About some representations of integer numbers

Maciej Ulas

IMUJ Preprint 2003/03

Abstract

We consider a problem of representation of integer numbers in the form $k = \pm 1^s \pm 2^s \pm ... \pm m^s$, where $s \in \mathbb{N}_+$. We prove that, for each s, an integer k can be represented in this form, where $m \in \mathbb{N}_+$ and signs $+$ and $-$ are suitably selected, we next give some generalizations of this fact. [1]

Introduction

Problems considered in this paper arise from lecture of Wacław Sierpiński's book *„250 problems in elementary theory of numbers"* ([2]). One of problems stated there is:

Prove, that every integer k can be represented in infinitely many ways as $k = \pm 1^2 \pm 2^2 \pm ... \pm m^2$, where m is an integer and signs $+$ and $-$ are suitable selected.

The questions arises, whether a similar result can be obtained for higher powers? We ask if the following statement is true:

For each integer k and exponent s, k can be presented in infinitely many ways as $k = \pm 1^s \pm 2^s \pm ... \pm m^s$, where m is an integer, and signs $+$ and $-$ are suitable selected .

Of course, while moving forward we may state a more general question:

Whether for each polynomial $f(x)$ with integer values for $x \in \mathbb{N}$ and for each $k \in \mathbb{N}$, can k be presented in infinitely many ways as

[1]2000 *Mathematic Subject Classification.* 11A99, 11P05

Key words and phrases. Integer numbers, representations of integer numbers

$k = \pm f(1) \pm f(2) \pm ... \pm f(m)$, for some integer m, and suitable signs $+$ and $-$ selection ?

In this work we are going to show that some generalizations of the problem stated in Sierpiński's book are true. More precisely we prove, that the answer to the above question is "yes" for polynomials $f_s(x) = x^s$ (**Theorem 3.1.**) and more generally for $f \in \mathbb{Z}[X]$, fulfilling additional condition $f(1) = 1$ (**Theorem 3.2.**)

We start with some special cases which are rather easy to see and can give a good introduction to this problematic and to be a good presentation of main tools we use in general cases.

1. First theorems - special cases

Theorem 1.1.

Each integer k can be represented in infinitely many ways as $k = \pm 1^2 \pm 3^2 \pm ... \pm (2m-1)^2$, where $m \in \mathbb{N}$, and signs $+$ and $-$ are suitably selected.

Proof: First, we are going to prove the existence of at least one representation for each $k \in \mathbb{N}$. To do this we will use the following identity:

$$(*) \quad (2m+1)^2 - (2m+3)^2 - (2m+5)^2 + (2m+7)^2 = 16.$$

On the basis of $(*)$ we can see, that it is enough to show, that natural number $k \leq 16$ can be obtained in demanded form.

Indeed, if there exist $m \in \mathbb{N}$ and the arrangement of signs $+$ and $-$ such as

$$k = \pm 1^2 \pm 3^2 \pm ... \pm (2m-1)^2,$$

then

$$k + 16 = \pm 1^2 \pm 3^2 \pm ... \pm (2m-1)^2 + (2m+1)^2 - (2m+3)^2 - (2m+5)^2 + (2m+7)^2.$$

Next we check directly that suitable representations exist for $k \leq 16$.

$0 = -1^2 + 3^2 + 5^2 - 7^2 + 9^2 - 11^2 - 13^2 + 15^2$
$1 = 1^2$
$2 = 1^2 + 3^2 + 5^2 - 7^2 + 9^2 - 11^2 - 13^2 + 15^2$
$3 = 1^2 + 3^3 + 5^2 + 7^2 - 9^2$
$4 = -1^2 - 3^2 - 5^2 - 7^2 + 9^2 - 11^2 - 13^2 + 15^2 - 17^2 + 19^2$
$5 = 1^2 + 3^2 + 5^2 + 7^2 - 9^2 + 11^2 + 13^2 + 15^2 + 17^2 - 19^2 - 21^2$
$6 = -1^2 - 3^2 + 5^2 - 7^2 - 9^2 + 11^2$
$7 = 1^2 + 3^2 + 5^2 - 7^2 + 9^2 + 11^2 + 13^2 + 15^2 + 17^2 + 19^2 + 21^2 + 23^2$
$\quad -25^2 - 27^2 - 29^2$

$8 = -1^2 + 3^2$
$9 = -1^2 - 3^2 - 5^2 - 7^2 - 9^2 + 11^2 - 13^2 - 15^2 - 17^2 - 19^2 - 21^2 - 23^2 + 25^2$
$+27^2 + 29^2 + 31^2 - 33^2$
$10 = 1^2 + 3^2$
$11 = -1^2 - 3^2 + 5^2 - 7^2 - 9^2 - 11^2 - 13^2 - 15^2 + 17^2 - 19^2 - 21^2 + 23^2 + 25^2$
$12 = -1^2 - 3^2 - 5^2 - 7^2 - 9^2 + 11^2 - 13^2 + 15^2$
$13 = -1^2 - 3^2 - 5^2 - 7^2 + 9^2 + 11^2 - 13^2 - 15^2 + 17^2$
$14 = -1^2 - 3^2 - 5^2 + 7^2$
$15 = -1^2 - 3^2 + 5^2$
$16 = 1^2 - 3^2 - 5^2 + 7^2$

We have thus proved, that each $k \in \mathbb{N}$ can be for at least one way represented as $k = \pm 1^2 \pm 3^2 \pm ... \pm (2m-1)^2$.

It rest only to show, that we can do this in infinitely many ways. For this purpose, in $(*)$ we input $m+4$ in the place of m and substracting by sides we get the following identity:

$$(**) \qquad (2m+1)^2 - (2m+3)^2 - (2m+5)^2 + (2m+7)^2$$
$$-(2m+9)^2 + (2m+11)^2 + (2m+13)^2 - (2m+15)^2 = 0.$$

By this property we get, that if k can be represented in considered form with some m then we will achieve a good form taking $m+8, m+16$ and so on. This proves, that k can be presented in infinitely many ways in this form. $\square$

Now we denote $n-$th triangular number by t_n, this means $t_n = \frac{n(n+1)}{2}$. As we know $t_n = \sum_{i=1}^{n} i$.

Theorem 1.2.

Each integer k can be represented in infinitely many ways as $k = \pm t_1 \pm t_2 \pm ... \pm t_m$, where $m \in \mathbb{N}$, and signs $+$ and $-$ are suitably selected.

Proof: Justification of this fact we ground on the following identity

$$(*) \quad t_{m+1} - t_{m+2} - t_{m+3} + t_{m+4} = 2$$

and equalities:
$0 = -t_1 - t_2 - t_3 + t_4$
$1 = t_1$
$2 = -t_1 + t_2$

Applying similar reasoning as in the proof of previous theorems we obtain our result. $\square$

Now we denote $n-$th tetrahedron numbers by T_n , this means $T_n = \frac{n(n+1)(n+2)}{6}$. As we know $T_n = \sum_{i=1}^{n} t_i$.

Theorem 1.3.

Each integer k can be represented in infinitely many ways as $k = \pm T_1 \pm T_2 \pm \ldots \pm T_m$, where $m \in \mathbb{N}$, and signs $+$ and $-$ are suitably selected.

Proof: Our considerations in this case are carried out as in the proofs of previous theorems, therefore we sketch the proof presenting a suitable identity and considering the necessary special cases. We have

$$(*) \quad -T_{m+1} + T_{m+2} + T_{m+3} - T_{m+4} + T_{m+5} - T_{m+6} - T_{m+7} + T_{m+8} = 8$$

and equalities:

$0 = -T_1 - T_2 - T_3 - T_4 + T_5$
$1 = T_1$
$2 = T_1 - T_2 - T_3 - T_4 + T_5$
$3 = -T_1 + T_2$
$4 = -T_1 - T_2 + T_3 + T_4 + T_5 - T_6$
$5 = T_1 + T_2$
$6 = -T_1 - T_2 + T_3 - T_4 - T_5 + T_6$
$7 = T_1 - T_2 + T_3$
$8 = -T_1 + T_2 - T_3 - T_4 + T_5$

Repeating the same reasoning as in the previous theorem we obtain our result. □

Note that a similar result can be obtained for polynomial $f(x) = x^3$. In the proof we must check, that this fact is true for the $k \in \mathbb{N}$, and $k < 48$ (direct calculations) and apply the identity

$$\begin{aligned} &-(x+1)^3 + (x+2)^3 + (x+3)^3 - (x+4)^3 \\ &+(x+5)^3 - (x+6)^3 - (x+7)^3 + (x+8)^3 = 48. \end{aligned}$$

2. Reduction of the general situation

As we can see analyzing carefully the proofs from the first paragraph in each case the ground of the result is an algebraic identity that allows a reduction of the problem just to checking finite number of cases.

The basic question is:

Is it possible to find an identity of the type

$$\varepsilon_1 g(x+1) + \varepsilon_2 g(x+2) + \ldots + \varepsilon_r g(x+r) = C(g)$$

where $r \in \mathbb{N}$, $\varepsilon_i \in \{-1, 1\}$, and $C(g)$ is a constant for each $g \in \mathbb{R}[x]$?

The following theorem gives an answer for this question.

Theorem 2.1.

Let $g(x) = a_s x^s + a_{s-1}x^{s-1} + ... + a_0 \in \mathbb{R}[x]$ *non-zero polynomial of degree* s. *Then there exist* $r \in \mathbb{N}$, $C(g) \in \mathbb{R}$, $\varepsilon_i \in \{-1, 1\}$ *such that*

$$(*) \quad \varepsilon_1 g(x+1) + \varepsilon_2 g(x+2) + ... + \varepsilon_r g(x+r) = C(g).$$

Proof: First we notice, that applying Newton's binomial theorem for a polynomial $f(x) = b_n x^n + ... + b_0 \in \mathbb{R}[x]$ of degree $n > 0$ and for $c, d \in \mathbb{R}$ we obtain that $h(x) := -f(x+c) + f(x+d)$ is a polynomial of degree lower then n.

Applying now this remark to our polynomial $g(x) = a_s x^s + a_{s-1}x^{s-1} + ... + a_0 \in \mathbb{R}[x]$, we construct a sequence of polynomials of decreasing degrees in the following way:

$$\begin{aligned}
h_1(x) &:= -\ g(x+1) + g(x+2),\\
h_2(x) &:= -\ h_1(x) + h_1(x+2),\\
h_3(x) &:= -\ h_2(x) + h_2(x+2^2),\\
&\ \ \vdots\\
h_s(x) &:= -\ h_{s-1}(x+1) + h_{s-1}(x+2^{s-1}).
\end{aligned}$$

We notice, that this sequence fulfills the following properties:

(1) $\deg(h_{i+1}) < \deg(h_i)$ for $i \in \{1, ..., s-1\}$.

(2) $h_i(x) = \varepsilon_1 g(x+1) + \varepsilon_2 g(x+2) + ... + \varepsilon_{2^i} g(x+2^i)$, for $i \in \{1, ..., s\}, \varepsilon_i \in \{-1, 1\}$

The first property can be seen directly from the remark made at the beginning of the proof.

The second one we prove by induction:

For $i = 1$ it is enough to take $\varepsilon_1 := -1$ and $\varepsilon_2 := 1$.

Now, if $h_i(x) = \varepsilon_1 g(x+1) + \varepsilon_2 g(x+2) + ... + \varepsilon_{2^i} g(x+2^i)$, for $i \in \{2, ..., s-1\}$, then

$$h_{i+1}(x) = -h_i(x) + h_i(x+2^i) = -\varepsilon_1 g(x+1) - \varepsilon_2 g(x+2) - ...$$
$$-\varepsilon_{2^i} g(x+2^i) + \varepsilon_1 g(x+1+2^i) + \varepsilon_2 g(x+2+2^i) + ... + \varepsilon_{2^i} g(x+2^{i+1})$$

so it has a demanded form.

According to the above properties we can see, that polynomial $h_s(x)$ has a degree zero, so we can take $C(g) := h_s(x)$ and $r := 2^s$. $\square$

Now an obvious question appears:

Is there any possibility to calculate explicitly the constant $C(g)$*?*

It turns out that the answer to this question is affirmative.

First we introduce some notations and recall a theorem which will be useful in the proof.

For $h_1 \in \mathbb{R}$ and a function $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ define:

$$\triangle_{h_1}\varphi : \mathbb{R} \longrightarrow \mathbb{R}$$

as

$$(\triangle_{h_1}\varphi)(x) = \varphi(x + h_1) - \varphi(x).$$

Next, inductively having $\triangle_{h_1} ... \triangle_{h_{n-1}} \varphi$ and $h_n \in \mathbb{R}$, we define

$$(\triangle_{h_1} ... \triangle_{h_n} \varphi)(x) := \triangle_{h_1} ... \triangle_{h_{n-1}} \varphi(x + h_n) - \triangle_{h_1} ... \triangle_{h_{n-1}} \varphi(x)$$

Theorem 2.2. ([1])

Let $\varphi \in \mathbb{R}[x]$ be a polynomial of degree n, $\varphi = \varphi_0 + ... + \varphi_n$ where $\varphi_i : \mathbb{R} \longrightarrow \mathbb{R}$ is the homogeneous part of φ of degree $i = 0, ..., n$. $g_n : \mathbb{R}^n \longrightarrow \mathbb{R}$ – a symmetric polynomial function such, that

$$\varphi_n(x) = g_n(x, ..., x).$$

Then for $(h_1, \ldots , h_n) \in \mathbb{R}^n$:
(1)The first difference: $\triangle_{h_1}\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ is a polynomial of the degree $\leq n - 1$.
(2) The n-th difference $\triangle_{h_1} ... \triangle_{h_n} \varphi$ is a constant equal to:

$$(\triangle_{h_1} ... \triangle_{h_n} \varphi)(x) = n! g_n(h_1, ..., h_n)$$

(doesn't depend on $x \in \mathbb{R}$.)

We are ready to formulate and prove the following theorem.

Theorem 2.3.

Let $f_s(x) = a_s x^s + a_{s-1}x^{s-1} + ... + a_0 \in \mathbb{R}[x]$, where $s \in \mathbb{N}_+$, $a_s \neq 0$. Then the constant $C(f_s) = a_s s!\, 2^{\frac{s(s-1)}{2}}$.

Proof: We apply **Theorem 2.2** to $\varphi(x) = f_s(x)$, $g_s(x_1, ..., x_s) = a_s x_1 ... x_s$ and $h_i = 2^{i-1}$ for $i = 1, 2, ..., s$. Then $\varphi_s(x) = a_s x^s$ and we get q that

$$\begin{aligned} C(f_s) = \triangle_{h_1} ... \triangle_{h_s} \varphi = a_s\ s!\ g_s(h_1, ..., h_s) = a_s\ s!\ 2^{0+1+...+s-1} \\ = a_s\ s!\ 2^{\frac{s(s-1)}{2}}. \end{aligned}$$

$\square$

3. Polynomial $f_s(x) = x^s$

Now, we can prove the theorem which was the aim of this work.

Theorem 3.1.

Each integer k can be represented in infinitely many ways as $k = \pm 1^s \pm 2^s \pm ... \pm m^s$, where $m, s \in \mathbb{N}$, and signs $+$ and $-$ are suitably selected.

Proof: Before we step forward to the proof, we notice, that is enough to prove the following fact:
For each natural number l, there is $n \in \mathbb{N}$ and a sequence $\varepsilon_i^{'} \in \{-1, 1\}$, $i = 1, ..., n$, and a constant C such that:

$$\sum_{i=1}^{n} \varepsilon_i^{'} i^s \equiv l \; mod(C). \tag{1}$$

This will mean, that

$$\sum_{i=1}^{n} \varepsilon_i^{'} i^s + aC = l \tag{2}$$

for $a \in \mathbb{Z}$. In this case $C := C(f_s) = s! \; 2^{\frac{s(s-1)}{2}}$.

Why does it make our theorem true? It is enough to go back to the identity from thesis of **Theorem 2.1.** Without loss of generality we can assume, that $a \in \mathbb{N}_+$. When in this identity we replace x by $n + j2^s$ for $j = 0, ..., a-1$ and add by sides we get, that

$$aC = \sum_{i=1}^{2^s} \varepsilon_i (n+i)^s + ... + \sum_{i=1}^{2^s} \varepsilon_i (n + (a-1)2^s + i)^s \tag{3}$$

We have represented aC as a sum of following s-th powers of the following natural numbers starting with the expression $\varepsilon_1 (n+1)^s$. But the sum

$$\textstyle\sum_{i=1}^{n} \varepsilon_i^{'} i^s$$

ends up with the expression $\varepsilon_n^{'} n^s$, so we have a representation of the integer l in the form: $l = \varepsilon_i 1^s + \varepsilon_i 2^s + ... + \varepsilon_i m^s$, where $\varepsilon_i^{'} \in \{-1, 1\}$.

It rest only to prove a congruence (1). Let $l = 2p + \varepsilon$, where $\varepsilon \in \{0, 1\}$. The following congruences take place :

$$\textstyle\sum_{i=1}^{C} i^s + (C+1)^s - \sum_{i=2}^{C} (C+i)^s \equiv 2 \; mod(C),$$

$$\textstyle\sum_{i=1}^{C} (2C+i)^s + (3C+1)^s - \sum_{i=2}^{C} (3C+i)^s \equiv 2 \; mod(C),$$

$$...$$

$$\textstyle\sum_{i=1}^{C} ((2(p-1)C+i)^s + ((2p-1)C+1)^s - \sum_{i=2}^{C} ((2p-1)C+i)^s \equiv 2 \; mod(C).$$

Summing by sides, on the left side we get an expression of the form $\sum_{i=1}^{n} \varepsilon_i^{'} i^s$ for some $n \in \mathbb{N}$, and on the right side $2p$, so a congruence

$$\sum_{i=1}^{n} \varepsilon_i^{'} i^s \equiv 2p \; mod(C).$$

This proves the theorem in the case when l is even. In the case when l is odd, we consider additionally the following congruence

$$(2pC + 1)^s \equiv 1 \; mod(C)$$

and we obtain our result. □

Now, we will prove a generalization of the **Theorem 3.1.**

Theorem 3.2.

Let $f(x) = a_s x^s + ... + a_0 \in \mathbb{Z}[x]$, where $a_s \neq 0, s \in \mathbb{N}_+$, and $f(1) = 1$. Then every integer k can be represented in infinitely many ways as $k = \pm f(1) \pm f(2) \pm ... \pm f(m)$, where $m, s \in \mathbb{N}$, and signs $+$ and $-$ are suitably selected.

Proof: We will use the same method as in the proof of **Theorem 3.1.** From the **Theorem 2.1** results the existence of the constant $C = C(f)$ for this polynomial, which is an integer. So we will show, that

For each integer l, there is $n \in \mathbb{N}$ and the sequence $\varepsilon_i^{'} \in \{-1, 1\}$, $i = 1, ..., n$, such that

$$\sum_{i=1}^{n} \varepsilon_i^{'} f(i) \equiv l \; mod(C).$$

Let then $l = 2p + \varepsilon$, where $\varepsilon \in \{0, 1\}$. We have the following congruences:

$$f(1) \equiv 1 \; mod(C),$$
$$aC + i \equiv i \; mod(C),$$
$$f(aC + i) \equiv f(i) \; mod(C).$$

From this links we obtain the following relation:

$$\sum_{i=1}^{C} f(aC + i) + f((a+1)C + 1) - \sum_{i=2}^{C} f((a+1)C + i) \equiv \; 2 mod(C)$$

This relation is true for each $a \in \mathbb{Z}$.

Inserting now in turn $a = 0$, $a = 1$, ... , $a = 2(p-1)$ and summing by sides above congruences, we get an expression of the form

$$\sum_{i=1}^{n} \varepsilon_i^{'} f(i) \equiv 2p \; mod(C)$$

what gives our result for even numbers. If l is odd, we add to above congruence

$$f(2pC + 1) \equiv 1 \; mod(C)$$

and we get the theorem in general. □

4. Final remarks

It would be interesting to know what conditions must fulfill a polynomial $f \in \mathbb{Z}[x]$ with the property considered in the introduction. Of course, it would be perfect to find an equivalent condition for this property depending only on polynomial coefficients.

Now, I will present a necessary condition – which is unfortunately no efficient one.

Theorem 4.1.

Let $g(x) = a_s x^s + a_{s-1}x^{s-1} + ... + a_0 \in \mathbb{Z}[x]$ *a polynomial of degree* $s \in \mathbb{N}_+$*, such, that each integer* k *can be presented as*

$$k = \varepsilon_1 g(1) + \varepsilon_2 g(2) + ... + \varepsilon_m g(m),$$

where $m \in \mathbb{N}$ *and* $\varepsilon_i \in \{-1, 1\}$*. Then* $\mathrm{GCD}(a_0, a_1, ..., a_s) = 1$.

Proof: For the proof assume, that $\mathrm{GCD}(a_0, a_1, ..., a_s) = d$ i $d > 1$. Then, it means, that $d \mid g(i)$ for $i \in \mathbb{N}$. We get, that

$$d \mid \varepsilon_1 g(1) + \varepsilon_2 g(2) + ... + \varepsilon_m g(m)$$

for each natural number m and each $\varepsilon_i \in \{-1, 1\}$. This contradicts the assumption, because it would be, that $d \mid 1$, but $d > 1$. The contradiction ends the proof of the theorem. $\square$

It is easy to give an example showing that in above theorem the implication cannot be replaced by equivalence. We consider the polynomial $g(x) = x(x+1)$. We have $\mathrm{GCD}(1, 1) = 1$, so the assumed condition is fulfilled, but it is clear, that $2 \mid g(m)$ for each $m \in \mathbb{N}$. It results, that no odd numbers can be represented in considered form.

We can therefore ask, if to the **Theorem 4.1.** we add the condition, that there is no such number p, that, $p \mid g(m)$ for $m \in \mathbb{N}$ is enough to get the equivalence? I think that the answer for this question is YES. Unfortunately I cannot present a proof for an arbitrary polynomial. The strong argument for the truth of this fact is the **Theorem 3.2.**

We can notice, that the problem can be generalized to polynomials with real coefficients. Let $f(x) = \frac{1}{4}x^2 + ax + b$, $g(x) = \frac{1}{48}x^3 + cx^2 + dx + e$ where $a, b, c, d, e \in \mathbb{R}$. For this polynomials there are following links

$$f(x+1) - f(x+2) - f(x+3) + f(x+4) = 1$$

and

$$\begin{aligned} &-g(x+1) + g(x+2) + g(x+3) - g(x+4) \\ &+g(x+5) - g(x+6) - g(x+7) + g(x+8) = 1 \end{aligned}$$

Substituting in f, $a = \sqrt{2}$, $b = 0$, and in g, $c = \sqrt{2}$, $d = 0$, $e = 0$ we obtain polynomials having considered form and without integer value for $x \in \mathbb{N}$.

Acknowledgments

I am especially grateful to Ewa Cygan, Piotr Tworzewski for every suggestions and their help during preparation of this work, and Witold Jarnicki for showing the right way to the proof of **Theorem 3.1.**

References

1. H. Cartan, *Calcul differential, formes differentielles,* Hermann, Paris, 1967.

2. W. Sierpiński, *250 Problems in Elementary Number Theory* , Elsevier, New York, 1970.

e-mail: maciejulas@poczta.onet.pl