A POLYNOMIAL WITH $2k$ CRITICAL VALUES AT INFINITY

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Abstract

We construct a polynomial $f : \mathbb{C}^2 \to \mathbb{C}$ of degree $4k+2$ with no critical points in $\mathbb{C}^2$ and with $2k$ critical values at infinity.

Let $f : \mathbb{C}^2 \to \mathbb{C}$ be a polynomial. We call $\lambda \in \mathbb{C}$ a critical value at infinity of f if there exists a sequence $z_n \in \mathbb{C}^2$ such that $z_n \to \infty$, $\mathrm{grad} f(z_n) \to (0,0)$ and $f(z_n) \to \lambda$ as $n \to \infty$. Several equivalent definitions of critical values at infinity are given in [Durfee]. The definition we use is due to Ha (see [Durfee]).

When studying polynomial automorphisms of $\mathbb{C}^2$ we encounter polynomials with no critical points in $\mathbb{C}^2$ (see, for example [Ephraim], [Gwoździewicz–Płoski]). It is natural to ask if a polynomial with no critical points in $\mathbb{C}^2$ can have many critical values at infinity.

For given integer $k > 0$ we construct a polynomial $f : \mathbb{C}^2 \to \mathbb{C}$ of degree $4k+2$ such that

(i) f has no critical points in $\mathbb{C}^2$,

(ii) f has exactly $2k$ critical values at infinity.

Let $A(x) = x^k - 1$, $B(x) = \frac{x^{k+1}}{k+1} - x$ and

$$f(x,y) = \left(yA^2 - x\right)^2 + B.$$

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To prove (i) let us recall that critical points of f are solutions of the system of equations $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$, so we have

$$\begin{cases} 2(yA^2 - x)(2yAA' - 1) + A = 0, \\ 2(yA^2 - x)A^2 = 0. \end{cases}$$

The second equality gives $A = 0$ or $yA^2 - x = 0$. If $A = 0$ then from the first equality we get $x = 0$ which is impossible because $A(0) \neq 0$. If we assume that $yA^2 - x = 0$ then $A = 0$ and we get a contradiction again.

Now we will show that all critical values at infinity of f are of the form $-\frac{k}{k+1}a$ and $a^2 - \frac{k}{k+1}a$ for a to be any root of the polynomial A. Assume that $z_n = (x_n, y_n)$ is a sequence which satisfies the four conditions

$$z_n \to \infty, \tag{1}$$

$$\frac{\partial f}{\partial x}(z_n) = 2(y_n A(x_n)^2 - x_n)(2y_n A(x_n)A'(x_n) - 1) + A(x_n) \to 0, \tag{2}$$

$$\frac{\partial f}{\partial y}(z_n) = 2(y_n A(x_n)^2 - x_n)A(x_n)^2 \to 0, \tag{3}$$

$$f(z_n) = \big(y_n A(x_n)^2 - x_n\big)^2 + B(x_n) \to \lambda \in \mathbb{C}. \tag{4}$$

Replacing z_n by a subsequence we may additionally assume that $x_n \to \infty$ or $x_n \to b \in \mathbb{C}$.

If x_n tends to infinity then by (3) $y_n A(x_n)^2 - x_n \to 0$. Hence $f(z_n) \to \infty$ (because $B(x_n) \to \infty$) which contradicts (4).

Now let us suppose that x_n converges to $b \in \mathbb{C}$ and b is not a root of the polynomial A. From (3) we have $y_n A(x_n)^2 - x_n \to 0$ which gives $y_n \to b/A(b)^2$. Hence the sequence z_n is bounded contrary to (1). So we have proven that $x_n \to a$ where a is a root of A.

¿From (4) we see that the sequence $y_n A(x_n)^2 - x_n$ is convergent. There are two possibilities. The first one is $y_n A(x_n)^2 - x_n \to 0$. Then $f(z_n) \to B(a) = -\frac{k}{k+1}a$. The second one is $y_n A(x_n)^2 - x_n \to \alpha \neq 0$. Then according to (2) we have $2y_n A(x_n)A'(x_n) - 1 \to 0$ otherwise $\frac{\partial f}{\partial x}(z_n)$ would not be convergent to zero. But we have

$$y_n A(x_n)^2 - x_n = \frac{A(x_n)}{2A'(x_n)}[(2y_n A(x_n)A'(x_n) - 1) + 1] - x_n \to -a.$$

So we see from (4) that

$$f(z_n) \to a^2 + B(a) = a^2 - \frac{k}{k+1}a.$$

Finally we obtain that the critical values at infinity of the the polynomial f are of the form $-\frac{k}{k+1}a$ and $a^2 - \frac{k}{k+1}a$ for $a \in A^{-1}(0)$. Taking $x_n \to a$, $y_n = x_n/A(x_n)^2$ we get $f(z_n) \to -\frac{k}{k+1}a$ and taking $x_n \to a$, $y_n = 1/(2A(x_n)A'(x_n))$ we get $f(z_n) \to a^2 - \frac{k}{k+1}a$.

It suffices to show that the above numbers are pairwise different. We will show that if $a \neq b$ and $a, b \in A^{-1}(0)$ then

$$a^2 - \frac{k}{k+1}a \neq b^2 - \frac{k}{k+1}b,$$
$$a^2 - \frac{k}{k+1}a \neq -\frac{k}{k+1}b.$$

The first inequality can be rewritten as $(a-b)(a+b-\frac{k}{k+1}) \neq 0$ so it is enough to show that the equality $a+b = \frac{k}{k+1}$ does not hold. If the above equality is true then a and b must be conjugate so we get the equality $a+\bar{a} = \frac{k}{k+1}$. Multiplying it by $(k+1)a$ we get

$$(k+1)a^2 - ka + k + 1 = 0,$$
$$a^k - 1 = 0.$$

We will show that these equalities can not hold. To do this let us consider the resultant R of the polynomials $(k+1)x^2 - kx + k + 1$ and $x^k - 1$ i.e.

$$R = \begin{vmatrix} k+1 & -k & k+1 & 0 & \dots & 0 \\ 0 & k+1 & -k & k+1 & \dots & 0 \\ \vdots & & \ddots & & & \vdots \\ 0 & \dots & k+1 & -k & k+1 & 0 \\ 0 & \dots & 0 & k+1 & -k & k+1 \\ 1 & 0 & \dots & & -1 & 0 \\ 0 & 1 & 0 & \dots & & -1 \end{vmatrix}$$

and the natural homomorphism $\phi : \mathbb{Z} \to \mathbb{Z}/(k+1)\mathbb{Z}$ of the ring of integers to the ring of rests of modulo $k+1$. Then

$$\phi(R) = \begin{vmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & & \ddots & & & \vdots \\ 0 & \dots & 0 & 1 & 0 & 0 \\ 0 & \dots & 0 & 0 & 1 & 0 \\ 1 & 0 & \dots & & -1 & 0 \\ 0 & 1 & 0 & \dots & & -1 \end{vmatrix} = (-1)^{k+1}$$

hence we see that $R \neq 0$.

To end the proof we need to show that the equality

$$a^2 - \frac{k}{k+1}a = -\frac{k}{k+1}b \tag{5}$$

is false for any $a, b \in A^{-1}(0)$. For that purpose let us notice that if $k = 1$ then $a = b = 1$ and $a^2 - \frac{1}{2}a = \frac{1}{2} \neq -\frac{1}{2} = -\frac{1}{2}b$. Let us suppose that $k > 1$. The equality (5) can be written as

$$\frac{k+1}{k} = a^{-1} - ba^{-2} = a^{-1} + (-ba^{-2}).$$

We see that the numbers a^{-1} and $-ba^{-2}$ must be conjugate hence $-ba^{-2} = a$. Multiplying the above equality by ka we get

$$ka^2 - (k+1)a + k = 0.$$

Of course $a^k - 1 = 0$. By using the same method as earlier ie. using a resultant modulo k we can show that the system of equations

$$\begin{aligned} &kx^2 - (k+1)x + k = 0, \\ &x^k - 1 = 0 \end{aligned}$$

has no solutions for $k > 1$. This ends the proof.

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