FACTORIZATION OF THE POLAR CURVE AND THE NEWTON POLYGON

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Abstract

Using the Newton polygon we prove a factorization theorem for the local polar curves. Then we give some applications to the polar invariants and pencils of plane curve singularities.

Introduction

Let $\mathbf{C}\{X, Y\}$ be the ring of complex power series in two variables X, Y. We denote by $\operatorname{ord} f$ and $\operatorname{in} f$ respectively the order and the initial form of a nonzero power series $f \in \mathbf{C}\{X, Y\}$. By definition $\operatorname{ord} 0 = +\infty$ and $\operatorname{in} 0 = 0$. Let f be a nonzero power series without constant term. If $f = f_1^{m_1} \dots f_r^{m_r}$ is a decomposition of f into irreducible pairwise different factors $f_i \in \mathbf{C}\{X, Y\}$ then we put $f_{\text{red}} = f_1 \dots f_r$. Let $t(f) = \operatorname{ord}(\operatorname{in} f)_{\text{red}}$. Then $t(f)$ is the number of tangents to the local curve $f = 0$.

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Write

$$\operatorname{in} f = (\text{a monomial}) \prod_{i=1}^{s} (X - c_i Y)^{m_i}$$

with pairwise different c_i. We put

$$d(f) = \sum_{i=1}^{s} (m_i - 1)$$

and call $d(f)$ *degeneracy* of f. If $s = 0$ then $\operatorname{in} f$ reduces to a monomial and $d(f) = 0$. Note that $d(f) = 0$ if and only if all tangents to $f = 0$ different from the axes are of multiplicity 1.

Assume that f has an isolated singularity at $(0,0) \in \mathbf{C}^2$ (this is equivalent to the conditions $\operatorname{ord} f > 1$ and $f = f_{\text{red}}$) and suppose that the line $bX - aY = 0$ is not tangent to $f = 0$. The generic polar of f is by definition the series $\partial f = a(\partial f/\partial X) + b(\partial f/\partial Y)$.

Let us consider the factorization $\partial f = \prod_{i=1}^{u} h_i$ with irreducible $h_i \in \mathbf{C}\{X, Y\}$ and put $(f, h)_0 = \dim_{\mathbf{C}} \mathbf{C}\{X, Y\}/(f, h)$. According to Teissier [Te1] the quotients $(f, h_i)_0/\operatorname{ord} h_i$ are called polar invariants of the singularity f. The multiplicity m_q of the polar quotient q is defined to be $m_q = \sum_{i \in I_q} \operatorname{ord} h_i$ where $I_q = \{i : (f, h_i)_0/\operatorname{ord} h_i = q\}$.

Teissier's collection $\{(q, m_q)\}$ of polar invariants and their multiplicities is a topological invariant of the singularity (see [Te1] and [Te2]). There are several theorems on the factorization of the polar curve that enable calculation of Teissier's collection (see [M], [D], [G], [LP]). The aim of this note is to study the factorization of the polar curve ∂f associated with the Newton polygon $\mathcal{N}_f$. The main result is a refinement of the factorization theorem given in [LP]. Using our theorem we prove a bound on the number of special values of the pencil $(f - tl^N : t \in \mathbf{C})$. This bound is analogous to the estimation due to Le Van Thanh and Mutsuo Oka (see [LO], Main result) given in the global affine context.

1 Main result

We use the notation introduced in [LP]. In particular we denote by $\mathcal{N}_f = \mathcal{N}(f)$ the set of all segments of the Newton polygon of the series $f = \sum c_{\alpha\beta} X^\alpha Y^\beta \in \mathbf{C}\{X, Y\}$. For every $S \in \mathcal{N}_f$ we denote by $|S|_1$ and $|S|_2$

the lenghts of the projection of S on the horizontal and vertical axes. We call $|S|_1/|S|_2$ *inclination* of the segment S. The power series f is elementary if $\mathcal{N}_f$ contains only one segment with vertices on the axes. Let $||S|| = \min\{|S|_1, |S|_2\}$ and denote a_S, b_S the distances from S to the axes. Thus the vertices of S are $(a_S,\ |S|_2+b_S)$ and $(|S|_1+a_S,\ b_S)$. Let $\alpha/\alpha(S)+\beta/\beta(S)=1$ be the equation of the line containing S. Clearly $\alpha(S)$, $\beta(S)$ are rational numbers and $\alpha(S)/\beta(S) = |S|_1/|S|_2$. A segment $S \in \mathcal{N}_f$ is *exceptional* if $1 = |S|_1 < |S|_2$ and $a_S = 0$ or $1 = |S|_2 < |S|_1$ and $b_S = 0$. A segment $S \in \mathcal{N}_f$ (necessarily unique) is principal if $|S|_1 = |S|_2$. We set $\mathcal{N}_f^* = \mathcal{N}_f \setminus \{\text{exceptional segments}\}$ and $\mathcal{N}_f^{**} = \mathcal{N}_f^* \setminus \{\text{principal segment}\}$. For every segment $S \in \mathcal{N}_f^*$ we define $\varepsilon(S) \in \{-1,0,1\}$ by putting $\varepsilon(S) = -1$ is $|S|_1 < |S|_2$ and $a_S = 0$ or $|S|_2 < |S|_1$ and $b_S = 0$. If $|S|_1 = |S|_2$ then $\varepsilon(S) = 1 -$ (number of verticies of S lying on the axes). We put $\varepsilon(S) = 0$ for all remaining cases. A segment $S \in \mathcal{N}_f^{**}$ is of the first kind if $\varepsilon(S) = 0$, it is of the second kind if $\varepsilon(S) = -1$.

Let $\operatorname{in}(f,S) = \sum_{(\alpha,\beta)\in S} X^\alpha Y^\beta$. Clearly $X^{a_S}Y^{b_S}$ is the monomial of the highest degree dividing $\operatorname{in}(f,S)$. Thus we can write $\operatorname{in}(f,S) = X^{a_S}Y^{b_S}\widehat{\operatorname{in}(f,S)}$ in $\mathbf{C}\{X,Y\}$. Note that $\mathcal{N}(\widehat{\operatorname{in}(f,S)}) = \{S'\}$ where S' is the segment with vertices $(|S|_1, 0)$ and $(0, |S|_2)$. We define the degeneracy $d(f,S)$ of f on S by putting $d(f,S) = \operatorname{ord}\widehat{\operatorname{in}(f,S)} - \operatorname{ord}\widehat{\operatorname{in}(f,S)}_{\text{red}}$. Note that $d(f,S) = 0$ if and only if f is nondegenerate on S that is if the polynomial $\operatorname{in}(f,S)$ has no critical points in the set $(\mathbf{C}\setminus\{0\})\times(\mathbf{C}\setminus\{0\})$. If $S \in \mathcal{N}_f^{**}$ is of the second kind then we let $v_S = X$ if $|S|_1 < |S|_2$ and $v_S = Y$ if $|S|_2 < |S|_1$. Let S be a segment of a Newton polygon. We call a power series S-elementary if it is elementary and its unique segment is parallel to S. Let us state the main result

Theorem 1.1 *Let $f = f(X,Y) \in \mathbf{C}\{X,Y\}$ be a power series with an isolated singularity at $(0,0) \in \mathbf{C}^2$. Then for every line $bX - aY = 0$ not tangent to the curve $f = 0$ there is a factorization of the polar $\partial f = a(\partial f/\partial X) + b(\partial f/\partial Y)$:*

$$\partial f = A\,B \prod_{S\in\mathcal{N}_f^{**}} A_S B_S \quad \textit{in } \mathbf{C}\{X,Y\}$$

such that

(i) $\operatorname{ord} A = t(f) - 1$, $\operatorname{ord} B = d(f)$. *If h is an irreducible factor of $A\,B$ then $(f,h)_0/\operatorname{ord} h \geq \operatorname{ord} f$ with equality if and only if h divides A.*

(ii) $\operatorname{ord} A_S = ||S|| + \varepsilon(S) - d(f,S)$, $\operatorname{ord} B_S = d(f,S)$. *If h is an irreducible factor of $A_S B_S$ then $(f,h)_0/\operatorname{ord} h \geq \max(\alpha(S),\beta(S))$ with equality if and only if h divides A_S.*

(iii) *If $\operatorname{ord} B_S > 0$ then B_S is S-elementary. If $\operatorname{ord} A_S > 0$ and S is of the first kind then A_S is S-elementary.*

(iv) *If $\operatorname{ord} A_S > 0$ and S is of the second kind then there is a factorization $A_S = A'_S A''_S$ such that*

- · *if $\operatorname{ord} A'_S > 0$ then A'_S is S-elementary.*
- ·· *If $\operatorname{ord} A''_S > 0$ then every barier of the Newton diagram of f parallel to a segment of $\mathcal{N}(A''_S)$ passes through the vertex of S lying on the axis $v_S = 0$. If $|S|_1 < |S|_2$ (resp. $|S|_2 < |S|_1$) then for every $T \in \mathcal{N}(A''_S)$: $|T|_1/|T|_2 < |S|_1/|S|_2$ (resp. $|S|_1/|S|_2 < |T|_1/|T|_2$).*

The proof of Theorem 1.1 we give in Section 6 of this note.

Corollary 1.2 *Let $f = f(X,Y) \in \mathbf{C}\{X,Y\}$ be a power series with an isolated singularity at $(0,0) \in \mathbf{C}^2$ such that $\mathcal{N}_f^{**} \neq \emptyset$. Then*

(i) *Let $S \in \mathcal{N}_f^{**}$ be of the first kind. Then $\max(\alpha(S),\beta(S))$ is a polar invariant of the curve $f = 0$. Its multiplicity is at least $||S|| - d(f,S)$.*

(ii) *If $S \in \mathcal{N}_f^{**}$ is of the second kind then $\max(\alpha(S),\beta(S))$ is a polar invariant of f if and only if $\operatorname{ord} \operatorname{in}\widehat{(f,S)}_{\mathrm{red}} > 1$. Its multiplicity is at least $||S|| - d(f,S) - 1$. If $\operatorname{ord} \operatorname{in}\widehat{(f,S)}_{\mathrm{red}} = 1$ then there is a polar invariant strictly greater than $\max(\alpha(S),\beta(S))$.*

Proof. Fix a segment $S \in \mathcal{N}_f^{**}$. It is easy to check that $\operatorname{ord} A_S = ||S|| + \varepsilon(S) - d(f,S) = \operatorname{ord} \operatorname{in}\widehat{(f,S)}_{\mathrm{red}} + \varepsilon(S)$. If S is of the first kind then $\varepsilon(S) = 0$ and (i) follows. If S is of the second kind then $\max(\alpha(S),\beta(S))$ is a polar invariant if and only if $\operatorname{ord} A_S = \operatorname{ord} \operatorname{in}\widehat{(f,S)}_{\mathrm{red}} - 1 > 0$.

Example 1.3 Let $f(X,Y) = (Y - X^2)^2 + X^5$. Then $\mathcal{N}_f = \{S\}$ where S is the segment with vertices $(0,2)$ and $(4,0)$. Clearly $\mathrm{in}(\widehat{f,S})_{\mathrm{red}} = Y - X^2$ is of order 1. According to Corollary 1.2 we can only say that the curve $f = 0$ has a polar invariant greater than $\max(\alpha(S), \beta(S)) = \max(2,4) = 4$. Taking the new system of coordinates $X_1 = X$, $Y_1 = Y - X^2$ we get $f_1(X_1, Y_1) = Y_1^2 + X_1^5$ and using 1.2 to f_1 in coordinates (X_1, Y_1) we get that there is a unique polar invariant equal to 5.

Example 1.4 Let $f(X,Y) = Y^{11} + XY^8 - 2X^2Y^6 + X^3Y^4 - 2X^4Y^3 + X^5Y^2 - 2X^7Y + X^9$. Then $\mathcal{N}_f = \{E, S, U, T\}$ where $|E|_1/|E|_2 < |S|_1/|S|_2 < |U|_1/|U|_2 < |T|_1/|T|_2$. Here E is exceptional, U is principal and $\mathcal{N}_f^{**} = \{S, T\}$ where S is of the first kind ($\varepsilon(S) = 0$) and T is of the second kind ($\varepsilon(T) = -1$). According to Theorem 1.1 there is a factorization $\partial f = ABA_SB_SA_TB_T$ in $\mathbf{C}\{X,Y\}$ where $\mathrm{ord}\, A = t(f) - 1 = 2$, $\mathrm{ord}\, B = d(f) = 1$, $\mathrm{ord}\, A_S = ||S|| - d(f,S) = 1$, $\mathrm{ord}\, B_S = d(f,S) = 1$, $\mathrm{ord}\, A_T = ||T|| - 1 - d(f,T) = 0$, $\mathrm{ord}\, B_T = d(f,T) = 1$. We may assume that $A_T = 1$ in $\mathbf{C}\{X,Y\}$ for A_T is a unit. The polar $\partial f = 0$ consists of the curve $A = 0$ of order 2 transverse to the curve $f = 0$ and of four nonsingular branches $A_S = 0$, $B = 0$, $B_S = 0$ and $B_T = 0$. The polar invariants are $\mathrm{ord}\, f = 7$ (of multiplicity 2), $(f, A_S)_0 = \max(\alpha(S), \beta(S)) = 10$ and the numbers $(f,B)_0 > 7$, $(f, B_S)_0 > 10$ and $(f, B_T) > \max(\alpha(T), \beta(T)) = 9$. The theorem does not give information if the invariants $(f,B)_0$, $(f,B_S)_0$ and $(f,B_T)_0$ are equal or not.

2 Contact exponent and minimal polar invariant

Let $f = f_1 \ldots f_r$ be an isolated singularity with branches $f_i = 0$ and $l = 0$ be a smooth curve (that is l is a series of order 1). Then we consider the contact exponent of $l = 0$ with $f = 0$

$$\delta(f,l) = \min_{i=1}^{r} \left\{ \frac{(f_i, l)_0}{\mathrm{ord}\, f_i} \right\}$$

and the contact exponent of $f=0$:

$$\delta(f)=\max\left\{\delta(f,l):\begin{array}{l}l=0 \text{ runs over the set of nonsingular}\\ \text{curves different from the branches } f_i=0\end{array}\right\}$$

(see [BK] for Hironaka's theory of maximal contact).

Clearly $\delta(f)\geq 1$ and $\delta(f)=1$ if and only if $t(f)>1$.

Theorem 2.1 *Let $f=f(X,Y)\in \mathbf{C}\{X,Y\}$ be a power series with an isolated singularity at $(0,0)\in\mathbf{C}^2$. Then*

(i) *if $t(f)>1$ then the minimal polar invariant of $f=0$ is equal to* $\operatorname{ord} f$ *and its multiplicity is $t(f)-1$.*

(ii) *Suppose that $t(f)=1$ and $\delta(f,Y)=\delta(f)$. Let F be the first segment of the Newton polygon $\mathcal{N}_f$. Then the minimal polar invariant of $f=0$ is equal to $\alpha(F)$ and its multiplicity is $||F||-\varepsilon(F)-d(F,f)$.*

(iii) *The minimal polar invariant of the singularity $f=0$ is equal to* $(\operatorname{ord} f)\delta(f)$.

Proof. Part (i) of the theorem follows from Theorem 1.1 (i). To check (ii) observe that from the assumptions it follows that the axis $X=0$ is transverse to the curve $f=0$. The Newton diagram of f has the vertex $(0,\operatorname{ord} f)$ and lies strictly above the line $\alpha+\beta=\operatorname{ord} f$. Hence all segments of $\mathcal{N}_f$ have the inclination strictly greater than 1. In particular $|F|_1>|F|_2$. Recall that $|F|_1/|F|_2=\delta(f,Y)=\delta(f)$ and consider two cases.

Case 1. The power series f is not elementary. Then the segment F is of the first kind and $\alpha(F)$ is a polar invariant of $f=0$. Using Theorem 1.1 we check that all polar invariants of f different from $\alpha(F)$ are strictly greater than $\alpha(F)$. Thus $\alpha(F)$ is the minimal polar invariant of f and its multiplicity equals $||F||+\varepsilon(F)-d(f,F)$.

Case 2. The power series f is elementary. Then F is the unique segment of $\mathcal{N}_f$. Using the criterion of maximal contact (see [BK], Lemma 5, p. 649) we get that $\operatorname{in}(f,F)$ is not of the form $(bY-aX^k)^m$, $ab\neq 0$. Therefore by our main result $\alpha(F)=\max(\alpha(F),\beta(F))$ is the minimal polar invariant of f and its multiplicity is $||F||+\varepsilon(F)-d(f,F)$.

To check (iii) we note that $\alpha(F)/\operatorname{ord} f=\alpha(F)/\beta(F)=|F|_1/|F|_2=\delta(f)$ and use (ii).

Example 2.2 Suppose f is an irreducible power series with characteristic $\beta_0, \beta_1, \ldots, \beta_g$ (see for example [M]). If the axis $Y = 0$ has the maximal contact with $f = 0$ then $\mathcal{N}_f = \{F\}$ where F is the segment with vertices $(0, \beta_0)$ and $(\beta_1, 0)$. Let $e_1 = \mathrm{GCD}(\beta_0, \beta_1)$. Then $\mathrm{in}(f, S) = (bX^{\beta_1/e_1} - cY^{\beta_0/e_1})^{e_1}$ with $bc \neq 0$ and an easy calculation shows that the minimal polar invariant equals $\max(\beta_0, \beta_1) = \beta_1$ and is of multiplicity $\beta_1/e_1 - 1$ (here $\varepsilon(F) = -1$). Thus we have got the first of Merle's formulas [M].

The reasoning like that in the proof of Theorem 2.1 shows

Theorem 2.3 *Suppose that $f = 0$ has exactly one polar invariant. If $\delta(f, Y) = \delta(f)$ and $(f, X)_0 = \mathrm{ord}\, f$ then $\mathcal{N}_f = \{F\}$ and f is nondegenerate on F. The segment F has vertices $(0, n)$ and $(m, 0)$ or $(0, n)$ and $(m, 1)$ with $m \geq n$.*

Corollary 2.4 ([E], p. 16) *If $f = 0$ has exactly one polar invariant then $f = 0$ is topologically equivalent to a plane curve singularity of type $Y^n - X^m = 0$ or of the type $Y^n - YX^m = 0$.*

3 Special values of plane curve pencils

When studying the singularities at infinity of a polynomial in two complex variables of degree $N > 0$ one considers the pencils of plane curves of the form $f_t = f - tl^N, t \in \mathbf{C}$ where f and $l = bX - aY$ are coprime (such pencils are called in [C] Iomdin Lê deformations). Let $\mu_0(f) = (\partial f/\partial X, \partial f/\partial Y)_0$ be the Milnor number of the local curve $f = 0$. Recall that $\mu_0(f) = +\infty$ if and only if f has a multiple factor. The number $t_0 \in \mathbf{C}$ is a special value of the pencil $(f_t, t \in \mathbf{C})$ if $\mu_0(f_{t_0}) > inf\{\mu_0(f_t) : t \in \mathbf{C}\}$. The set of special values is finite. Using our main result we will give a bound on the number of special values in terms of the Newton diagram of the series. Let $r(f, S)$ be the number of irreducible factors of $\mathrm{in}\widehat{(f, S)}$ and put $r(S) = \mathrm{GCD}(|S|_1, |S|_2)$.

Lemma 3.1 *One has $r(S) - r(f, S) = (r(S)/||S||)d(f, S)$. In particular $r(f, S) \leq r(S)$ with equality if and only if f is nondegenerate on S.*

Proof. Write

$$\mathrm{in}\widehat{(f, S)} = \prod_{i=1}^{r} (b_i X^{|S|_1/r(S)} - a_i Y^{|S|_2/r(S)})^{m_i}$$

with pairwise linearly independent $(a_i, b_i) \in \mathbf{C}$. Then $r(f,S) = r$ and $r(S) = \sum_{i=1}^{r} m_i$. Now

$$\begin{aligned} d(f,S) &= \operatorname{ord} \operatorname{in}\widehat{(f,S)} - \operatorname{ord} \operatorname{in}\widehat{(f,S)}_{\mathrm{red}} \\ &= \sum_{i=1}^{r} m_i \frac{||S||}{r(S)} - \sum_{i=1}^{r} \frac{||S||}{r(S)} = \frac{||S||}{r(S)}(r(S) - r(f,S)) \end{aligned}$$

and the lemma follows.

The following result is a local counterpart of the Le Van Thanh and Oka theorem giving an estimation for the number of critical values at infinity (see [LO], Main Theorem).

Let $q(S) = \max(\alpha(S), \beta(S))$ for any $S \in \mathcal{N}_f^*$. We put $l = bX - aY$ and suppose that the line $l = 0$ is not tangent to $f = 0$.

Theorem 3.2 *Let $N \neq \operatorname{ord} f$ be a strictly positive integer. The number of the nonzero special values of the pencil $(f - tl^N : t \in \mathbf{C})$ is less than or equal to*

$$\sum_{S:\, q(S)<N} (r(S) - r(f,S)) + \sum_{S:\, q(S)=N} r(f,S) \ .$$

We use the convention that a sum over the empty set equals zero. If f is a nondegenerate power series then the sum above reduces to

$$\sum_{S:\, q(S)=N} r(S) \ .$$

Note also the bound for all series with the given Newton polygon.

Corollary 3.3 *The number of the nonzero special values of $(f - tl^N : t \in \mathbf{C})$ $(N \neq \operatorname{ord} f)$ is less than or equal to*

$$\sum_{S:\, q(S)\leq N} r(S) \ .$$

In connection with the above corollary recall the following well-known fact: the number of branches of the curve $f = 0$ different from the axes is less or equal to $\sum_S r(S)$ (with equality for nondegenerate curves).

To get Theorem 3.2 from the main result we need a few lemmas. The lemma below is a local version of the description of critical values at infinity given in [LO] (pp. 410–411). Let f/h be a meromorphic fraction with coprime

$f, h \in \mathbf{C}\{X, Y\}$ and let $p = p(X, Y) \in \mathbf{C}\{X, Y\}$ be irreducible power series such that p does not divide h. Let $(x(u), y(u)) \in \mathbf{C}\{u\}^2$, $(x(0), y(0)) = (0, 0)$ be a parametrization of the branch $p = 0$. Then we put

$$\left(\frac{f}{h}\right)(p) = \left.\frac{f(x(u), y(u))}{l(x(u), y(u))}\right|_{u=0} \in \mathbf{C} \cup \{\infty\}.$$

Lemma 3.4 *The set of nonzero special values of the pencil* $(f - tl^N : t \in \mathbf{C})$ *is equal to the set*

$$\{(f/l^N)(p) : \ p \textit{ is irreducible factor of } j(f, l) \textit{ such that } (f, p)_0/(l, p)_0 = N\}\ .$$

Proof. See [MM] Théorème 1 or [GB-P] Proposition 2.2.

Let $r_0(\phi)$ be the number of irreducible factors of the series ϕ.

Lemma 3.5 *Suppose that* ϕ *is* S*-elementary. Then*

$$r_0(\phi) \leq (\operatorname{ord} \phi) \frac{r(S)}{||S||}\ .$$

Proof. Let $r = r_0(\phi)$. Then $\phi = \prod_{i=1}^{r} \phi_i$ with irreducible ϕ_i. The power series ϕ_i are S-elementary. Therefore the unique segment of $\mathcal{N}(\phi_i)$ joins the points $(k_i|S|_1/r(S), 0)$ and $(0, k_i|S|_2/r(S))$ for an integer $k_i \geq 1$. Consequently

$$\operatorname{ord} \phi_i = \min\left(\frac{|S|_1}{r(S)} k_i, \frac{|S|_2}{r(S)} k_i\right) \geq \frac{||S||}{r(S)}$$

for all $i = 1, \ldots, r$. We get

$$\operatorname{ord} \phi = \sum_{i=1}^{r} \operatorname{ord} \phi_i \geq \frac{||S||}{r(S)} r_0(\phi)$$

and the lemma follows.

Lemma 3.6 *Let us keep the notation from Theorem 1.1. Then*

(i) *If* $\operatorname{ord} B > 0$ *then* $\mathcal{N}_f$ *has the principial segment* U *and* $\operatorname{ord} B = r(U) - r(f, U)$.

(ii) *If* $\operatorname{ord} B_S > 0$ *then* $r_0(B_S) \leq r(S) - r(f, S)$.

(iii) *If* $\operatorname{ord} A_S > 0$ *and* S *is of the first kind then* $r_0(A_S) \leq r(f, S)$.

Proof.

(i) It is easy to see that if $d(f) > 0$ then $\mathcal{N}_f$ has the principal segment U and $d(f) = r(U) - r(f, U)$. Use Theorem 1.1 (i).

(ii) Suppose that $\mathrm{ord}\, B_S > 0$. Then by Theorem 1.1 (ii) we get $\mathrm{ord}\, B_S = d(f, S)$. Now Lemmas 3.5 and 3.1 give

$$r_0(B_S) \leq (\mathrm{ord}\, B_S)\frac{r(S)}{||S||} = d(f, S)\frac{r(S)}{||S||} = r(S) - r(f, S)\ .$$

(iii) Suppose that $\mathrm{ord}\, A_S > 0$ and S is of the first kind. Then $\mathrm{ord} A_S = ||S|| - d(f, S)$ by Theorem 1.1 (ii) and using Lemmas 3.5 and 3.1 we get

$$r_0(A_S) \leq (\mathrm{ord}\, A_S)\frac{r(S)}{||S||} = (||S|| - d(f, S))\frac{r(S)}{||S||} = r(f, S)\ .$$

Lemma 3.7 *Suppose that* $\mathrm{ord}\, A_S > 0$ *for a segment* $S \in \mathcal{N}_f^{**}$ *of the second kind. Let* $N = \max(\alpha(S), \beta(S))$. *Let* $A_S = A'_S A''_S$ *be the factorization of* A_S *such that in Theorem 1.1 (iv). Then*

(i) $r_0(A'_S) \leq r(f, S) - 1$,

(ii) *for every prime factor* p *of* A''_S: $(f/l^N)(p) = (f/l^N)(v_S)$.

Proof. By Theorem 1.1 (ii) we get $\mathrm{ord}\, A_S = ||S|| - 1 - d(f, S)$ ($\varepsilon(S) = -1$ for the segments of second kind) and consequently, like in the proof of Lemma 3.7 we obtain

$$r_0(A'_S) \leq (\mathrm{ord}\, A'_S)\frac{r(S)}{||S||} \leq (\mathrm{ord}\, A_S)\frac{r(S)}{||S||} = r(f, S) - \frac{r(S)}{||S||} < r(f, S)\ .$$

Since $r_0(A'_S)$ and $r(f, S)$ are integers we get $r_0(A'_S) \leq r(f, S) - 1$. To prove the second part of Lemma 3.7 assume that $S = F$ is the first segment of $\mathcal{N}_f$ (if $S = L$ is the last segment the proof is similar). Then $v_S = v_F = X$. Let p be a prime factor of A''_F. We may assume that the branch $p = 0$ is different from the axis $X = 0$. Note that $\frac{|F_1|}{|F_2|} < 1$ and $N = \beta(F)$. Let $(x(u), y(u))$ be the injective parametrization of the branch $p = 0$. Put $m = \mathrm{ord}\, x(u)$ and $n = \mathrm{ord}\, y(u)$. The series p is elementary, the unique segment of $\mathcal{N}_p$ joins the points $(n,0)$ and $(0,m)$ and is of inclination $n/m \leq |F|_1/|F|_2 < 1$ by Theorem 1.1 (iv). The line supporting the Newton diagram of f of slope $-m/n$ passes through the point $(0,\beta(F)) = (0,N)$ and, consequently, has the equation $m\alpha + n\beta = nN$. It intersects the Newton diagram of f exactly at point $(0,N)$. Therefore

$$f(X,Y) = c_{0N}Y^N + \sum_{m\alpha+n\beta>nN} c_{\alpha\beta}X^\alpha Y^\beta \quad \text{with } c_{0N} \neq 0\ .$$

The line $l(X,Y) = bX - aY$ is not tangent to $f = 0$. Then $a \neq 0$ and

$$\begin{aligned} f(x(u),y(u)) &= c_{0N}y(u)^N + \text{ terms of order } > nN \\ l(x(u),y(u)) &= (-a)^N y(u)^N + \text{ terms of order } > nN \end{aligned}$$

Consequently

$$\left(\frac{f}{l^N}\right)(p) = \frac{c_{0N}}{(-a)^N} = \left(\frac{f}{l^N}\right)(X)\ .$$

Now we give the proof of Theorem 3.2.

Let

$$\partial f = AB \prod_{S \in \mathcal{N}_f^{**}} A_S B_S$$

be a factorization of ∂f such that in Theorem 1.1.

According to Lemmas 3.4 and 3.7 the number of nonzero special values of $(f - tl^N : t \in \mathbf{C})$ is equal to

$$\begin{aligned} &\#\{(f/l^N)(p) :\ p \text{ is a prime factor of } \partial f \text{ and } (f,p)_0/\operatorname{ord} p = N\} \\ &\leq \operatorname{ord} B + \sum_{S:\, q(S)<N} r_0(B_S) + \sum_{S:\, q(S)=N}^{I} r_0(A_S) + \sum_{S:\, q(S)=N}^{II} (r_0(A'_S) + 1) \end{aligned}$$

where the symbols $\sum^I$ resp ($\sum^{II}$) mean that the summation is carried over the segments of the first kind (of the second kind). The theorem follows from Lemmas 3.6 and 3.7.

Remark 3.8 *An obvious modification of the above proof shows that the pencil* $(f - tl^{\operatorname{ord} f} :\ t \in \mathbf{C})$ *has at most* $t(f) - 1$ *nonzero special values.*

4 Preliminary lemmas

Let $\varphi = \varphi(X,Y) \in \mathbf{C}\{X,Y\}$ be a nonzero power series without constant term and $\lambda = bX - aY$ be a linear form. Put $\partial\varphi = a(\partial\varphi/\partial X) + b(\partial\varphi/\partial Y)$ (we do not assume that $\mathrm{in}\varphi(a,b) \neq 0$!) and note that $\mathrm{ord}\,\partial\varphi = \mathrm{ord}\,\varphi - 1$ if and only if $\mathrm{in}\,\varphi \neq \mathrm{const}.\lambda^{\mathrm{ord}\,\varphi}$.

Moreover $\partial\varphi = 0$ in $\mathbf{C}\{X,Y\}$ if and only if $\varphi = \varphi^0(\lambda)$ where φ^0 is a power series of one variable. In particular if $\partial\varphi = 0$ then λ divides φ.

Lemma 4.1 *Suppose that* $\mathrm{ord}\,\partial\varphi = \mathrm{ord}\,\varphi - 1$ *and let* $k \geq 0$ *be the greatest integer such that* λ^k *divides* φ. *Let* $\varphi = \lambda^k \varphi_1^{m_1} \dots \varphi_s^{m_s}$ *with irreducible and pairwise coprime* $\varphi_i \in \mathbf{C}\{X,Y\}$. *Then* $\partial\varphi = \lambda^k \varphi_0 \varphi_1^{m_1-1} \dots \varphi_s^{m_s-1}$ *in* $\mathbf{C}\{X,Y\}$ *and* φ, φ_0 *are coprime.*

Proof. It is easy to check that $\lambda^k, \varphi_1^{m_1-1}, \dots, \varphi_s^{m_s-1}$ divide $\partial\varphi$ and are pairwise coprime. Thus we can write $\partial\varphi = \lambda^k \varphi_0 \varphi_1^{m_1-1} \dots \varphi_s^{m_s-1}$ in $\mathbf{C}\{X,Y\}$. If φ_i were a factor of φ_0 then we would get that φ_i didides $\partial\varphi_i$. This implies $\partial\varphi_i = 0$. Impossible because λ does not divide φ_i. Similarly we check that λ does not divide φ_0.

Remark
It is easy to check that $\mathrm{ord}\,\varphi_0 = \sum_{i=1}^{s} \mathrm{ord}\,\varphi_i - 1$.

The following is well-known

Lemma 4.2 *If* $\mathrm{in}\, f = \varphi_1^{m_1} \dots \varphi_t^{m_t}$ *with* φ_i *linear pairwise linearly independent then* $f = f_1 \dots f_t$ *in* $\mathbf{C}\{X,Y\}$ *and* $\mathrm{in}\, f_i = \varphi_i^{m_i}$ *for* $i = 1, \dots, t$.

Lemma 4.3 *Let* $\partial f = a(\partial f/\partial X) + b(\partial f/\partial Y)$ *with* $(a,b) \in \mathbf{C}^2$ *such that* $\mathrm{in}\, f(a,b) \neq 0$. *Then* $\partial f = A\tilde{A}$ *in* $\mathbf{C}\{X,Y\}$ *where* $\mathrm{ord}\, A = t(f) - 1$, $\mathrm{ord}\,\tilde{A} = \mathrm{ord}\, f - t(f)$ *and for every irreducible factor* h *of* ∂f: $(f,h)_0/\mathrm{ord}\, h = \mathrm{ord}\, f$ *if and only if* h *divides* A.

Proof. Let $\mathrm{in}\, f = \varphi_1^{m_1} \dots \varphi_t^{m_t}$, φ_i linear and $t = t(f)$. Then $\mathrm{in}(\partial f) = \partial(\mathrm{in}\, f) = \varphi_0 \varphi_1^{m_1-1} \dots \varphi_t^{m_t-1}$ in $\mathbf{C}\{X,Y\}$ with coprime φ_0, $\mathrm{in}\, f$. By Lemma 4.2 we get the factorization $\partial f = g_0 g_1 \dots g_t$ where $\mathrm{in}\, g_0 = \varphi_0$ and $\mathrm{in}\, g_i = \varphi_i^{m_i-1}$ for $i = 1, \dots, t$. By Remark to Lemma 4.1 we get

$$\mathrm{ord}\, g_0 = \sum_{i=1}^{t} \mathrm{ord}\,\varphi_i - 1 = t - 1 = t(f) - 1 \,.$$

Put $A = g_0$, $\tilde{A} = g_1 \ldots g_t$.

5 Newton polygon and factorization of polar series

Let us keep the notation introduced above. The following two lemmas are well-known.

Lemma 5.1 *Let $f = f(X, Y)$ be a nonzero power series without constant term. Then there is a factorization*

$$f = uX^{d_1}Y^{d_2} \prod_{S \in \mathcal{N}_f} f_S \quad \textit{in } \mathbf{C}\{X, Y\}$$

where u is a unit, such that

(i) $\mathcal{N}(f_S) = \{S'\}$ *where S' is the segment with vertices* $(|S|_1, 0)$ *and* $(0, |S|_2)$,

(ii) $\mathrm{in}(f_S, S') = \mathrm{const.in}(\widehat{f, S})$

Lemma 5.2 *Suppose that $\mathcal{N}(f) = \{S\}$ where S is a segment with vertices on the axes. Suppose that* $\mathrm{in}(f, S) = \psi_1 \ldots \psi_m$ *with coprime ψ_i. Then there is a factorization*

$$f = f_1 \ldots f_m$$

such that

(i) $\mathcal{N}(f_i) = \{S^{(i)}\}$ *where $S^{(i)}$ is a segment parallel to S,*

(ii) $\mathrm{in}(f_i, S^{(i)}) = \psi_i$ *for $i = 1, \ldots, m$.*

6 Proof of the main result

We will prove our theorem for polars $\partial f = a(\partial f/\partial X) + b(\partial f/\partial Y)$ such that $ab \,\mathrm{in}\, f(a, b) \neq 0$. If $a = 0$ or $b = 0$ but $\mathrm{in}\, f(a, b) \neq 0$ then the proof needs some modifications (see [LP],p. 318). By Lemma 5.1 we may write

(1) $\partial f = uX^{\delta_1}Y^{\delta_2} \prod_{T \in \mathcal{N}(\partial f)} (\partial f)_T$ in $\mathbf{C}\{X, Y\}$ where u is a unit and

(2) $(\partial f)_T$ is an elementary power series; $\mathcal{N}((\partial f)_T) = \{T'\}$ where T' is the segment with vertices $(|T|_1, 0)$ and $(0, |T|_2)$,

(3) $\mathrm{in}((\partial f)_T, T') = \mathrm{const.}\, \mathrm{in}(\widehat{\partial f}, T)$.

The proposition below is already proved in [LP] but it is not stated there explicitly.

Proposition 6.1 *Suppose that $f \in \mathbf{C}\{X, Y\}$ has an isolated singularity at $(0,0) \in \mathbf{C}^2$. Then there is a factorization*

$$\partial f = g \prod_{S \in \mathcal{N}_f^{**}} g_S \ \textit{in}\ \mathbf{C}\{X, Y\}$$

such that

(i) $\mathrm{ord}\, g = d(f) + t(f) - 1$, $\mathrm{ord}\, g_S = ||S|| + \varepsilon(S)$ *for* $S \in \mathcal{N}_f^{**}$,

(ii) *If $S \in \mathcal{N}_f^{**}$ is a segment of the first kind then there is a segment $T \in \mathcal{N}_{\partial f}$ (necessarily unique) parallel to S. We have $g_S = (\partial f)_T$.*

(iii) *Suppose that $S \in \mathcal{N}_f^{**}$ is a segment of the second kind. Then*

  (α) *for every $T \in \mathcal{N}(g_S)$ the power series $(\partial f)_T$ divides g_S.*

  (β) *If $|S|_1 < |S|_2$ (resp. $|S|_2 < |S|_1$) then for every $T \in \mathcal{N}(g_S)$: $|T|_1/|T|_2 \leq |S|_1/|S|_2$ (resp. $|S|_1/|S|_2 \leq |T|_1/|T|_2$).*

  (γ) *Every barier of the Newton diagram of f parallel to a segment of $\mathcal{N}(g_S)$ passes through the vertex of S lying on the axis $v_S = 0$.*

  (δ) *If there is no segment of $\mathcal{N}(g_S)$ parallel to a segment of $\mathcal{N}_f^{**}$ then $d(f, S) = 0$.*

Proof. We may assume that $\mathcal{N}_f^{**} \neq \emptyset$. According to [LP], Theorem 1.1 p. 310 there is a factorization

(4) $\partial f = v \prod_{S \in \mathcal{N}_f^{**}} g_S$ in $\mathbf{C}\{X, Y\}$
such that

(5) $\mathrm{ord}\, g_S = ||S|| + \varepsilon(S)$ for every $S \in \mathcal{N}_f^{*}$,

(6) if $\mathrm{ord}\, v > 0$ then $\mathrm{ord}\, v = 1$ and $(f, v)_0 = \mathrm{ord}\, f$.

Moreover, by the definition of v given in [LP], p. 317 we have

(7) $\mathrm{ord} v > 0$ if and only if $\mathrm{in} f = \mathrm{const.}\, X^{\alpha_0} Y^{\beta_0}$ for some $\alpha_0, \beta_0 > 0$.

To define the power series g we consider two cases.

Case 1. The initial form $\mathrm{in} f$ is not a monomial. Then there exists the principal segment $U \in \mathcal{N}_f^*$ and $\mathrm{in} f = \mathrm{in}(f, U)$. It is easy to see that $\mathrm{ord}\,\mathrm{in}(\widehat{f,U})_{red} = t(f) - 1 - \varepsilon(U)$. Consequently $d(f) = d(f, U) = ||U|| - \mathrm{ord}\,\mathrm{in}(\widehat{f,U})_{red} = ||U|| + \varepsilon(U) - (t(f) - 1)$ and $\mathrm{ord}\, g_U = ||U|| + \varepsilon(U) = d(f) + t(f) - 1$ by (5). Put $g = v g_U$. Note that v is a unit by (7) hence $\mathrm{ord}\, g = \mathrm{ord}\, g_U = d(f) + t(f) - 1$.

Case 2. The initial form $\mathrm{in} f$ is a monomial. If $\mathrm{in}\, f = \mathrm{const.} X^{\mathrm{ord} f}$ or $\mathrm{in}\, f = \mathrm{const.}\, Y^{\mathrm{ord}\, f}$ then $d(f) + t(f) - 1 = 0$. We put $g = v$. By (7) v is a unit and consequently $\mathrm{ord}\, g = 0$. If $\mathrm{in}\, f = \mathrm{const.}\, X^{\alpha_0} Y^{\beta_0}$ with $\alpha_0 > 0$ and $\beta_0 > 0$ then $d(f) + t(f) - 1 = 1$. We put $g = v$. By (6) and (7) we get $\mathrm{ord}\, g = 1$.

By the definition of the series g we can rewrite (4) in the form

(8) $\partial f = g \prod_{S \in \mathcal{N}_f^{**}} g_S$, $\mathrm{ord} g = d(f) + t(f) - 1$

The conditions (ii) and (iii) (α), (β), (γ) follow immediately from the definition of g_S given in [LP] p. 317. To check (iii) (δ) we observe that by [LP], Lemma 2.3 the initial form $\mathrm{in}(f, S)$ is the sum of two monomials. Thus $d(f, S) = 0$.

Proposition 6.2 *Let $S \in \mathcal{N}_f^{**}$ and $T \in \mathcal{N}_{\partial f}$ be parallel. Then there is a factorization*

$$(\partial f)_T = A_S B_S \ \ in\ \ \mathbf{C}\{X, Y\}$$

such that

(i) *If* $\mathrm{ord}\, A_S > 0$ *then A_S is S-elementary. Suppose* $\mathrm{ord}\, A_S > 0$ *and let $\mathcal{N}(A_S) = \{\tilde{S}\}$. Then the power series* $\mathrm{in}(A_S, \tilde{S})$ *and* $\mathrm{in}(f, S)$ *are coprime.*

(ii) *We have* $\mathrm{ord}\, B_S = d(f, S)$. *If* $\mathrm{ord}\, B_S > 0$ *then B_S is S-elementary. Suppose* $\mathrm{ord}\, B_S > 0$ *and let $\mathcal{N}(B_S) = \{\tilde{\tilde{S}}\}$. Then the power series* $\mathrm{in}(B_S, \tilde{\tilde{S}})$ *divides* $\mathrm{in}(f, S)$.

Proof. Let $\partial = a(\partial/\partial X) + b(\partial/\partial X)$ with $ab\,\mathrm{in} f(a,b) \neq 0$. We may assume that $|S|_1 < |S|_2$. By [LP], Theorem 2.1 (5), p. 313 we get

(9) $\mathrm{in}(\partial f, T) = a\dfrac{\partial}{\partial X}\mathrm{in}(f,S)$.

Let us consider the factorization

(10) $\mathrm{in}(f,S) = X^{a_S}Y^{b_S}\psi_1^{k_1}\ldots\psi_m^{k_m}$ with irreducible, pairwise coprime $\psi_i \in \mathbf{C}\{X,Y\}$ and let $\lambda = Y$.

We have assumed that $|S|_1 < |S|_2$ thus the lower vertex of S does not lie on the vertical axis and we may apply Lemma 3.1 to $\mathrm{in}(f,S)$ and λ:

(11) $\dfrac{\partial}{\partial X}\mathrm{in}(f,S) = Y^{b_S}\psi_0 X^{\max(a_S-1,0)}\psi_1^{k_1-1}\ldots\psi_m^{k_m-1}$
where ψ_0 and $\mathrm{in}(f,S)$ are coprime.

By (2), (3) and (11) we get

(12) $\mathrm{in}((\partial f)_T, T') = \psi_0\psi_1^{k_1-1}\ldots\psi_m^{k_m-1}$.

Clearly the power series $\psi_1,\ldots,\psi_m$ and ψ_0 (if $\mathrm{ord}\psi_0 > 0$) are $S-$elementary. According to Lemma 4.2 we get the factorization

(13) $(\partial f)_T = g_0 g_1 \ldots g_m$ in $\mathbf{C}\{X,Y\}$ such that

- · $\mathrm{ord} g_0 > 0$ if and only if $\mathrm{ord}\,\psi_0 > 0$. If $\mathrm{ord}\,g_0 > 0$ then g_0 is $S-$ elementary, $\mathcal{N}(g_0) = \{S^{(0)}\}$ and $\mathrm{in}(g_0, S^{(0)}) = \psi_0$,
- ·· $\mathrm{ord}\,g_j > 0$ if and only if $k_j > 0$ for $(j = 1,\ldots,m)$. If $\mathrm{ord}\,g_j > 0$ then g_j is S-elementary, $\mathcal{N}(g_j) = \{S^{(j)}\}$ and $\mathcal{N}(g_j, S^{(j)}) = \psi_j^{k_j-1}$.

Let $A_S = g_0$ and $B_S = g_1\ldots g_m$. By (13) we get $(\partial f)_T = A_S B_S$. Suppose that $\mathrm{ord} A_S > 0$. Then $\mathcal{N}(A_S) = \{\tilde{S}\}$ where $\tilde{S} = S^{(0)}$ and $\mathrm{in}(A_S, \tilde{S}) = \mathrm{in}(g_0, S^{(0)}) = \psi_0$, consequently $\mathrm{in}(f,S)$ are coprime. On the other hand

$$\begin{aligned}\mathrm{ord}\,B_S &= \sum_{j=1}^{m}\mathrm{ord}\,g_j = \sum_{j=1}^{m}(k_j-1)\mathrm{ord}\,\psi_j\\ &= \sum_{j=1}^{m}k_j\mathrm{ord}\,\psi_j - \sum_{j=1}^{m}\psi_j\\ &= \mathrm{ord}\,\mathrm{in}\widehat{(f,S)} - \mathrm{in}\widehat{(f,S)}_{\mathrm{red}} = d(f,S)\ .\end{aligned}$$

Suppose that $\operatorname{ord} B_S > 0$. The power series B_S is S-elementary as a product of S-elementary power series. If $\mathcal{N}(B_S) = \{\tilde{\tilde{S}}\}$ then $\operatorname{in}(B_S, \tilde{\tilde{S}}) = \prod_{j=1}^{m} \psi_j^{k_j - 1}$ divides $\operatorname{in}(f, S)$.

Proposition 6.3 *Let $S \in \mathcal{N}_f^{**}$ and let h be an irreducible factor of g_S. Then $(f,h)_0/\operatorname{ord} h \geq \max(\alpha(S), \beta(S))$. The inequality $(f,h)_0/\operatorname{ord} h > \max(\alpha(S), \beta(S))$ holds if and only if the following two conditions are fulfiled*

(a) *the power series h is S-elementary,*

(b) *if $\tilde{S}$ is the unique segment of $\mathcal{N}_h$ then the system of equations $\operatorname{in}(h, \tilde{S}) = 0, \operatorname{in}(f, S) = 0$ has a solution in $(\mathbf{C} \setminus \{0\}) \times (\mathbf{C} \setminus \{0\})$.*

Proof. Proposition 6.3 follows from [LP], Lemma 3.2, p. 316 which remains true when we replace the sentence "if the pair f, h is nondegenerate" by "if and only if the pair f, h is nondegenerate".

Now we can give the proof of Theorem 1.1. Let us consider the factorization

$$\partial f = g \prod_{S \in \mathcal{N}_f^{**}} g_S \text{ in } \mathbf{C}\{X, Y\}$$

such that in Proposition 6.1.

Let $S \in \mathcal{N}_f^{**}$ be a segment of the first kind. Then by Proposition 6.1 (ii) we have $g_S = (\partial f)_T$ where $T \in \mathcal{N}_{\partial f}$ is a segment parallel to S. Let $g_S = (\partial f)_T = A_S B_S$ be the factorization of $(\partial f)_T$ such that in Proposition 6.2. Then $\operatorname{ord} B_S = d(f, S)$ by Proposition 6.2 (ii) and consequently $\operatorname{ord} A_S = \operatorname{ord} g - \operatorname{ord} B_S = ||S|| + \varepsilon(S) - d(f, S)$ by Proposition 6.1 (i). Moreover if $\operatorname{ord} A_S > 0$ (resp. $\operatorname{ord} B_S > 0$) then A_S (resp. B_S) is S-elementary. Let h be an irreducible factor of $A_S B_S = g_S$. Then $(f,h)_0/\operatorname{ord} h \geq \max(\alpha(S), \beta(S))$ by Proposition 6.3. Using Proposition 6.2 we check that h divides B_S if and only if h fulfils the conditions (a) and (b) from Proposition 6.3. Thus $(f,h)_0/\operatorname{ord} h > \max(\alpha(S), \beta(S))$ if and only if h is a divisor of B_S. Summing up, we have checked that the factorization $g_S = A_S B_S$ where $S \in \mathcal{N}_f^{*}$ is of the first kind, satisfies all conditions stated in Theorem 1.1. Now suppose that $S \in \mathcal{N}_f^{**}$ is of the second kind. We consider two cases.

Case 1. There is no segment $T \in \mathcal{N}_{\partial f}$ parallel to S. Then we put $A_S = A'_S = g_S$, $A''_S = 1$ and $B_S = 1$.

Case 2. There is a segment $T \in \mathcal{N}_{\partial f}$ parallel to S. Then by Proposition 6.2 we get

$$(\partial f)_T = A'_S B_S \text{ in } \mathbf{C}\{X,Y\}\ .$$

On the other hand $(\partial f)_T$ divides g_S by Proposition 6.1 (iii) and we can write

$$g_S = A''_S (\partial f)_T \text{ in } \mathbf{C}\{X,Y\}\ .$$

Thus we get $g_S = A_S B_S$ with $A_S = A'_S A''_S$. As in the case of the segment of the first kind we check that the factorizations $g_S = A_S B_S$, $A_S = A'_S A''_S$ have all properties stated in Theorem 1.1. To finish the proof it suffices to check that there is a factorization $g = AB$ in $\mathbf{C}\{X,Y\}$ with $\operatorname{ord} A = t(f) - 1$ such that h is an irreducible factor of g with property $(f,h)_0/\operatorname{ord} h = \operatorname{ord} f$ if and only if h divides A. To this purpose we apply Lemma 4.3 to the series ∂f and observe that if h is an irreducible factor of $\prod_{S \in \mathcal{N}_f^{**}} g_S$ then $(f,h)_0/\operatorname{ord} h \geq \max(\alpha(S), \beta(S))$ for a segment $S \in \mathcal{N}_f^{**}$ and $\max(\alpha(S), \beta(S)) > \operatorname{ord} f$.

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