The norms of minimal projections onto some subspaces of $\ell_1^{(n)}$

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We calculate the norms of minimal projections onto some symmetric subspaces of $\ell_1^{(n)}$ (Th. 3.1). This extends Th. 2.6 from [1].

1. Introduction. Let X be a normed space and let V be a linear subspace of X. Denote by $\mathcal{P}(X, V)$ the set of all continuous linear projections from X onto V, i.e.,

$$\mathcal{P}(X, V) = \{P \in \mathcal{L}(X, V) : P|_V = \mathrm{id}_V\}.$$

A projection $P \in \mathcal{P}(X, V)$ is called *minimal* if

$$\|P\| = \lambda(V, X) \stackrel{\mathrm{df}}{=} \inf\{\|P\| : P \in \mathcal{P}(X, V)\}.$$

The constant $\lambda(V, X)$ is called the *relative projection constant.*

Observe that the problem of finding minimal projections is strictly related to the Hahn-Banach theorem, because we look for an extension of the identity operator id : $V \to V$ onto X of minimal norm.

For more information concerning minimal projections (existence, effective formulas, uniqueness or estimates of the norm) the reader is referred to the papers enlisted in references.

The aim of this paper is to determine the norms of minimal projections onto some symmetric subspaces of $\ell_1^{(n)}$ (i.e. $\mathbb{R}^n$ equipped with the ℓ_1 norm), (see Definition 2.1 and Theorem 3.1). This extends Theorem 2.6 from [1]. We also give a new example of regular symmetric subspaces. This notion has been introduced in [4].

Now we introduce some notation which will be used later. We denote by S_V the unit sphere in a normed space V. The symbol $\mathrm{ext}(S_V)$ stands for the set of all extremal points

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of S_V. Note that if V is a k-dimensional subspace of $\ell_1^{(n)}$, then each $P \in \mathcal{P}(\ell_1^{(n)}, V)$ has the form

$$Px = \sum_{i=1}^{k} u^i(x)v^i,$$

where $v^1, \dots, v^k$ is a fixed basis of V and $u^1, \dots, u^k \in \ell_\infty^{(n)}$ satisfy

$$u^j(v^i) = \sum_{l=1}^{n} u_l^i v_l^i = \delta_{ij}. \tag{1.1}$$

A point $x \in X$ is called a *norming* point for $f \in X^*$ if

$$x \in S_X \text{ and } f(x) = \|f\|.$$

Definition 1.1. Let V be a finite dimensional, real Banach space. V is *symmetric* if there exists a basis $v^1, \dots, v^k \in V$ such that

$$\|\sum_{i=1}^{k} |\alpha_i| v^i\| = \|\sum_{i=1}^{k} \alpha_{\pi(i)} v^i\|$$

for any $\alpha_1, \dots, \alpha_k \in \mathbb{R}$ and any permutaton π of indices.

Now let $P = \sum_{i=1}^{k} u^i(\cdot)v^i \in \mathcal{P}(\ell_1^{(n)}, V)$. Define

$$\mathrm{crit}(P) = \{j \in \{1, \dots, n\} : \|Pe_j\| = \|P\|\},$$

where e_j is the j-th unit vector from the canonical basis of $\mathbb{R}^n$ and for $j = 1, \dots, n$

$$V_j = (v_j^1, \dots, v_j^k), \quad U_j = (u_j^1, \dots, u_j^k). \tag{1.2}$$

Theorem 1.2 ([5], Th. 3, p. 294). *Let $P = \sum_{i=1}^{k} u^i(\cdot)v^i \in \mathcal{P}(\ell_1^{(n)}, V)$. Then P is minimal if and only if there exists a nonzero $k \times k$ matrix M such that for every $j \in \mathrm{crit}(P)$,*

$$U_j = (u_j^1, \dots, u_j^k) = \|P\| a^j,$$

where $a^j = (a_1^j, \dots, a_k^j)$ are the coordinates with respect to $v^1, \dots, v^k$ of a norming point for a functional on V associated with MV_j, i.e.,

$$(MV_j)(x) = \sum_{i=1}^{k} (MV_j)_i x_i. \tag{1.3}$$

Here $x = \sum_{i=1}^{k} x_i v^i$.

Remark 1.3 (see e.g. [17]). If V is a symmetric space, then the identity matrix is one of possible choices for M.

Remark 1.4. By [6, Th.1] it easy to see that if M is invertible and $V_j \neq 0$ for $j = 1, \dots, n$, then $\mathrm{crit}(P) = \{1, \dots, n\}$ for any minimal projection.

Let Σ_k denote the set of all permutations of $\{1,\dots,k\}$ and let ε, σ be the operators from $\mathbb{R}^k$ onto $\mathbb{R}^k$ defined by

$$\varepsilon(x) = (\varepsilon_1 x_1, \dots, \varepsilon_k x_k), \text{ where } \varepsilon_j \in \{-1,1\}, j = 1,\dots,k,$$
$$\sigma(x) = (x_{\sigma(1)}, \dots, x_{\sigma(k)}).$$

Definition 1.5 ([4], Def. 2.1). Let $x \in \mathbb{R}^k$. Suppose $J : \Sigma_k \times \{-1,1\}^k \to \{1,\dots,k!2^k\}$ is a fixed bijection such that $J(\mathrm{id},(1,\dots,1)) = 1$. By $[[x]]$ we denote the $k \times k!2^k$-matrix with the columns $x^{(j)}$, $j = 1,\dots,k!2^k$, where

$$x^{(j)} = (\varepsilon \circ \sigma)(x) = (\varepsilon_1 x_{\sigma(1)}, \dots, \epsilon_k x_{\sigma(k)}).$$

Here $\varepsilon \in \{-1,1\}^k$, $\sigma \in \Sigma_k$ are chosen so that $j = J(\sigma,\epsilon)$. We will refer to the matrix $[[x]]$ as the *block generated by* x.

Definition 1.6 ([4], Def. 2.2). Let $k, N \in \mathbb{N}$. Put $n = Nk!2^k$. Let $x^1,\dots,x^N \in \mathbb{R}^k$. It is said that a linear subspace $V \subset \ell_1^{(n)}$ is *generated by* $(x^1,\dots,x^N)$ if and only if the rows $v^1,\dots,v^k$ of the $k \times Nk!2^k$ matrix $[[x^1]],\dots,[[x^N]]$ form a basis of V, where $[[x^i]]$ is the block generated by x^i for $i = 1,\dots,N$. It is easy to check that V is a symmetric space with respect to $v^1,\dots,v^k$. If for $i = 1,...,N$ x^i has at most two coordinates different from zero, then we can treat V as a subspace of $l_1^{(n)}$ with $n = 4Nk(k-1)$.

Definition 1.7. Let $k, N \in \mathbb{N}$. Set $n = Nk!2^k$. Suppose $x^1,\dots,x^N \in \mathbb{R}\setminus\{0\}$. Let V be the space generated by $x^1,\dots,x^N$. It is said that V is a *regular symmetric subspace* if and only if for any $j \in \{1,\dots,n\}$ the vector

$$\frac{\sum_{i=1}^k v_j^i v^i}{\|\sum_{i=1}^k v_j^i v^i\|}$$

is a norming point for a functional on V associated with V_j by (1.3). (By Remark 1.3 we put $M =$ identity matrix).

2. Technical lemmas. We start with

Definition 2.1. Let $k \in \mathbb{N}$ and $k \geq 2$. Put $n = 8k(k-1)$. Let $V \subset \ell_1^{(n)}$ be a linear space generated by (x^1, x^2), where $x^1, x^2 \in \mathbb{R}^k$, $x^1 = (1,b,0,\dots,0)$, $x^2 = (d,c,0,\dots,0)$, $0 \leq b,c,d \leq 1$, $d \geq c$ and $d^2 + c^2 > 0$. Now fix $\epsilon > 0$. Let us extend matrix $[[x^1]],[[x^2]]$ to the matrix M_ϵ by adding k columns of the form ϵe_i, where e_i is the i-th vector from the canonical basis of $\mathbb{R}^k$. By V^ϵ we denote a k-dimensional subspace of $\ell_1^{(n+k)}$ spanned by the rows of M_ϵ. Observe that V is k-dimensional symmetric subspace of $\ell_1^{(n)}$ and V^ϵ is k-dimensional symmetric subspace of $\ell_1^{(n+k)}$.

Example 2.2. If $k = 2$ and V is as in Definition 2.1, then

$$v^1 = (1,b,-1,-b,-1,-b,1,b,d,c,-d,-c,-d,-c,d,c),$$
$$v^2 = (b,1,b,1,-b,-1,-b,-1,c,d,c,d,-c,-d,-c,-d).$$

Analogously, if $\epsilon > 0$, $k = 2$ and V^ϵ is as in Definition 2.1, then

$$\begin{aligned} v^1 &= (1, b, -1, -b, -1, -b, 1, b, d, c, -d, -c, -d, -c, d, c, \epsilon, 0), \\ v^2 &= (b, 1, b, 1, -b, -1, -b, -1, c, d, c, d, -c, -d, -c, -d, 0, \epsilon). \end{aligned}$$

Remark 2.3. Let V and V^ϵ be as in Definition 2.1. It is easy to check that

$$\begin{aligned} \|v^1\| &= 4(k-1)(1+b+c+d), \\ \|v^1\|^\epsilon &= \|v^1\| + \epsilon, \\ \|v^1 + bv^2\| &= 4(k-2)b^2 + 4(2k-3)b + 4(k-1) + \\ &\quad 4(c+d)(kb - 2b + k - 1) - 2c + 2bd + 2|c - bd|, \\ \|v^1 + bv^2\|^\epsilon &= \|v^1 + bv^2\| + (1+b)\epsilon, \\ \|dv^1 + cv^2\| &= 4(k-2)c^2 + 4(2k-3)cd + 4(c+d)(kb - 2b + k - 1) + \\ &\quad 4kd^2 - 4d^2 + 2bd - 2c + 2|c - bd|, \\ \|dv^1 + cv^2\|^\epsilon &= \|dv^1 + cv^2\| + (d+c)\epsilon. \end{aligned}$$

Here and subsequently $\|\cdot\|$ (resp. $\|\cdot\|^\epsilon$) stands for the norm in V (resp. V^ϵ) induced from $\ell_1^{(n)}$ (resp. $\ell_1^{(n+k)}$).

The proofs of the following two lemmas appeared in [1] and [2].

Lemma 2.4. *Let $V = \mathrm{Span}[v^1, \dots, v^k]$ be a k-dimensional ($k \geq 2$) subspace of $\ell_1^{(n)}$. Then $x = \sum_{i=1}^k x_i v^i \in \mathrm{ext}(S_V)$ if and only if the matrix W consisting of all vectors V_j (see (1.2)) orthogonal to x has rank $k-1$ and $\|x\| = 1$. We understand that V_j is orthogonal to x if*

$$V_j(x) = \sum_{i=1}^k (V_j)_i x_i = \sum_{i=1}^k v_j^i x_i = 0.$$

Corollary 2.5. *Let $V = \mathrm{Span}[v^1, v^2]$ be a 2-dimensional subspace of $\ell_1^{(n)}$. Then $x = x_1 v^1 + x_2 v^2 \in \mathrm{ext}(S_V)$ if and only if there exists $V_j \neq 0$ which is orthogonal to x and $\|x\| = 1$.*

Lemma 2.6. *Let $\|\cdot\|$ be a norm on $\mathbb{R}^k$ satisfying*

$$\|(x_1, \dots, x_k)\| = \|(x_1, \dots, x_{k-1}, -x_k\|.$$

Let $u = (u_1, \dots, u_{k-1}, 0) \in (\mathbb{R}^k, \|\cdot\|)^*$. If $x = (x_1, \dots, x_k)$ is a norming point for u then so is $(x_1, \dots, x_{k-1}, 0)$.

The next lemma is related to Lemma 2.5 from [1].

Lemma 2.7. *Let V^ϵ be the space given by Definition 2.1. If $x = \sum_{j=1}^k x_j v^j \in \mathrm{ext}(S_{V^\epsilon})$ is a norming point for V_j ($j = 1, \dots, n+k$), then $(x_1, \dots, x_k) = cV_l$ for some $l \in \{1, \dots, n+k\}$ and $c \in \mathbb{R}$ chosen so that $\|x\|^\epsilon = 1$.*

Proof. If $k = 2$, the result follows from Corollary 2.5. So suppose that $k \geq 3$. Let $x \in \mathrm{ext}(S_{V^\epsilon})$ be a norming point for V_j. Because of the symmetry, without loss of generality we can assume that $V_j = (d, c, 0, \dots, 0)$, $V_j = (1, b, 0, \dots, 0)$ or $V_j = (1, 0, \dots, 0)$. First we consider the case $V_j = (d, c, 0, \dots, 0)$. Observe that

$$\|x\|^\epsilon = \|x\| + \epsilon \sum_{i=1}^{k} |x_i|. \tag{2.1}$$

Applying Lemma 2.6 $k-2$ times we see that $y = x_1 v^1 + x_2 v^2$ is also a norming point for V_j. In particular $\|y\|^\epsilon = 1$. Since $y = \frac{x+z}{2}$ where $z = y - \sum_{j=3}^{k} x_j v^j$ and $\|x\| = \|z\|$ we have $\|y\| \leq \|x\|$. By (2.1) we get $x_i = 0$ for $i = 3, \dots, k$, which shows that $x = y$. Observe that $(x_1, \dots, x_k)$ is orthogonal to e_i for $i = 3, \dots, k$. Since $x \in \mathrm{ext}(S_{V^\epsilon})$, by Lemma 2.4, the result follows. The same argument applies to the case $V_j = (1, b, 0, \dots, 0)$.

The case $V_j = (1, 0, \dots, 0)$ may be handled in much the same way (Lemma 2.6 should be applied $k-1$ times). □

Lemma 2.8. *Let $k, N \in \mathbb{N}$, $k \geq 2$. Put $n = Nk!2^k$. Let $x^1, \dots, x^N \in \mathbb{R}^k$ be such that x^i and x^j are linearly independent for all $i \neq j$. Let $V \subset \ell_1^{(n)}$ be generated by $(x^1, \dots, x^N)$. Assume that*

$$\frac{\|x^1\|_e}{\|\sum_{i=1}^{k} x_i^1 v^i\|} \leq \cdots \leq \frac{\|x^N\|_e}{\|\sum_{i=1}^{k} x_i^N v^i\|}, \tag{2.2}$$

here $\|\cdot\|_e$ stands for the Euclidean norm in $\mathbb{R}^k$.

Then none of the points of the form $\frac{\sum_{i=1}^{k} x_i^l v^i}{\|\sum_{i=1}^{k} x_i^l v^i\|}$, where $l < p$, is norming for each functional on V associated with the first column of $[[x^p]], p = 1, \dots, N$ (see Definition 1.5, (1.2) and (1.3)).

Proof. Fix $p, l \in \{1, \dots, N\}$ such that $l < p$. Using classical Schwarz inequality and (2.2) we have

$$\frac{x_1^p x_1^l + \cdots + x_k^p x_k^l}{\|\sum_{i=1}^{k} x_i^l v^i\|} < \frac{\|x^p\|_e \|x^l\|_e}{\|\sum_{i=1}^{k} x_i^l v^i\|} \leq \frac{(x_1^p)^2 + \cdots + (x_k^p)^2}{\|\sum_{i=1}^{k} x_i^p v^i\|}.$$

□

The next lemma follows from Theorem 3.1.6 from [12].

Lemma 2.9. *Let V and V^ϵ be as in Definition 2.1. Then*

$$(a) \quad \lambda(V, \ell_1^{(n)}) = \lim_{\epsilon \to 0} \lambda(V^\epsilon, \ell_1^{(n+k)}),$$

$$(b) \quad \lambda(V, \ell_1^{(n)}) = \lim_{l \to \infty} \lambda(V_l, l_1^{(n)}),$$

where $V_l \subset l_1^{(n)}$ is generated by $(1, b_l, 0, \dots, 0)$, $(d_l, c_l, 0, \dots, 0)$ and $b_l \to b$, $c_l \to c$ and $d_l \to d$ $(l \to \infty)$.

3. The main result.

Theorem 3.1. *Let V be a k-dimensional ($k \geq 2$) subspace of $\ell_1^{(n)}$ given by Definition 2.1. Consider the following cases:*

(A)
$$\frac{1+b^2}{\|v^1+bv^2\|} \geq \frac{d+bc}{\|dv^1+cv^2\|}, \quad \frac{1+b^2}{\|v^1+bv^2\|} \geq \frac{1}{\|v^1\|},$$
$$\frac{d^2+c^2}{\|dv^1+cv^2\|} \geq \frac{d+bc}{\|v^1+bv^2\|}, \quad \frac{d^2+c^2}{\|dv^1+cv^2\|} \geq \frac{d}{\|v^1\|},$$

(B)
$$\frac{1}{\|v^1\|} \geq \frac{1+b^2}{\|v^1+bv^2\|}, \quad \frac{1}{\|v^1\|} \geq \frac{d+bc}{\|dv^1+cv^2\|},$$
$$\frac{d}{\|v^1\|} \geq \frac{d+bc}{\|v^1+bv^2\|}, \quad \frac{d}{\|v^1\|} \geq \frac{d^2+c^2}{\|dv^1+cv^2\|},$$

(C)
$$\frac{1+b^2}{\|v^1+bv^2\|} \geq \frac{1}{\|v^1\|}, \quad \frac{1+b^2}{\|v^1+bv^2\|} \geq \frac{d+bc}{\|dv^1+cv^2\|},$$
$$\frac{d+bc}{\|v^1+bv^2\|} \geq \frac{d}{\|v^1\|}, \quad \frac{d+bc}{\|v^1+bv^2\|} \geq \frac{d^2+c^2}{\|dv^1+cv^2\|},$$

(D)
$$\frac{d+bc}{\|dv^1+cv^2\|} \geq \frac{1+b^2}{\|v^1+bv^2\|}, \quad \frac{d+bc}{\|dv^1+cv^2\|} \geq \frac{1}{\|v^1\|},$$
$$\frac{d^2+c^2}{\|dv^1+cv^2\|} \geq \frac{d+bc}{\|v^1+bv^2\|}, \quad \frac{d^2+c^2}{\|dv^1+cv^2\|} \geq \frac{d}{\|v^1\|},$$

(E)
$$\frac{1}{\|v^1\|} \geq \frac{1+b^2}{\|v^1+bv^2\|}, \quad \frac{1}{\|v^1\|} \geq \frac{d+bc}{\|dv^1+cv^2\|},$$
$$\frac{d+bc}{\|v^1+bv^2\|} \geq \frac{d^2+c^2}{\|dv^1+cv^2\|}, \quad \frac{d+bc}{\|v^1+bv^2\|} \geq \frac{d}{\|v^1\|},$$

(F)
$$\frac{1}{\|v^1\|} \geq \frac{1+b^2}{\|v^1+bv^2\|}, \quad \frac{1}{\|v^1\|} \geq \frac{d+bc}{\|dv^1+cv^2\|},$$
$$\frac{d^2+c^2}{\|dv^1+cv^2\|} \geq \frac{d}{\|v^1\|}, \quad \frac{d^2+c^2}{\|dv^1+cv^2\|} \geq \frac{d+bc}{\|v^1+bv^2\|},$$

(G)
$$\frac{d+bc}{\|dv^1+cv^2\|} \geq \frac{1}{\|v^1\|}, \quad \frac{d+bc}{\|dv^1+cv^2\|} \geq \frac{1+b^2}{\|v^1+bv^2\|},$$
$$\frac{d}{\|v^1\|} \geq \frac{d+bc}{\|v^1+bv^2\|}, \quad \frac{d}{\|v^1\|} \geq \frac{d^2+c^2}{\|dv^1+cv^2\|},$$

(H)
$$\frac{1+b^2}{\|v^1+bv^2\|}\geq\frac{1}{\|v^1\|},\quad\frac{1+b^2}{\|v^1+bv^2\|}\geq\frac{d+bc}{\|dv^1+cv^2\|},$$
$$\frac{d}{\|v^1\|}\geq\frac{d+bc}{\|v^1+bv^2\|},\quad\frac{d}{\|v^1\|}\geq\frac{d^2+c^2}{\|dv^1+cv^2\|}.$$

If A is satisfied, then V is a regular symmetric subspace and

$$\lambda(V,\ell_1^{(n)})=\frac{1}{\frac{4(k-1)(1+b^2)}{\|v^1+bv^2\|}+\frac{4(k-1)(d^2+c^2)}{\|dv^1+cv^2\|}}.$$

If is B satisfied, then

$$\lambda(V,\ell_1^{(n)})=\frac{\|v^1\|}{4(k-1)(1+d)}.$$

If is C satisfied, then

$$\lambda(V,\ell_1^{(n)})=\frac{\|v^1+bv^2\|}{4(k-1)(1+b^2+bc+d)}.$$

If is D satisfied, then

$$\lambda(V,\ell_1^{(n)})=\frac{\|dv^1+cv^2\|}{4(k-1)(c^2+d^2+bc+d)}.$$

If is E satisfied, then

$$\lambda(V,\ell_1^{(n)})=\frac{1}{\frac{4(k-1)}{\|v^1\|}+\frac{4(k-1)(bc+d)}{\|v^1+bv^2\|}}.$$

If is F satisfied, then

$$\lambda(V,\ell_1^{(n)})=\frac{1}{\frac{4(k-1)}{\|v^1\|}+\frac{4(k-1)(d^2+c^2)}{\|dv^1+cv^2\|}}.$$

If is G satisfied, then

$$\lambda(V,\ell_1^{(n)})=\frac{1}{\frac{4(k-1)(d+bc)}{\|dv^1+cv^2\|}+\frac{4d(k-1)}{\|v^1\|}}.$$

If is H satisfied, then

$$\lambda(V,\ell_1^{(n)})=\frac{1}{\frac{4(k-1)(1+b^2)}{\|v^1+bv^2\|}+\frac{4d(k-1)}{\|v^1\|}}.$$

Proof. Let

$$Z_1=\{(b,c,d):b\neq 1\},$$
$$Z_2=\{(b,c,d):c\neq bd\},$$
$$Z_3=\{(b,c,d):c\neq d\},$$
$$Z_4=\{(b,c,d):b\neq 0\},$$
$$Z_5=\{(b,c,d):c\neq 0\}.$$

Put

$$Z = \bigcap_{j=1}^{5} Z_j.$$

Since all the cases can be shown in a similar way, we will only give the proof of the case (A). We divide our proof into two parts.
PART I. Assume that $(b,c,d) \in Z$. Hence there exists a sequence ϵ_l such that $\epsilon_l > 0, \epsilon_l \to 0$ $(l \to \infty)$ and exactly one of the cases (A) – (H) for $\|\cdot\|^{\epsilon_l}$ instead of $\|\cdot\|$ is satisfied with strict inequalities. This means that for any functional V_j there exists exactly one norming point in V^{ϵ_l}. Fix $l \in \mathbb{N}$. Without loss of generality we can assume that

$$\frac{1+b^2}{\|v^1+bv^2\|^{\epsilon_l}} > \frac{d+bc}{\|dv^1+cv^2\|^{\epsilon_l}}, \quad \frac{1+b^2}{\|v^1+bv^2\|^{\epsilon_l}} > \frac{1}{\|v^1\|^{\epsilon_l}},$$
$$\frac{d^2+c^2}{\|dv^1+cv^2\|^{\epsilon_l}} > \frac{d+bc}{\|v^1+bv^2\|^{\epsilon_l}}, \quad \frac{d^2+c^2}{\|dv^1+cv^2\|^{\epsilon_l}} > \frac{d}{\|v^1\|^{\epsilon_l}}.$$

By the symmetry of V^{ϵ_l} it is necessary to find norming points for $(1,0,\dots,0)$, $(1,b,0,\dots,0)$ and $(d,c,0,\dots,0)$. By Lemma 2.7, $\frac{v^1+bv^2}{\|v^1+bv^2\|^{\epsilon_l}}$ is the only norming point for $(1,b,\dots,0)$, $\frac{dv^1+cv^2}{\|dv^1+cv^2\|^{\epsilon_l}}$ is the only norming point for $(d,c,0,\dots,0)$ and $\frac{v^1}{\|v^1\|^{\epsilon_l}}$ is the only norming point for $(1,0,\dots,0)$. Let P_{ϵ_l} denotes a minimal projection onto V^{ϵ_l} (since V^{ϵ_l} is finite dimensional such a projection exists, see e.g. Corollary 3.1.2 in [12], by the way this P_{ϵ_l} is the only minimal projection, which follows from Theorem 1.2). By Theorem 1.2, Remark 1.4 and (1.1) applied to $i=j=1$, we get

$$\sum_{s=1}^{n+k} u_s^1 v_s^1 = \sum_{s=1}^{n+k} \|P_{\epsilon_l}\| a_s^1 v_s^1 = 1.$$

Hence

$$\|P_{\epsilon_l}\| = \frac{1}{\frac{4(k-1)(1+b^2)}{\|v^1+bv^2\|^{\epsilon_l}} + \frac{4(k-1)(d^2+c^2)}{\|dv^1+cv^2\|^{\epsilon_l}} + \frac{\epsilon_l}{\|v^1\|^{\epsilon_l}}}.$$

Taking the limit on both sides as $l \to \infty$ and applying Lemma 2.9(a), we complete the proof of Part I.
PART II. Suppose $(b,c,d) \notin Z$. Observe that for any nontrivial polynomial p of three variables the set $Z_p = \{z \in \mathbb{R}^3 : p(z) = 0\}$ has empty interior in $\mathbb{R}^3$. Hence there exists a sequence $(b_l, c_l, d_l) \in Z$ such that $b_l \to b, c_l \to c, d_l \to d, (l \to \infty)$. By Part I and Lemma 2.9(b) we get the result. □

Example 3.2. According to Lemma 2.8, for every choice of b, c, d (as in Definition 2.1) at least one of the cases (A)-(H) holds. Note that, by Lemma 2.7, V is a regular symmetric subspace if and only if (A) is satisfied.
Moreover,
(A) is satisfied e.g. when $c = d = 1$, $b \in (0, \frac{1}{3})$ for $k=2$, $b \in (\frac{1}{2}, \frac{\sqrt{41}-5}{2})$ for $k=3$, $b \in (\frac{k-2}{k-1}, \frac{2k-5}{2k-4})$ for $k > 3$.
(B) is satisfied e.g. when $d=1$, $c = \frac{1}{4}$, $b \in (0, \frac{1}{4})$ for $k \geq 3$ (for $k=2$ the case (B) never happens).

(C) is satisfied e.g. when $d = 1$, $c = \frac{3}{100}$, $b = \frac{1}{100}$ for $k = 2$; $d = 1$, $c = 1 - \frac{1}{k}$, $b = 1 - \frac{1}{k} - \frac{1}{k^2}$ for $k \geq 3$.
(D) is satisfied e.g. when $d = c = 1$, $b \in (\frac{1}{3}, 1)$ for $k = 2$, $b \in (\frac{4k-9}{4k-8}, 1)$ for $k \geq 3$.
(E) is satisfied e.g. when $d = 1$, $c = 1 - \frac{1}{k}$, $b = \frac{1}{k^2}$ for $k \geq 3$ (for $k = 2$ the case (E) never happens).
(F) is satisfied e.g. when $c = d = 1$, $b \in (0, \frac{k-2}{k-1})$ for $k \geq 3$ (for $k = 2$ the case (F) never happens).
(G) is satisfied e.g. when $d = 1$, $b = 1 - \frac{1}{k}$, $c = \frac{1}{k^2}$ for $k \geq 3$ (for $k = 2$ the case (G) never happens).
(H) is satisfied e.g. when $b = d = 1$, $c \in (0, \frac{k-2}{k-1})$ for $k \geq 3$ (for $k = 2$ the case (H) never happens).

Remark 3.3. If $k = 2$, then the proof of Theorem 3.1 may be simplified. We may then apply Corollary 2.5, by which we do not have to use the space V^ϵ in the proof.

Example 3.4. Let $k = 2$, $d = 1$, $b < c$ $(0 \leq b, c \geq 1)$. Applying Remark 3.3 it is easily seen that it is sufficient to consider only the cases A,C and D, which take the simple form

$$\text{(A)} \qquad 1 \geq b(c+2), \quad c \geq \sqrt{2} - 1 \text{ and } \lambda(V, \ell_1^{(16)}) = \frac{1}{\frac{b^2+1}{b+c+2} + \frac{c^2+1}{2(c+1)}},$$

$$\text{(C)} \qquad \sqrt{2} - 1 \geq c \text{ and } \lambda(V, \ell_1^{(16)}) = \frac{b+c+2}{b^2+bc+2},$$

$$\text{(D)} \qquad b(c+2) \geq 1 \text{ and } \lambda(V, \ell_1^{(16)}) = \frac{2(c+1)}{c^2+bc+2}.$$

Remark 3.5. Similar method can be applied for any finite mumber of blocks $[[x^1]], \ldots, [[x^p]]$, where $x^j = (a_j, b_j, 0, \ldots, 0)$ for $a_j, b_j \in \mathbb{R}$, $j = 1, \ldots, p$. However, it is necessary to consider much more cases then in Theorem 3.1.

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