ON THE CONTACT EXPONENT OF PLANE CURVE SINGULARITIES

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Abstract

We give an explicit formula for the contact exponent (in the sense of Hironaka) of plane curve singularities in terms of Puiseux exponents and intersection multiplicities of branches.

1 Introduction

Let $(C, 0)$ be the germ of a complex-analytic curve in $\mathbb{C}^2$ with multiplicity $m(C)$. To calculate the number of infinitely near $m(C)$-fold points Hironaka (see [H] or [BK]) introduced the concept of contact exponent $d(C)$ which, in the case of irreducible germ, is equal to the (first) Puiseux exponent. Let $(C, 0) = \bigcup_{i=1}^{r}(C_i, 0)$ be a many-branched singularity. In this note we give a formula for $d(C)$ in terms of Puiseux exponents and intersection multiplicities of branches $(C_i, 0)$. Then we prove the inequality $\mu(C) \geq (m(C)d(C) - 1)(m(C) - 1)$ for the Milnor number $\mu(C)$ of $(C, 0)$ and characterize the singularities for which the equality holds true.

2 Formula for the contact exponent

Let $(C, D)_0$ denote the intersection multiplicity of two curves germs $(C, 0)$ and $(D, 0)$ in $(\mathbb{C}^2, 0)$. If $(C, 0)$ and $(D, 0)$ are irreducible curve germs (branches)

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then we put $d(C,D)=\frac{(C,D)_0}{m(C)m(D)}$ and call $d(C,D)$ the order of contact of branches $(C,0)$ and $(D,0)$. For any branches $(C,0)$, $(D,0)$ and $(E,0)$ we have:
(d_1) $d(C,D)=\infty$ if and only if $(C,0)=(D,0)$,
(d_2) $d(C,D)=d(D,C)$,
(d_3) $d(C,D)\geq \inf\{d(C,E),d(E,D)\}$ with equality if $d(C,E)\neq d(E,D)$.

The proof of (d_3) is given in [Ch-P].

If $(C,0)=\bigcup_{i=1}^r(C_i,0)$ is a reduced germ with $r>1$ branches $(C_i,0)$ then we put $d(C,D)=\inf{}_{i=1}^r\{d(C_i,D)\}$ for any branch $(D,0)$.

Lemma 1 *If $(C,0)=\bigcup_{i=1}^r(C_i,0)$ is a germ with r branches $(C_i,0)$ and $(D,0)$ is a branch then*

1. *$d(C,D)\leq \inf_{i,j}\{d(C_i,C_j)\}$,*
2. *$d(C,D)\leq \frac{(C,D)_0}{m(C)m(D)}$ with equality if $d(C,D)<\inf_{i,j}\{d(C_i,C_j)\}$.*

Proof.- Let i_0,j_0 be such that $\inf\{d(C_i,C_j)\}=d(C_{i_0},C_{j_0})$. Then $d(C,D)=\inf_{1\leq i\leq r}\{d(C_i,D)\}\leq \inf\{d(C_{i_0},D),d(C_{j_0},D)\}$ and using $(d3)$ we get $d(C,D)\leq d(C_{i_0},C_{j_0})$ which proves the first part of the Lemma.
In order to check the second part let us observe that

$$(C,D)_0=\sum_{i=1}^r(C_i,D)_0=\sum_{i=1}^r d(C_i,D)m(C_i)m(D)\geq \sum_{i=1}^r d(C,D)m(C_i)m(D)$$

so $d(C,D)\leq \frac{(C,D)_0}{m(C)m(D)}$ with equality if and only if $d(C,D)=d(C_i,D)$ for all $i\in\{1,\ldots,r\}$.
Suppose that $d(C,D)<\inf{}_{i,j}\{d(C_i,C_j\}$. By definition there is an $i_0\in\{1,\ldots,r\}$ such that $d(C,D)=d(C_{i_0},D)$, so $d(C_{i_0},D)<d(C_{i_0},C_j)$ for all $j\in\{1,\ldots,r\}$. Using $(d3)$ to C_{i_0}, D and C_j we get

$$d(C_j,D)=\inf\{d(C_{i_0},D),d(C_j,C_{i_0})\}=d(C_{i_0},D)=d(C,D)\quad \text{for all } j,$$

so $d(C,D)=\frac{(C,D)_0}{m(C)m(D)}$.

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Now we put for any reduced plane curve germ $(C,0)$:

$$d(C)=\sup\{d(C,W)\ :\ (W,0)\ \text{ runs over all smooth branches}\}$$

and call $d(C)$ the contact exponent of $(C,0)$ (see [H], Definition 1.5 where the term *characteristic exponent* is used). We say that a smooth germ $(W,0)$ has maximal contact with $(C,0)$ if $d(C,W)=d(C)$.
Observe that $d(C)=+\infty$ if and only if $(C,0)$ is a smooth branch.

Let $(C,0)$ be a branch with characteristic $(\beta_0,\beta_1,\dots,\beta_g)$ (see [Z]). Recall that $\beta_0 = m(C)$ and $\beta_1 = \sup\{(C,W)_0 \,:\, (W,0) \text{ runs over all smooth branches}\}$ (the supremum is attained). Clearly $d(C) = \frac{\beta_1}{\beta_0}$ (the first Puiseux exponent) and a smooth branch $(W_0,0)$ has maximal contact with $(C,0)$ if $(C,W_0)_0 = \beta_1$. Therefore we have

Lemma 2 *Let $(C,0)$ be a branch. Then there exists a smooth branch which has maximal contact with $(C,0)$.*

Using the order of contact we can tell if for two branches $(C,0)$ and $(D,0)$ there is a smooth branch which has the maximal contact with $(C,0)$ and $(D,0)$.

Lemma 3 *Let $(C,0)$ and $(D,0)$ be two branches. Then*

1. *If there exists a smooth branch which has maximal contact with $(C,0)$ and $(D,0)$ then $d(C,D) \geq \inf\{d(C),d(D)\}$ with equality if $d(C) \neq d(D)$.*

2. *If there does not exist such a branch then $d(C,D)$ is an integer and $d(C,D) < \inf\{d(C),d(D)\}$.*

Proof.-
If there exists a smooth branch $(W,0)$ such that $d(W,C) = d(C)$ and $d(W,D) = d(D)$ then we get the first part of the lemma by using Property $(d3)$ to the branches $(C,0)$, $(D,0)$ and $(W,0)$. In order to check the second part suppose that such a branch does not exist. Let $(U,0)$ and $(V,0)$ be smooth branches such that $d(U,C) = d(C)$ and $d(V,D) = d(D)$. By hypothesis $d(C,V) < d(C) = d(C,U)$ and $d(D,U) < d(D) = d(D,V)$. According to $(d3)$ we get $d(U,V) = \inf\{d(C,V),d(C,U)\} = d(C,V)$ and $d(U,V) = \inf\{d(D,U),d(D,V)\} = d(D,U)$ thus

$$d(C,V) = d(D,U) = d(U,V) \tag{1}$$

Without lost of generality we can suppose that $d(C) \leq d(D)$. Therefore $d(C,V) < d(C)$ so $d(C,V) < d(D)$ and using $(d3)$ we get

$$d(C,D) = \inf\{d(C,V),d(D,V)\} = d(C,V) \tag{2}$$

From (1) and (2) it follows that $d(C,D) = d(U,V)$ which is an integer because U and V are smooth branches. Moreover $d(C,D) < \inf\{d(C),d(D)\}$ and we are done.

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Our main result is

Theorem 1 *Let* $(C,0) = \bigcup_{i=1}^r (C_i,0)$ *be a reduced germ with* $r > 1$ *branches* $(C_i,0)$. *Then*

$$d(C) = \inf\{\inf_i\{d(C_i)\}, \inf_{i,j}\{d(C_i,C_j)\}\}.$$

Moreover, there exists an $i_0 \in \{1,\ldots,r\}$ *such that if a smooth branch* $(W,0)$ *has maximal contact with the branch* $(C_{i_0},0)$ *then it has maximal contact with the germ* $(C,0)$.

The proof of the above theorem is given in the third section of this Note. It is based on Properties $(d1)$, $(d2)$, $(d3)$ of the order of contact and on Lemma 2. This allows to prove the stability of contact exponent (see [H], Main Theorem 1.7) without appealing to the Newton diagrams (see Remark 1 in this Section).
Using Theorem 1 and Lemma 3 we get

Proposition 1 *Let* $(C,0) = \bigcup_{i=1}^r (C_i,0)$ *be a reduced plane curve germ with* r *branches* $(C_i,0)$. *Then* $d(C)$ *is a rational number greater than or equal to 1. Moreover*

1. $d(C) = 1$ *if and only if* $(C,0)$ *has at least two tangents.*
2. $d(C) < \inf{}_{i=1}^{r}\{d(C_i)\}$ *if and only if* $d(C)$ *is an integer.*

Proof.-
The first part follows immediately from Theorem 1. In order to check the second part suppose that $d(C) \in \mathbb{N}$. Then $d(C) \neq \inf_i\{d(C_i)\}$ and by Theorem 1 we get the inequality $d(C) < \inf_i\{d(C_i)\}$.
Suppose that $d(C) < \inf_i\{d(C_i)\}$. We have to check that $d(C) \in \mathbb{N}$. By Theorem 1 we get $d(C) = \inf_{i,j}\{d(C_i,C_j)\} = d(C_{i_0},C_{j_0})$ for some i_0, j_0. Thus $d(C)$ is an integer by Lemma 3 because $d(C_{i_0},C_{j_0}) < \inf\{d(C_{i_0}), d(C_{j_0})\}$.

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Proposition 2 *Let* $(C,0) = \cup_{i=1}^r (C_i,0)$ *be a reduced plane curve germ. A smooth branch* $(W,0)$ *does not have maximal contact with* $(C,0)$ *if and only if* $(C,W)_0 < d(C)m(C)$. *Moreover in this case* $(C,W)_0 \equiv 0$ *(mod* $m(C)$*).*

Proof.- Let us suppose that $(W,0)$ is a smooth branch which does not have maximal contact with $(C,0)$. We will check that $(C,W)_0 < d(C)m(C)$ and $\frac{(C,W)_0}{m(C)} \in \mathbb{N}$. By Theorem 1 we get $d(C,W) < \inf{}_{i,j}\{d(C_i,C_j)\}$ since $d(C,W) < d(C)$. According to the second part of Lemma 1 we can write $d(C,W) = \frac{(C,W)_0}{m(C)}$, thus $(C,W)_0 = d(C,W)m(C) < d(C)m(C)$. We claim that $\frac{(C,W)_0}{m(C)} = d(C,W)$ is an integer. Indeed, we have the following property: if $d(C,W) \notin \mathbb{N}$ then $d(C,W) = d(C)$.

The above property is obvious if $(C,0)$ is a branch. In the general case $d(C,W) = \inf{}_i\{d(C_i,W)\} = d(C_{i_0},W)$ for some $i_0 \in \{1,\dots,r\}$. If $d(C,W) \notin \mathbb{N}$ then $d(C_{i_0},W) \notin \mathbb{N}$ and $d(C_{i_0},W) = d(C_{i_0})$ since C_{i_0} is a branch. Consequently, we get $d(C,W) = d(C_{i_0})$ which implies, by Theorem 1, $d(C) = d(C_{i_0}) = d(C,W)$.
Now, suppose that $(C,W)_0 < d(C)m(C)$. Then by the second part of Lemma 1 we get $d(C,W) \leq \frac{(C,W)_0}{m(C)}$ and consequently $d(C,W) < d(C)$ which means that $(W,0)$ does not have maximal contact with $(C,0)$.

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Remark 1 *Let $(\overline{C},\overline{0})$ be the proper image of a reduced unitangent germ $(C,0)$ under blowing up the point $0 \in \mathbb{C}^2$. Then the following hold*
(i) if $1 < d(C) < 2$ then $m(\overline{C}) < m(C)$,
(ii) if $d(C) \geq 2$ then $d(C) = d(\overline{C}) + 1$.
To prove (i) and (ii) we check the above properties in the case of branches and then we use Theorem 1. Using (ii) and the Max Noether formula for the intersection multiplicity we get
(iii) if $d(C) \geq 2$ and $(D,0)$ is a branch then $d(C,D) = d(\overline{C},\overline{D}) + 1$.
Properties (i), (ii) and (iii) imply the stability of maximal contact (see [H] and [BK] Chapter III).

3 Proof of the result

Let $(C,0)$ be a reduced plane curve germ in $(\mathbb{C}^2,0)$. For every non-empty family of branches $\mathcal{B}$ we put

$$d(C,\mathcal{B}) = \sup\{d(C,W) \ : \ (W,0) \in \mathcal{B}\}.$$

Note that $d(C,\mathcal{B}) = +\infty$ if $(C,0) \in \mathcal{B}$. We will prove the following

Theorem 2 *Let $(C,0) = \bigcup_{i=1}^{r}(C_i,0)$ be a reduced germ with $r > 1$ branches $(C_i,0)$ and let $\mathcal{B}$ be a family of branches such that for every $i \in \{1,\dots,r\}$ there exists a branch $(W_i,0) \in \mathcal{B}$ for which $d(C_i,\mathcal{B}) = d(C_i,W_i)$. Then*

$$d(C,\mathcal{B}) = \inf\{\inf_i\{d(C_i,\mathcal{B})\}, \inf_{i,j}\{d(C_i,C_j)\}\}.$$

Moreover, there exists an $i_o \in \{1,\dots,r\}$ such that for every $(W,0) \in \mathcal{B}$ with $d(C_{i_0},W) = d(C_{i_0},\mathcal{B})$ one has $d(C,W) = d(C,\mathcal{B})$.

Proof.-
Set $d^*(C,\mathcal{B}) = \inf\{\inf_i d(C_i,\mathcal{B}), \inf_{i,j} d(C_i,C_j)\}$.

The inequality $d(C,\mathcal{B}) \leq d^*(C,\mathcal{B})$ follows from the first part of Lemma 1 and from the definition of $d(C_i,\mathcal{B})$. Thus to prove the result let us consider two cases:

First case: $\inf_i\{d(C_i,\mathcal{B})\} \le \inf_{i,j}\{d(C_i,C_j)\}$.
Let $i_0 \in \{1,\dots,r\}$ be such that $d(C_{i_0},\mathcal{B}) = \inf_i\{d(C_i,\mathcal{B})\}$. Then, we have

$$d(C_{i_0},\mathcal{B}) = d^*(C,\mathcal{B}). \tag{3}$$

Let $W \in \mathcal{B}$ be a branch such that $d(C_{i_0},W) = d(C_{i_0},\mathcal{B})$. We claim that

$$d(C_{i_0},W) \le d(C_i,W) \text{ for all } i \in \{1,\dots,r\}. \tag{4}$$

To obtain a contradiction suppose that (4) does not hold. Thus there is an $i_1 \in \{1,\dots,r\}$ such that

$$d(C_{i_1},W) < d(C_{i_0},W). \tag{5}$$

Using Property (d_3) to the branches $(C_{i_0},0),(C_{i_1},0)$ and $(W,0)$ we get

$$d(C_{i_1},W) = d(C_{i_0},C_{i_1}). \tag{6}$$

On the other hand, in the case under consideration we have

$$d(C_{i_0},\mathcal{B}) = \inf_i\{d(C_i,\mathcal{B})\} \le d(C_{i_0},C_{i_1}). \tag{7}$$

Therefore by (7),(6) and (5) we get $d(C_{i_0},\mathcal{B}) \le d(C_{i_1},W) < d(C_{i_0},W)$ which contradicts the definition of $d(C_{i_0},\mathcal{B})$.

Now, using (4) and (3), we calculate

$$d(C,W) = \inf\{d(C_{i_0},W), \inf_{i\ne i_0} d(C_i,W)\} = d(C_{i_0},W) = d(C_{i_0},\mathcal{B}) = d^*(C,\mathcal{B})$$

which proves the theorem in the first case.

Second case: $\inf_i\{d(C_i,\mathcal{B})\} > \inf_{i,j}\{d(C_i,C_j)\}$.
Let i_0, j_0 be such that $d(C_{i_0},C_{j_0}) = \inf_{i\ne j} d(C_i,C_j) = d^*(C,\mathcal{B})$.
Let $(W,0)$ be a branch such that $d(C_{i_0},W) = d(C_{i_0},\mathcal{B})$. We claim that

$$d(C_{i_0},C_{j_0}) \le d(C_i,W) \text{ for all } i \in \{1,\dots,r\} \text{ with equality for } i = j_0. \tag{8}$$

First observe that in the case under consideration we have

$$d(C_{i_0},C_{j_0}) < d(C_{i_0},\mathcal{B}) = d(C_{i_0},W). \tag{9}$$

Fix an $i \in \{1,\dots,r\}$. If $d(C_{i_0},W) \le d(C_i,W)$ then (8) follows from (9). If $d(C_i,W) < d(C_{i_0},W)$ then by Property (d_3) applied to the branches $(C_i,0),(C_{i_0},0)$ and $(W,0)$ we get $d(C_i,W) = d(C_i,C_{i_0}) \ge \inf_{i,j}\{d(C_i,C_j)\} = d(C_{i_0},C_{j_0})$. In particular for $i = j_0$, $d(C_{j_0},W) = d(C_{j_0},C_{i_0}) = d(C_{i_0},C_{j_0})$.
Now, we get by the definition of $d(C,W)$ and properties (8) and (9):

$$\begin{aligned} d(C,W) &= \inf\{d(C_{i_0},W), d(C_{j_0},W), \inf_{i\ne i_0,j_0} d(C_i,W)\} \\ &= d(C_{j_0},W) = d(C_{i_0},C_{j_0}) = d^*(C,\mathcal{B}) \end{aligned}$$

which proves the theorem in the second case.

■

Proof of Theorem 1

Theorem 1 follows immediately from Theorem 2 and Lemma 2 by taking $\mathcal{B}$ to be the set of all smooth branches.

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4 Bound for the Milnor number

Let $\mu(C)$ be the Milnor number of the germ $(C,0)$ (for basic properties of the Milnor number see [M] and [Ph]). In this section we give a lower bound for $\mu(C)$ and characterize the singularities for which the bound is exact.

Proposition 3 *Let $(C,0)=\bigcup_{i=1}^{r}(C_i,0)$ be a singular reduced germ with r branches $(C_i,0)$. Then $\mu(C)\geq(d(C)m(C)-1)(m(C)-1)$ with equality if and only if the following two conditions are satisfied:*

(e_1) $d(C_i,C_j)=d(C)$ for all $i\neq j$,

(e_2) if the branch $(C_i,0)$ is singular then $(C_i,0)$ has exactly one Puiseux pair (i.e. the characteristic of $(C_i,0)$ has exactly two terms) and $d(C_i)=d(C)$.

Proof.-

First let us suppose that $(C,0)$ is a branch with characteristic $(\beta_0,\beta_1,\dots,\beta_g)$. Then the inequality takes the form $\mu(C)\geq(\beta_1-1)(\beta_0-1)$ and Proposition 3 follows from the well-known formula $\mu(C)=\sum_{k=1}^{g}(l_{k-1}-l_k)(\beta_k-1)$ where $l_k=\gcd(\beta_0,\dots,\beta_k)$ for $k\in\{0,\dots,g\}$.

Now suppose that the germ $(C,0)$ has $r>1$ branches $(C_i,0)$ and let $n_i=m(C_i)$ for $i\in\{1,\dots,r\}$. From Theorem 1 we get $d(C_i)\geq d(C)$ and $d(C_i,C_j)\geq d(C)$ for all $i,j\in\{1,\dots,r\}$. By the first part of the proof $\mu(C_i)\geq(d(C_i)n_i-1)(n_i-1)$ for the singular branches with equality if and only if $(C_i,0)$ is a singular branch satisfying condition (e_2).

Let $I:=\{i\ :\ C_i \text{ is singular}\}$. Now, we get

$$\begin{aligned}
\mu(C)+r-1 &= \sum_{i=1}^{r}\mu(C_i)+2\sum_{1\le i<j\le r}(C_i,C_j)_0 \\
&= \sum_{i=1}^{r}\mu(C_i)+2\sum_{1\le i<j\le r}d(C_i,C_j)n_in_j \\
&\ge \sum_{i\in I}(d(C_i)n_i-1)(n_i-1)+2\sum_{1\le i<j\le r}d(C_i,C_j)n_in_j \\
&\ge \sum_{i=1}^{r}(d(C)n_i-1)(n_i-1)+2\sum_{1\le i<j\le r}d(C)n_in_j \\
&= d(C)(m(C)^2-m(C))-m(C)+r
\end{aligned}$$

with equality if and only if the conditions (e_1) and (e_2) are satisfied.

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Lemma 4 *Let $(C,0)$ be a germ with exactly one tangent. We have:*

1. $d(C)\ge 1+\frac{1}{m(C)}$. *Moreover* $d(C)=1+\frac{1}{m(C)}$ *if and only if* $(C,0)$ *is a branch of characteristic* $(m(C),m(C)+1)$.
2. $\mu(C)\ge m(C)(m(C)-1)$ *with equality if and only if* $d(C)=1+\frac{1}{m(C)}$.

Proof.- Let $\{(C_i,0)\}_i$ be the set of the branches of $(C,0)$. To check the first part of the Lemma we may assume that $d(C)$ is not an integer. Then by Theorem 1 and the second part of Proposition 1 there is an i_0 such that $d(C)=d(C_{i_0})$. The contact exponent $d(C_{i_0})$ is a fraction with the denominator less than or equal to $m(C_{i_0})$. Therefore we get $d(C)=d(C_{i_0})\ge 1+\frac{1}{m(C_{i_0})}\ge 1+\frac{1}{m(C)}$ and the equality $d(C)=1+\frac{1}{m(C)}$ implies $m(C_{i_0})=m(C)$ and consequently $C_{i_0}=C$. Moreover $(C,0)$ is of characteristic $(m(C),m(C)+1)$ since $m(C)$ and $m(C)+1$ are coprime.

In order to prove the second part we get, by Proposition 3 and the first part of this Lemma,

$$\begin{aligned}
\mu(C) &\ge (d(C)m(C)-1)(m(C)-1) \\
&\ge ((1+\frac{1}{m(C)})m(C)-1)(m(C)-1)=m(C)(m(C)-1).
\end{aligned}$$

If $\mu(C)=m(C)(m(C)-1)$ then from the above calculation it follows that $d(C)=1+\frac{1}{m(C)}$.

On the other hand if $d(C)=1+\frac{1}{m(C)}$ then by the first part of this lemma C is a branch of characteristic $(m(C),m(C)+1)$. According to Proposition 3 $\mu(C)=(d(C)m(C)-1)(m(C)-1)=m(C)(m(C)-1)$.

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Let $t=t(C)$ be the number of tangents to the germ $(C,0)$.

Proposition 4 *For every germ* $(C,0)=\cup_{i=1}^{r}(C_i,0)$ *we have* $\mu(C)\geq m(C)(m(C)-1)-t(C)+1$ *with equality if and only if the branches of* $(C,0)$ *are pairwise transverse (i.e.* $d(C_i,C_j)=1$ *for* $i\neq j$*) and for every singular branch* $(C_i,0)$ *of* $(C,0)$ *one has* $d(C_i)=1+\frac{1}{m(C_i)}$.

Proof.-
We can write $(C,0)=\bigcup_{i=1}^{t}(\tilde{C}_i,0)$ where $\{(\tilde{C}_i,0)\}_i$ are germs with one tangent and pairwise transverse. Let $\tilde{n}_i=m(\tilde{C}_i)$. Then by Lemma 4

$$\begin{aligned}
\mu(C)+t(C)-1 &= \sum_{i=1}^{t}\mu(\tilde{C}_i)+2\sum_{1\leq i<j\leq t}(\tilde{C}_i,\tilde{C}_j)_0 \\
&\geq \sum_{i=1}^{t}\tilde{n}_i(\tilde{n}_i-1)+2\sum_{1\leq i<j\leq t}\tilde{n}_i\tilde{n}_j \\
&= m(C)^2-m(C)
\end{aligned}$$

with equality if and only if $\mu(\tilde{C}_i)=\tilde{n}_i(\tilde{n}_i-1)$ for all $i\in\{1,\ldots,t\}$ which by the first part of the proof means that for every singular germ $(\tilde{C}_i,0)$ one has $d(\tilde{C}_i)=1+\frac{1}{m(\tilde{C}_i)}$. We use Lemma 4.

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Remark 2 *Let* $((\overline{C}_1,0_1),\ldots,(\overline{C}_t,0_t))$ *be the proper image of the germ* $(C,0)$ *under blowing up the point* 0*. Then we could get the Proposition 4 from the well-known formula* $\mu(C)+t(C)-1=m(C)(m(C)-1)+\sum_{k=1}^{t}\mu(\overline{C}_k)$.

5 Polar invariants and the contact exponent

Let $(C,0)$ be a reduced plane singular germ and let $(P(C),0)$ be the germ of a generic polar of C. Then $(P(C),0)$ is a reduced germ of multiplicity $m(P(C))=m(C)-1$. Let $(P(C),0)=\bigcup_{j=1}^{s}(D_j,0)$ be the decomposition of $(P(C),0)$ into branches $(D_j,0)$.

We put $\mathcal{Q}(C,0)=\{\frac{(C,D_j)_0}{m(D_j)}\ :\ j\in\{1,\ldots,s\}\}$ and call the elements of $\mathcal{Q}(C,0)$ the polar invariants of $(C,0)$. They are topological invariants of $(C,0)$ (see [T1], Théorème 6, p. 275). Let us consider the minimal polar invariant $\alpha(C):=\inf\mathcal{Q}(C,0)$.

Proposition 5 *For any singular reduced germ* $(C,0)$ *we have* $\alpha(C)=m(C)d(C)$.

Proof.- See [L-M-P].

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One could prove Proposition 5 by using Theorem 1 and the explicit formulae for the polar invariants given in [GP] (see Theorem 1.3) and [E].

We say that $(C,0)$ is an Eggers singularity if $\mathcal{Q}(C,0)$ has exactly one element.

Proposition 6 *Let $(C,0)$ be a singular reduced germ. Then $\mu(C) = (d(C)m(C)-1)(m(C)-1)$ if and only if $(C,0)$ is an Eggers singularity.*

Proof.-
By [T2] Proposition 1.2 we get

$$\begin{aligned}
\mu(C) &= (C, P(C))_0 - m(C) + 1 = \sum_{j=1}^{s} (C, D_j)_0 - m(C) + 1 \\
&\geq \alpha(C)m(P(C)) - m(C) + 1 = \alpha(C)(m(C)-1) - m(C) + 1 \\
&= (\alpha(C)-1)(m(C)-1)
\end{aligned}$$

with equality if and only if $(C,0)$ is an Eggers singularity. We use Proposition 5.

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If $(C,0)$ is an Eggers singularity then the pair $(m(C), m(C)d(C))$ determines the equisingularity class of $(C,0)$. More specifically, we have:

Proposition 7 (cf [E], Korollar 3, p. 16) *Let $(C,0)$ be a reduced plane singular germ. Then $(C,0)$ is an Eggers singularity if and only if one of the following three conditions hold*

1. *$d(C) \in \mathbb{N}$. All branches of $(C,0)$ are smooth and intersect pairwise with multiplicity $d(C)$.*
2. *$d(C) \notin \mathbb{N}$ and $m(C)d(C) \in \mathbb{N}$. The germ $(C,0)$ has $r = \text{g.c.d.}(m(C), m(C)d(C))$ branches, each of one of characteristic $(\frac{m(C)}{r}, \frac{m(C)d(C)}{r})$, intersecting pairwise with multiplicity $\frac{m(C)^2 d(C)}{r^2}$.*
3. *$m(C)d(C) \notin \mathbb{N}$. There is a smooth germ $(L,0)$ such that $(C,0) = (L,0) \cup (C',0)$ where $(C',0)$ is a germ of type (2) with $d(C') = d(C)$ and $m(C') = m(C) - 1$. The branch $(L,0)$ has maximal contact with any branch of $(C',0)$.*

Proof.-
We use Proposition 6. If one of conditions (1), (2) or (3) is satisfied then a direct calculation shows that $\mu(C) = (d(C)m(C)-1)(m(C)-1)$.
Suppose that $(C,0) = \cup_{i=1}^{r}(C_i,0)$ is an Eggers singularity. Then by Proposition 6 and Proposition 3 the conditions (e_1) and (e_2) are satisfied. Let us consider three cases:

Case 1: All branches $(C_i,0)$ are smooth. Then $(C,0)$ is of type (1) by (e_1).

Case 2: All branches $(C_i,0)$ are singular. Then the branches $\{(C_i,0)\}_i$ have the same characteristic (β_0,β_1) and according to (e_2) $d(C_i) = d(C)$ for all $i \in \{1,\dots,r\}$. Clearly, we have $m(C) = \sum_{i=1}^{r} m(C_i) = r\beta_0$ and $m(C)d(C) =$

$m(C)d(C_i) = r\beta_1$. Thus $m(C)d(C) \in \mathbb{N}$, $r = \text{g.c.d.}(m(C), m(C)d(C))$ and it is easy to see that $(C,0)$ is of type (2).

Case 3: Neither Case 1 nor Case 2 holds, thus $r > 1$. We may assume that $(C_1,0)$ is smooth and $(C_2,0)$ is singular. If $r > 2$ then all branches $(C_i,0)$ for $i \geq 3$ are singular. In fact, we have by (e_1): $d(C_1,C_i) = d(C_2,C_i) = d(C_1,C_2) = d(C)$ and by (e_2): $d(C) = d(C_2) \notin \mathbb{N}$. Thus $d(C_1,C_i) \notin \mathbb{N}$ and $(C_i,0)$ are singular for all $i \geq 3$. Let $(L,0) := (C_1,0)$ and $(C',0) := \cup_{i=2}^{r}(C_i,0)$. Then $(C,0) = (L,0) \cup (C',0)$ and we check using Theorem 1 that $(C,0)$ is of type (3).

■

6 Puiseux series and the contact exponent

Let $(C,0)$ be a reduced plane curve germ with multiplicity $n = m(C) > 1$. Let us take a system of coordinates x,y such that the axis $x = 0$ is transverse to the germ $(C,0)$. Then $(C,0)$ has the minimal equation $f(x,y) = 0$ where $f(x,y) = y^n + a_1(x)y^{n-1} + \cdots + a_n(x)$ is a Weierstrass polynomial.

Let $\mathbb{C}\{x\}^* = \bigcup_{N>0} \mathbb{C}\{x^{1/N}\}$ be the ring of Puiseux series. Write $f(x,y) = \prod_{i=1}^{n}(y - y_i(x))$ in $\mathbb{C}\{x\}^*[y]$.

Proposition 8 *Let us keep the notation introduced above. Then* $d(C) = \inf_{i,j}\{\text{ord}(y_i(x) - y_j(x))\}$.

Proof.-
If $(C,0)$ is irreducible then $d(C) = \frac{\beta_1}{\beta_0}$ and the formula follows from the definition of the characteristic (see [Z]). Suppose that $(C,0) = \bigcup_{k=1}^{r}(C_k,0)$ is a germ with $r > 1$ branches and let $f = f_1 \cdots f_r$ be a factorization of $f \in \mathbb{C}\{x,y\}$ into irreducible factors such that $f_k = 0$ is the minimal equation of $(C_k,0)$.

Let $\mathcal{R}(f_k)$ be the set of all roots of the equation $f_k(x,y) = 0$ in $\mathbb{C}\{x\}^*[y]$. Write $y_i = y_i(x)$ for $i \in \{1,\ldots,n\}$.
Lemma: With the notation introduced above

1. If there exists a smooth branch which has maximal contact with branches $(C_k,0)$ and $(C_l,0)$ then

   $$\inf\{\text{ord}(y_i - y_j) \ : \ y_i \in \mathcal{R}(f_k) \text{ and } y_j \in \mathcal{R}(f_l)\} \geq \inf\{d(C_k), d(C_l)\}$$

   with equality if $k = l$.

2. If there does not exist such a branch then $\text{ord}(y_i - y_j) = d(C_k,C_l)$ for all $y_i \in \mathcal{R}(f_k)$ and $y_j \in \mathcal{R}(f_l)$.

Proof of the Lemma: Let $(C_k, 0)$ be a singular branch. Then for every $y_i \in \mathcal{R}(C_k)$ there is a unique polynomial Y_k in x of degree less than $d(C_k)$ and a constant $c_i \neq 0$ such that

$$y_i = Y_k + c_i x^{d(C_k)} + \cdots \quad \text{(terms of order } > d(C_k)).$$

We will check the Lemma for two singular branches $(C_k, 0)$ and $(C_l, 0)$ (if one of the branches is smooth then the proof needs an obvious modification). We may assume than $k \neq l$. Let y_i, y_j be such that $y_i \in \mathcal{R}(f_k)$ and $y_j \in \mathcal{R}(f_l)$.

If there exists a smooth branch which has maximal contact with the branches $(C_k, 0)$ and $(C_l, 0)$ then $Y_k = Y_l$ and the order of $y_i - y_j = (c_i x^{d(C_k)} + \cdots) - (c_j x^{d(C_l)} + \cdots)$ is greater than or equal to $\inf\{d(C_k), d(C_l)\}$.

If there is not such a branch then $Y_k \neq Y_l$ and $\mathrm{ord}(y_i - y_j) = \mathrm{ord}(Y_k - Y_l)$. By Zeuthen's rule we get $(C_k, C_l)_0 = \sum_{i,j} \mathrm{ord}(y_i - y_j) = m(C_k)m(C_l)\mathrm{ord}(Y_k - Y_l)$ and the lemma follows.

Indeed, using the Lemma and Theorem 1 we get for any roots y_i, y_j of f: $\mathrm{ord}(y_i - y_j) \geq \inf\{\inf_k\{d(C_k)\}, \inf_{k,l}\{d(C_k, C_l)\}\} = d(C)$. If $d(C) = d(C_k)$ for a $k \in \{1, \ldots, r\}$ then, by the one branch case there exists $y_i, y_j \in \mathcal{R}(C_k)$ such that $\mathrm{ord}(y_i - y_j) = d(C_k) = d(C)$ and Proposition follows. If $d(C) < \inf_k\{d(C_k)\}$ then $d(C) = d(C_k, C_l)$ for some $k \neq l$ and by Lemma 3 there is no a smooth branch with maximal contact with $(C_k, 0)$ and $(C_l, 0)$. Thus $\mathrm{ord}(y_i - y_j) = d(C_k, C_l) = d(C)$ for $y_i \in \mathcal{R}(C_k)$ and $y_j \in \mathcal{R}(C_j)$ by the Lemma.

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Proposition 9 *Let $(C, 0)$ be a singular germ. Then $\mu(C) = (d(C)m(C) - 1)(m(C) - 1)$ if and only if there is a $d > 0$ such that $\mathrm{ord}(y_i(x) - y_j(x)) = d$ for all $i \neq j$.*

Proof.-
Let $\Delta_f(x) = \prod_{1 \leq i < j \leq n}(y_i - y_j)^2$ be the discriminant of the polynomial $f(x, y) \in \mathbb{C}\{x\}[y]$. Then $\mathrm{ord}\Delta_f(x) \geq n(n-1)\inf_{i,j}\{\mathrm{ord}(y_i - y_j)\} = n(n-1)d(C)$ by Proposition 8 with equality if and only if the orders $\{\mathrm{ord}(y_i - y_j)\}_{i,j}$ do not depend on $i \neq j$. It suffices to recall that $\mathrm{ord}\Delta_f(x) = \mu(C) + n - 1$.

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