ON ASYMPTOTIC SOLUTIONS OF ANALYTIC EQUATIONS

JACEK STASICA

IMUJ PREPRINT 2003/01

ABSTRACT. Sufficient conditions for the existence of analytic solution in complex and real cases are given.

1. INTRODUCTION.

For fixed holomorphic function $F(z,t)$ defined in a neighborhood of $(a,b) \in \mathbb{C}^{n+1}_{z,t}$ we show the existence of a number $N \in \mathbb{N}$ such that the existence of a continuous quasi-solution g of order at least N (i.e. $f(z,g(z)) = O|z-a|^N$) implies the existence of holomorphic solution (i.e. $f(z,h(z)) \equiv 0$ with some holomorphic function h defined in a neighborhood of a, and $h(a) = b$). We are also able to estimate the number N effectively.

The famous Artin theorem says that the existence of a formal solution of the system of analytic equations implies the existence of an analytic one. Our theorem implies Artin theorem in the case of one equation. Moreover we give some sufficient conditions for existence of Nash solution in real case. As an application we obtain a straightforward proof of the theorem saying that the set of smooth points of a semi-algebraic set is semi-algebraic as well. This theorem about semialgebraicity plays an important role in semialgebraic geometry. In [Ł1] S. Łojasiewicz proved that for locally semialgebraic sets the notion of a Nash smooth point coincides with analytic one. Moreover if $\Gamma \subset \mathbb{R}^n$ is an analytic submanifold, then for the germ Γ_a of Γ at $a \in \Gamma$ we have the equivalence:

$$\Gamma_a \text{ is a Nash } \Leftrightarrow \Gamma_a \text{ is semialgebraic.}$$

Hence the semialgebraicity of smooth points one can obtain also following Łojasiewicz's method for the analogous theorem for semianalytic sets.

1991 *Mathematics Subject Classification.* 14P10, 14P20, 32A10, 32A60.

Key words and phrases. Analytic function, holomorphic function, asymptotic solution, semialgebraic set, smooth point, topographic submanifold.

This paper is partially supported by the KBN grant number 159/P03/2001/21.

For the convenience of the reader we recall some basic definitions.

Consider a monic polynomial P in $t \in \mathbb{C}$ whose coefficients $a_1, \dots, a_n$ are holomorphic functions in an open subset Ω of $\mathbb{C}^n$, i.e. the function

$$P(z,t) = t^n + a_1(z)t^{n-1} + \cdots + a_n(z).$$

The function

$$D(P)(z) = \prod_{i<j}(t_i(z) - t_j(z))^2 = (-1)^{\binom{k}{2}} \prod_{j=1}^{k} \frac{\partial P}{\partial t}(z, t_j(z)),$$

where $t_1(z), \dots, t_k(z)$ is the complete sequence of roots of the polynomial $t \to P(z,t)$ is called *the discriminant* of the polynomial P.

We say that a holomorphic function $f(z,t)$ in a neighborhood of zero in $\mathbb{C}^n \times \mathbb{C}$ is t-regular if $f(0,t) \not\equiv 0$ in a neighborhood of zero in $\mathbb{C}$.

Let $P(z,t)$ be a distinguished polynomial for which $DP(z) \equiv 0$ in some neighborhood $0 \in \mathbb{C}^n$. In such a case there exists a distinguished polynomial $redP(z,t)$ for which $D(redP)(z) \not\equiv 0$ and $\{P = 0\} = \{redP = 0\}$ in some neighborhood of 0.

We say that a subset of $\mathbb{R}^n$ is semi-algebraic if it is described by polynomials on $\mathbb{R}^n$. Thus, the class of semi-algebraic subsets of $\mathbb{R}^n$ is the algebra of subset of $\mathbb{R}^n$ which is generated by the family of sets $\{P > 0\}$, where P is a polynomial. Equivalently, $E \subset \mathbb{R}^n$ is semi-algebraic if there are polynomials P_i and P_{ij}, $i = 1, \dots, p$, $j = 1, \dots, q$, such that

$$E = \bigcup_{i=1}^{p} \{x \mid P_i(x) = 0,\ P_{ij}(x) > 0,\ j = 1, \dots, q\}.$$

Let G be an open subset of $\mathbb{R}^n$. We say that an analytic function $f : G \longrightarrow \mathbb{R}$ is a Nash function at the point $a \in G$, if and only if $W(x, f(x)) = 0$ in a neighborhood of a with a polynomial $W \not\equiv 0$ in $\mathbb{R}^{n+1}_{x,t}$. A Nash function on G is an analytic function on G, which is Nash at each point of G.

2. Asymptotic solutions.

Lemma 1. *Let K_0, K be open discs in $\mathbb{C}$, $\overline{K_0} \subset K$ and $a \in K_0$. Let $P(z,t)$ be a monic polynomial with holomorphic coefficients in K. Let us assume that $D(P)(z) \neq 0$ in $K \setminus \{a\}$. Then each holomorphic in $K \setminus K_0$ function h for which $P(z, h(z)) \equiv 0$ extends holomorphically on K.*

Proof. Each of the open sets $B_\nu = K \cap \Pi_\nu$, where Π_1, Π_2 and Π_3, Π_4 are open halfplanes cut of from $\mathbb{C}$ by two different lines passing by a is homeomorphic with an open disc, hence each of the sets $W_{B_\nu} = W \cap (B_\nu \times \mathbb{C})$, where $W = \{P(z,t) = 0\}$, is a finite sum of graphs of holomorphic functions on B_ν. Therefore the restrictions of function h to the open connected sets $B_{0\nu} = (K \setminus \overline{K_0}) \cap \Pi_\nu$ has holomorphic extensions h_ν on B_ν, which are compatible. Thus $\bigcup h_\nu$ is the extension of function h, which from the Riemann Theorem extends holomorphically by a. $\square$

Lemma 2. *Let $P(z,t)$ be a monic polynomial of degree k with holomorphic coefficients for which $D(redP)(z) \neq 0$ in an open set $G \subset \mathbb{C}^n_z$. Let us define*

$$\tau(z) := \frac{1}{2} \min_{i \neq j} |t_i(z) - t_j(z)|$$

and

$$\eta(z,t) := \min_i |t - t_i(z)|,$$

where $t_1(z), \dots, t_l(z)$ is the sequence of all different roots of the polynomial $t \mapsto P(z,t)$. Then if there exists a continuous function g for which $\eta(z, g(z)) < \tau(z)$ in G, then there exists a holomorphic function h such that $P(z, h(z)) \equiv 0$ and $|h(z) - g(z)| \leq |P(z, g(z))|^{\frac{1}{k}}$ in G.

Proof. It follows from the assumption that for each $a \in G$ there exists the uniquely determined root $h(a)$ of the polynomial $t \mapsto P(a,t)$ such that $|h(a) - g(a)| < \tau(a)$. Hence $|h(a) - g(a)| = \eta(a, g(a)) \leq |P(a, g(a))|^{\frac{1}{k}}$. Since g is a continuous function, following implicit function theorem, the root $h(z)$ must coincide in some neighborhood of a with a holomorphic root $t(z)$ of the polynomial P. □

Theorem 1. *Let $F(z,t)$ be a holomorphic function in a neighborhood of $(a,b) \in \mathbb{C}^{n+1}_{z,t}$, t-regular in (a,b) and let $L > 0$. Then there exists $N \in \mathbb{N}$ such that, if $F(z, g(z)) = O(|z-a|^N)$ for $z \to a$ with some continuous in a neighborhood of a function g, $g(a) = b$, then there exists a holomorphic, in a neighborhood of a, function h, $h(a) = b$ such that $F(z, h(z)) \equiv 0$. Moreover, if in addition, the function g is holomorphic we have $h(z) - g(z) = O(|z-a|^L)$ for $z \to a$.*

Proof. We can assume that $a = b = 0$, and by the Weierstrass Preparation Theorem that $F(z,t) = t^k + a_1(z)t^{k-1} + \cdots + a_k(z)$ is a distinguished polynomial of degree $k > 0$ with bounded holomorphic coefficients a_j in some $\Omega \times \Delta$ open neighborhood of $0 \in \mathbb{C}^n$ where $\Omega \subset \mathbb{C}^{n-1}_u$ and $\Delta \subset \mathbb{C}^1_s$, $z = (u,s)$. We can also assume that polynomial $redF$ has bounded holomorphic coefficients in $\Omega \times \Delta$ and that $\{F = 0\} = \{redF = 0\}$. Moreover, by the Weierstrass Preparation Theorem we can assume that $D(redF) = HD_0$ in $\Omega \times \Delta$, where $D_0(u,s) = s^r + b_1(u)s^{r-1} + \cdots + b_r(u)$ is a distinguished polynomial with holomorphic, bounded in Ω coefficients b_j and $\inf |H| > 0$ in $\Omega \times \Delta$. Finally we can assume that $redD_0$ has holomorphic bounded coefficients in Ω. Let us take τ and η from Lemma 2 for the polynomial $redF$ in the set $D(redF) \neq 0$. Of course in this set we have $|D(redF)(z)| \leq M\tau(z)$ with some constant $M > 1$ and $|F(z,t)| \geq \eta(z,t)^k$. Hence, due to Lemma 2 for each open set $G \subset \{D(redF) \neq 0\}$ the inequality

$$(i) \qquad |F(z, g(z))| < M^{-k} |D(redF)(z)|^k$$

with some continuous function g on G gives us the existence of a holomorphic root h of the polynomial F in G, such that

$$(ii) \qquad |h(z) - g(z)| \leq |F(z, g(z))|^{\frac{1}{k}} \text{ in } G.$$

Put $|u| = \sigma$. The roots of the polynomial $s \mapsto D_0(u,s)$, are $O(\sigma^{\frac{1}{r}})$. Hence, taking M sufficiently big, there exists $\delta > 0$ such that for $\sigma > 0$ sufficiently small on the set

$$G_\sigma := B_\sigma \times (K'_\sigma \setminus \overline{K_\sigma}),$$

where $B_\sigma = \{|u| < 2\delta\}$, $K_\sigma = \{|s| < M\sigma^{\frac{1}{r}}\}$, $K'_\sigma = \{|s| < 2M\sigma^{\frac{1}{r}}\}$, we have the inequality $|D_0(u,s)| \geq \delta\sigma$ and $G_\sigma \subset \Omega \times \Delta$. Let $D(redD_0)(u) = \sum_{|s|\geq q} c_s u^s$. Hence there exists $\varepsilon \in (0,1)$ such that for sufficiently small $\sigma > 0$ we have

$$\max_{|u|=\sigma} |D(redD_0)(u)| \geq 3\varepsilon\sigma^q.$$

It follows that for some u_0, $|u_0| = \sigma$, we have $|D(redD_0)(u)| \geq 3\varepsilon\sigma^q$. But when $\sigma > 0$ is sufficiently small, the roots $s_i(u_0)$ of the polynomial $s \mapsto D_0(u_0, s)$ fulfils $|s_i(u_0)| \leq \frac{1}{2}$ therefore $|s_i(u_0) - s_j(u_0)| \geq 3\varepsilon\sigma^q$ for $s_i(u_0) \neq s_j(u_0)$. Observe that for

$$K_{\sigma_i} = \{|s - s_i(u_0)| < \varepsilon\sigma^k\}$$

we have $\overline{K_{\sigma_i}} \cap \overline{K_{\sigma_j}} = \emptyset$ (for different roots) and we have in the complement of $\bigcup_i \overline{K_{\sigma_i}}$ the inequality $|D_0(u,s)| \geq \varepsilon^r\sigma^{rq}$. Finally diminishing $\delta > 0$, when σ is sufficiently small, we have

$$|D(redF)(z)| \geq \delta\sigma^{rq}$$

in

$$G = G_\sigma \cup (U_\sigma \times (K_\sigma \setminus \bigcup_i \overline{K_{\sigma_i}})) \subset \Omega \times \Delta$$

for some open neighborhood $U_\sigma \subset B_\sigma$ of the point u_0. Taking $N > krq$, the condition $F(z, g(z)) = O(|z|^N)$ implies the inequality (i) for σ sufficiently small. Then there exists a holomorphic root h of the polynomial F in G. According to the Lemma 1, if $u \in U_\sigma$, the function $h(u,\cdot)$ extends holomorphically on $K_{\sigma'}$. Hence h extends holomorphically on $U_\sigma \times K_{\sigma'}$ (using Cauchy formula for $h(u,\cdot)$ on $\{|s| = \frac{3}{2}M\sigma^{\frac{1}{2}}\}$). Thus, from Hartogs Lemma, h extends holomorphically on $B_\sigma \times K_{\sigma'}$. In case g is holomorphic let us take moreover $N > kLr$. According to (ii) we have $|h(z) - g(z)| \leq \tilde{M}|z|^{Lr}$, with some $\tilde{M}$, in the set G_σ when σ is small enough. Hence in G_σ we have the inequality $|h(z) - g(z)| \leq M'\sigma^L$ with some constant M' and due to maximum principle this inequality holds also in $B_\sigma \times K_{\sigma'}$. So $h - g = O(|z|^L)$. $\square$

Remark. From the above proof it is easy to see that if $F(z,t)$ is a polynomial, monic in t, then there exists $N \in \mathbb{N}$ the same for all points $z \in \mathbb{C}^n$.

Theorem 2. *Let $Q(x,t)$ be a complex polynomial on $\mathbb{R}^{n+1}_{x,t}$, monic on t. Then there exists $m > 0$ such that if f, g are complex, continuous roots: $Q(x, f(x)) = Q(x, g(x)) = 0$ in a neighborhood of $a \in \mathbb{R}^n$, then the following implication holds:*

$$f(x) - g(x) = O(|x - a|^m) \Rightarrow f = g$$

in a neighborhood of a.

Proof. We can assume that $a = 0$ and the discriminant $D(Q)(z) \neq 0$ (replacing Q by $redQ$). Obviously, when $|x| \leq 1$ we have $|t''(x) - t'(x)| \geq M|D(Q)(x)|$ with some constant M for different roots $t''(x), t'(x)$ of the polynomial $t \mapsto Q(x,t)$. Let us take an open ball $B = \{|x| < R\}$, $R < 1$, in which f and g are defined. The number of connected components of the set $B \cap \{D(Q) \neq 0\}$ is finite and each connected component is semi-algebraic. Let S be one of them. Assume, that $0 \in \overline{S}$. The function

$$d : r \mapsto \sup_{S(r)} |D(r)|,$$

where $S(r) = S \cap \{|x| = r\}$, is positive semi-algebraic. Hence $d(r) \geq r^{m-1}$ in $(0, \delta)$ with common $m > 0$, $\delta > 0$ for all such components S. The assumption $f \not\equiv g$ in each neighborhood of zero implies $f \neq g$ on one of this components S, because the set $\{f = g\} \cap S$ is open-closed in S. But $|f(x) - g(x)| \geq M|D(Q)(x)|$ on S which implies $\sup_{S(r)} |f(x) - g(x)| \geq Mr^{m-1}$ for sufficiently small r. The last inequality contradicts our assumption that $f(x) - g(x) = O(|x|^m)$. □

Theorem 3. *Let Q be a real polynomial on $\mathbb{R}^{n+1}_{x,t}$, monic on t and let be $L > 0$. Then there exists $N \in \mathbb{N}$ such that the following implication holds: if $Q(x, \psi(x)) = O(|x - a|^N)$ for $x \to a$, with some real analytic function ψ defined in a neighborhood of $a \in \mathbb{R}^n$, then there exists an analytic function φ defined in a neighborhood of $a \in \mathbb{R}^n$ such that $Q(x, \varphi(x)) \equiv 0$ and $\varphi - \psi = O(|x - a|^L)$ for $x \to a$.*

Proof. Replacing Q and ψ by their complexification, and taking m from Theorem 2 we get (due to Theorem 1 with $L > m$) a holomorphic function φ defined in a neighborhood of a such that $Q(z, \overline{\varphi}(z)) \equiv 0$ and $\varphi - \psi = O(|z - a|^m)$. But then also $Q(z, \varphi(z)) \equiv 0$ holds. Moreover $\overline{\varphi} - \psi = O(|z - a|^m)$ which implies $\overline{\varphi} - \varphi = O(|z - a|^m)$. According to Theorem 2 we get $\overline{\varphi} = \varphi$ hence the function φ is real. □

3. Semialgebraicity of smooth points.

Lemma 3. *Let E be a semialgebraic subset of $\mathbb{R}^n$. If int $E = \emptyset$, then some nonzero polynomial on $\mathbb{R}^n$ vanishes on E.*

Proof. We have $E = \bigcup_{i=1}^q B_i$ with $B_i = \{P_i = 0\} \cap \{P_{ij} > 0\}$ with some polynomials P_i and P_{ij}. Now we can assume that each B_i is nonempty. Then $P_i \neq 0$, since otherwise B_i should be open. Hence $P_1 \cdot \ldots \cdot P_q$ is a required polynomial. □

Lemma 4. *Every semialgebraic set $E \subset \mathbb{R}^n$ is contained in an algebraic one $V \subset \mathbb{R}^n$ of the same dimension.*

Proof. Let $\pi_\alpha : \mathbb{R}^n \longrightarrow L_\alpha$ be the natural projections onto the spaces $L_\alpha = \mathbb{R}^{k+1}_{x_{\alpha_1}, \ldots, x_{\alpha_{k+1}}}$, where $\alpha = (\alpha_1, \ldots, \alpha_{k+1})$, $1 \leq \alpha_1 < \cdots < \alpha_{k+1} \leq n$, and $k = \dim E$. Then $\pi_\alpha(E)$ are semialgebraic of dimension at most k. Hence $\pi_\alpha(E) \subset \{P_\alpha = 0\}$ for a nonzero polynomial P_α. Therefore $E \subset V = \bigcap_\alpha \{P_\alpha \circ \pi_\alpha = 0\}$. Moreover $\dim V = k$, since otherwise V would contain a semialgebraic leaf Γ [1] of dimension $k + 1$ and so there would exists an α such that int $\pi_\alpha(\Gamma) \neq \emptyset$ that would imply $P_\alpha \equiv 0$. □

We say that a point $a \in E$ is a smooth point of dimension k of the set E if it has a neighborhood in E which is an analytic submanifold of dimension k. By definition the dimension of E is equal to the maximum of the dimension of its smooth points.

Note that every polynomial on $\mathbb{R}^n$ of degree $r > 0$ is monic of degree r with respect to each of its variables in some coordinate system.

[1] We call semialgebraic leaf any analytic submanifold which is a semialgebraic set.

Lemma 5. *Let $E \subset \mathbb{R}^n_x$ be a semialgebraic set of dimension $\leq k < n$. Then in some linear coordinate system E is contained in a Weierstrass set*

$$\{P_{k+1}(u, x_{k+1}) = \cdots = P_n(u, x_n) = 0\}, \qquad (*)$$

where $u = (x_1, \ldots, x_k)$, P_j is a monic polynomial on $\mathbb{R}^{k+1}_{u,x_j}$, $j = k+1, \ldots, n$. (Such a coordinate system we call a regular system for E).

Proof. By the previous lemma it suffices to give the proof for an algebraic set $V \supset E$ of dimension k. Let $\mathcal{P}_n \supset \cdots \supset \mathcal{P}_{k+1} \supset \mathcal{P}_k$ denote rings of polynomials on $\mathbb{R}^n_{u,x_{k+1},\ldots,x_n}, \ldots, \mathbb{R}^{k+1}_{u,x_{k+1}}$, $\mathbb{R}^k_u$ (after suitable identification). Let us denote by R_j the ring of restrictions of polynomials from $\mathcal{P}_j$ to V. Let $I = \{P \in \mathcal{P}_n \mid P_{|V} = o\}$. Changing in $\mathbb{R}^n, \ldots, \mathbb{R}^{k+1}$ successively coordinate system we find successively in $I \cap \mathcal{P}_j$ monic polynomials with respect to x_j. It means in particular that $w_j = x_{j|V}$ is integral over R_{j-1}. Evidently $R_j = R_{j-1}[w_j]$, it follows that $w_n, \ldots, w_{k+1}$ are integral over R_k, which means that there exist monic polynomials $P_j \in \mathcal{P}_k[x_j]$, $j = n, \ldots, k+1$, $P_{j|V} = 0$. □

We say that the submanifold $\Gamma \subset \mathbb{R}^n$ is *topographic* if it is the graph of an analytic mapping of an open subset of $\mathbb{R}^k$ into the space $\mathbb{R}^{n-k}$.

Theorem 4. *Let $E \subset \mathbb{R}^{k+1}_{u,v}$ be a semialgebraic set contained in the Weierstrass set $(*)$. Then the set*

$$\Lambda = \{x \in E \mid U \cap E \text{ is } k\text{-topographic submanifold for some nghb } U \text{ of } x\}$$

is semialgebraic.

Proof. For $a \in \mathbb{R}^k_u$, $b \in \mathbb{R}^l_v$ and $\delta, \varepsilon > 0$ let us denote $U_{ab\delta\varepsilon} = B(a,\delta) \times B(b,\varepsilon)$, where $B(a,\delta) = \{|u - a| < \delta\}$, $B(b,\varepsilon) = \{|v - b| < \varepsilon\}$. Let $(a, b) \in E$. The set $E_{ab\delta\varepsilon} = E \cap U_{ab\delta\varepsilon}$ is the graph of some continuous function $B(a,\delta) \longrightarrow \mathbb{R}^l_v$ if and only if $\overline{E} \cap U_{ab\delta\varepsilon} \subset E$ and for $u \in B(a,\delta)$ we have $(u, v) \in \overline{E} \cap U_{ab\delta\varepsilon}$ for exactly one v. Thus the set

$$F = \{(a, b, \delta, \varepsilon) \mid E_{ab\delta\varepsilon} \text{ is the graph of some continuous function on } B(a,\delta)\}$$

is semialgebraic.

Let us take m from Theorem 2, the same, for all polynomials P_j, and $N > m$ from Theorem 3 also the same for all polynomials P_j, and $L = m$. For $c = \{c_\alpha\}_{|\alpha| \leq N}$, where $c_\alpha \in \mathbb{R}^l$ and $\alpha \in \mathbb{N}^k$, we define the polynomial mapping $P_c : u \mapsto \sum_{|\alpha|} c_\alpha u^\alpha$. In the sequel we write $(c_0, c') = c$. For $a \in \mathbb{R}^k_u$, $c, C, N, \delta > 0$ we define the subset of $\mathbb{R}^{k+l}_{u,v}$:

$$W_{acCN\delta} = \{u \in B(a,\delta),\ |v - P_c(u - a)| \leq C|u - a|^N\}.$$

It is enough to show that

$$\Lambda = \{(a, b) \in \mathbb{R}^{k+l}_{u,v} \mid (a, b, \delta, \varepsilon) \in F \text{ and } E_{ab\delta\varepsilon} \subset W_{a(b,c')CN\delta} \text{ with some } \delta, \varepsilon, c', C\} \qquad (**)$$

because the latest set is semi-algebraic. Let $(a, b) \in \Gamma$. Obviously for some sufficiently small $\delta, \varepsilon > 0$ we have $(a, b, \delta, \varepsilon) \in F$ and the set $E_{ab\delta\varepsilon}$ is the graph of some Nash function φ; taking c' such that $x \mapsto P_{(b,c')}(x - a)$ is N-th Taylor polynomial for φ we have $E_{ab\delta\varepsilon} = \varphi \subset W_{a(b,c')CN\delta}$ with some C when δ is sufficiently small.

Now, let us assume, that the condition $(**)$ holds. Then $E_{ab\delta\varepsilon}$ is the graph of some continuous function $\psi : B(a,\delta) \longrightarrow \mathbb{R}^l_v$. Hence $Q(u,\psi(u)) = 0$ on $B(a,\delta)$, where $Q = (P_{k+1},\ldots,P_n)$. The graph of ψ is contained in the set $W_{acCN\delta}$. Thus $\psi(u) - P_c(u-a) = O(|u-a|^N)$. It follows that $Q(u, P_c(u-a)) = O(|u-a|^N)$. Due to the Theorem 3 there exists an analytic function φ defined in a neighborhood of a such that $Q(u,\varphi(u)) \equiv 0$ and $\varphi(u) - P_c(u-a) = O(|u-a|^m)$. Hence $\varphi - \psi = O(|u-a|^m)$ and according to the Theorem 2 we have $\varphi = \psi$ in a neighborhood of a. □

Theorem 5. *Let $E \subset \mathbb{R}^n$ be a semi-algebraic set. Then the set of its smooth points of dimension k $E^{(k)}$ is semi-algebraic as well.*

Proof. It is enough to prove the theorem for $k = \dim E$, because we can proceed by induction.

For each $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ isomorphism we define

$$\lambda(\varphi) = \varphi^{-1}(e_{k+1}) \wedge \cdots \wedge \varphi^{-1}(e_n) \in \Lambda^{n-k}\mathbb{R}^n,$$

where $e_1,\ldots,e_n$ is a canonical basis of $\mathbb{R}^n$. We say that a sequence of linear isomorphisms $\varphi_1,\ldots,\varphi_r$ of $\mathbb{R}^n$ is *complete* if $\lambda(\varphi_1),\ldots,\lambda(\varphi_r)$ generate $\Lambda^{n-k}\mathbb{R}^n$. One can easily prove that:

(1) there exists complete sequence,
(2) any sequence that is sufficiently closed (in the natural topology) to a complete one is complete,
(3) if $\varphi_1,\ldots,\varphi_r$ is a complete sequence and the set E is smooth of dimension k at a then $\varphi_\nu(E)$ is k-topographic at $\varphi_\nu(a)$ for some ν.

We find a complete sequence $\varphi_1,\ldots,\varphi_r$ such that each φ_i is regular system for E. According to the Theorem 5 the set R_ν of points in which $\varphi_\nu(E)$ is k-topographic is a semi-algebraic set. Hence

$$E^{(k)} = \bigcup \varphi_\nu^{-1}(R_\nu)$$

is semi-algebraic as finite sum of semi-algebraic sets. □

References

[A] M. Artin *On the solutions of analytic equations.* Invent. Math. **5**, 277-291 (1968)

[Ł1] S. Łojasiewicz *Ensembles semi-analitiques.* IHES, Bures-sur-Yvette (1965)

[Ł2] S. Łojasiewicz *Introduction to complex analytic geometry.* Birkhäuser (1991)

(Jacek Stasica) Instytut Matematyki, Uniwersytet Jagielloński, ul. Reymonta 4, 30-059 Kraków, Poland

E-mail address: jacek.stasica@im.uj.edu.pl