mathbb{L}_\infty} \right\}, \tag{1}$$

$$\begin{aligned} &\pounds_{p,t}(F) = \\ &= d - 1 - \sup\left\{ \inf_i \left\{ \tfrac{\operatorname{ord}_\gamma D_i}{\operatorname{ord}_\gamma \mathbb{L}_\infty} \right\} : \gamma \in \mathcal{B}_{\infty,p},\ \tfrac{\operatorname{ord}_\gamma C^t}{\operatorname{ord}_\gamma \mathbb{L}_\infty} > d \right\}. \end{aligned} \tag{2}$$

The property "if $\deg_\gamma F \neq 0$ then $\deg_\gamma F \leqslant \deg_\gamma \nabla F + \deg \gamma$" can be reformulated as follows

$$\text{if } \frac{\operatorname{ord}_\gamma C}{\operatorname{ord}_\gamma \mathbb{L}_\infty} \neq d \text{ then } \inf_i \{\operatorname{ord}_\gamma D_i\} \leqslant \operatorname{ord}_\gamma C. \tag{3}$$

Now, let us pass to the proof of the first part of Theorem 4.2. Fix $t \in \mathbb{C} \setminus \Lambda_p(F)$ and let $\gamma \in \mathcal{B}_{\infty,p}$ be a branch such that $\frac{\operatorname{ord}_\gamma C^t}{\operatorname{ord}_\gamma \mathbb{L}_\infty} > d$.

By Proposition 3.4 we get for every polar D generic at infinity

$$\frac{\operatorname{ord}_\gamma C^t}{\operatorname{ord}_\gamma \mathbb{L}_\infty} \leqslant \eta_p(C^t, \mathbb{L}_\infty) \text{ or } \frac{\operatorname{ord}_\gamma D}{\operatorname{ord}_\gamma \mathbb{L}_\infty} \leqslant \Theta_p(C^t, \mathbb{L}_\infty). \tag{4}$$

Since $t \in \mathbb{C} \setminus \Lambda_p(F)$ we have $\eta_p(C^t, \mathbb{L}_\infty) \leqslant d$ by Proposition 4.1. Therefore $\frac{\operatorname{ord}_\gamma C^t}{\operatorname{ord}_\gamma \mathbb{L}_\infty} > d \geqslant \eta_p(C^t, \mathbb{L}_\infty)$ and by (4) we get

$$\frac{\operatorname{ord}_\gamma D}{\operatorname{ord}_\gamma \mathbb{L}_\infty} \leqslant \Theta_p(C^t, \mathbb{L}_\infty) \text{ for every polar } D \text{ generic at infinity.} \tag{5}$$

In particular $\frac{\operatorname{ord}_\gamma D_i}{\operatorname{ord}_\gamma \mathbb{L}_\infty} \leqslant \Theta_p(C^t, \mathbb{L}_\infty)$ for $i = 1, 2$ and consequently, by (2) we get

$$\pounds_{p,t}(F) \geqslant d - 1 - \Theta_p(C^t, \mathbb{L}_\infty). \tag{6}$$

It is easy to check that

$$\operatorname{ord}_\gamma D_1 = \operatorname{ord}_\gamma D_2 \text{ for every branch } \gamma \in \mathcal{B}_{\infty,p} \text{ of } C^t. \tag{7}$$

Thus, for every branch γ of C^t:

$$\frac{\deg_\gamma \nabla F}{\deg \gamma} = d - 1 - \frac{\operatorname{ord}_\gamma D_1}{\operatorname{ord}_\gamma \mathbb{L}_\infty} \quad \text{by (1).} \tag{8}$$

By definition of Θ_p there is a branch $\gamma_0 \in \mathcal{B}_{\infty,p}$ of C^t such that $\frac{\operatorname{ord}_{\gamma_0} D_1}{\operatorname{ord}_{\gamma_0}\mathbb{L}_\infty} = \Theta_p(C^t, \mathbb{L}_\infty)$. Consequently

$$\frac{\deg_{\gamma_0} \nabla F}{\deg \gamma_0} = d - 1 - \Theta_p(C^t, \mathbb{L}_\infty) \tag{9}$$

and Theorem 4.2 *(i)* follows from (6) and (9).

To prove the second part of Theorem 4.2 fix $t \in \Lambda_p(F)$. We may assume that the germ (C^t, p) is reduced. Let γ be a branch such that $\frac{\operatorname{ord}_\gamma C^t}{\operatorname{ord}_\gamma \mathbb{L}_\infty} > d$. By Proposition 3.4 we get

$$\frac{\operatorname{ord}_\gamma C^t}{\operatorname{ord}_\gamma \mathbb{L}_\infty} \leqslant \eta_p(C^t, \mathbb{L}_\infty) \text{ or } \frac{\operatorname{ord}_\gamma D_1}{\operatorname{ord}_\gamma \mathbb{L}_\infty} \leqslant \Theta_p(C^t, \mathbb{L}_\infty). \tag{10}$$

Let us consider two cases

<u>First case</u> $\quad \dfrac{\operatorname{ord}_\gamma C^t}{\operatorname{ord}_\gamma \mathbb{L}_\infty} \leqslant \eta_p(C^t, \mathbb{L}_\infty)$.

Then, by (3) we get

$$\frac{\inf_i\{\operatorname{ord}_\gamma D_i\}}{\operatorname{ord}_\gamma \mathbb{L}_\infty} \leqslant \eta_p(C^t, \mathbb{L}_\infty). \tag{11}$$

<u>Second case</u> $\quad \dfrac{\operatorname{ord}_\gamma D_1}{\operatorname{ord}_\gamma \mathbb{L}_\infty} \leqslant \Theta_p(C^t, \mathbb{L}_\infty)$.

Then $\frac{\inf_i\{\operatorname{ord}_\gamma D_i\}}{\operatorname{ord}_\gamma \mathbb{L}_\infty} \leqslant \Theta_p(C^t, \mathbb{L}_\infty) < \eta_p(C^t, \mathbb{L}_\infty)$ by Proposition 2.4. Consequently, in both cases (11) holds and finally we get

$$\pounds_{p,t}(F) \geqslant d - 1 - \eta_p(C^t, \mathbb{L}_\infty) \tag{12}$$

No, let us fix a polar D generic at infinity. Then there is a branch γ_0 of D such that $\frac{\operatorname{ord}_{\gamma_0} C^t}{\operatorname{ord}_{\gamma_0}\mathbb{L}_\infty} = \eta_p(C^t, \mathbb{L}_\infty)$. Since $t \in \Lambda_p(F)$ then $\eta_p(C^t, \mathbb{L}_\infty) > d$ by Proposition 4.1 and consequently $\frac{\operatorname{ord}_{\gamma_0} C^t}{\operatorname{ord}_{\gamma_0}\mathbb{L}_\infty} > d$. On the other hand using Lemma 3.2 *(ii)* we check that $\inf\{\operatorname{ord}_{\gamma_0} D_i\} = \operatorname{ord}_{\gamma_0} C^t$.

Therefore

$$\pounds_{p,t}(F) \leqslant d - 1 - \frac{\inf\{\operatorname{ord}_{\gamma_0} D_i\}}{\operatorname{ord}_{\gamma_0}\mathbb{L}_\infty} = d - 1 - \frac{\operatorname{ord}_{\gamma_0} C^t}{\operatorname{ord}_{\gamma_0}\mathbb{L}_\infty} = d - 1 - \eta_p(C^t, \mathbb{L}_\infty) \tag{13}$$

and Theorem 4.2 *(ii)* follows from (12) and (13).

Now, we can give

Proof of Theorem 1.2. By Equisingularity at Infinity Property (Preliminaries 0.3) there exists a constant Θ_p^{gen} such that $\Theta_p(C^t, \mathbb{L}_\infty) = \Theta_p^{gen}$ for all $t \in \mathbb{C} \setminus \Lambda_p(F)$. We put $\pounds_p(F) = d - 1 - \Theta_p^{gen}$. For $t \notin \Lambda_p(F)$ we have $\eta_p(C^t, \mathbb{L}_\infty) \leqslant d$ by Proposition 4.1. Using Proposition 2.4 we get

$$\Theta_p^{gen} \leqslant \left(1 - \frac{1}{d_p}\right) d \leqslant d - 1. \tag{14}$$

Therefore $\text{ł}_p(F) = d - 1 - \Theta_p^{gen} \geqslant 0$ and Theorem 1.2 *(i)* follows from Theorem 4.2 *(i)*.

To check the second part of Theorem 1.2 suppose that $\text{ł}_p(F) = 0$. Then from (14) we get $d_p = d$ i.e. p is the only point at infinity of C. Moreover $\Theta_p(C^t, \mathbb{L}_\infty) = \Theta_p^{gen} = d - 1$ and $\eta_p(C^t, \mathbb{L}_\infty) = d$ for $t \in \mathbb{C} \setminus \Lambda_p(F)$. By Proposition 2.5 we get $\operatorname{ord}_p C^t = \deg C^t = d$. therefore C^t and consequently C are pencils of lines through p. This proves Theorem 1.2 *(ii)*.

Fix now $t \in \Lambda_p(F)$. If (C^t, p) is not reduced then $\text{Ł}_{p,t}(F) = -\infty$ and $\eta_p(C^t, \mathbb{L}_\infty) = +\infty$. Thus we may assume that (C^t, p) is reduced. Using Theorem 4.2 *(ii)* and Proposition 4.1 *(ii)* we get $\text{Ł}_{p,t}(F) = d - 1 - \eta_p(C^t, \mathbb{L}_\infty) < -1$. Moreover $\text{Ł}_{p,t}(F)$ is attained on every polar $\nabla_q C = 0$, $q \notin C$ by Theorem 4.2 *(ii)* and Theorem 1.2 *(iii)* follows.

In [CN-H] the authors calculated the total Łojasiewicz exponent $\text{Ł}_\infty(F)$ in terms of the Eisenbud and Neumann diagrams. Here is a reformulation of their result for polynomials F with $\Lambda(F) \neq \emptyset$.

Proposition 4.3 (cf. [CN-H], Proposition 6) *Let $F : \mathbb{C}^2 \to \mathbb{C}$ be a polynomial of degree $d > 1$ such that $\Lambda(F) \neq \emptyset$. Then*

$$\text{Ł}_\infty(F) = d - 1 - \sup\{\, \eta_p(C^t, \mathbb{L}_\infty) : (p, t) \in C'_\infty \times \mathbb{C} \,\}$$

Proof. We use Corollary 1.5 and Theorem 4.2(ii).

Let $\mu(F)$ be the total Milnor number of F defined to be the sum of all Milnor numbers of the curves $F = t$. Then $\mu(F) < +\infty$ if and only if all the curves $F = t$ are reduced.

Proposition 4.4 *Let F be a square–free polynomial of degree $d > 1$ such that the curve $F = 0$ has only one branch at infinity. Then*

$$\text{ł}_\infty(F) = \frac{\mu(F) - 1}{d} + 1.$$

In particular, if F is a component of a polynomial automorphism then $\text{ł}_\infty(F) = 1 - 1/d$.

Proof. Let p be the unique point at infinity of the curve C. Then $\text{ł}_\infty(F) = d - 1 - \Theta_p^{gen}$ by Theorem 4.2(i). On the other hand $\Theta_p^{gen} = (\mu_p^{gen} - 1)/d + 1$ by Remark 2.2(3) and we get $\text{ł}_\infty(F) = d - 2 - (\mu_p^{gen} - 1)/d$. According to the Ephraim–Moh theorem ([E], Theorem 3.4) we have $\Lambda(F) = \emptyset$. Using [CN] Proposition 12 we get $d^2 - 3d + 2 = \mu_p^{gen} + \mu(F)$ and the proposition follows.

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