# The equation of complex Monge-Ampère type and stability of solutions

by

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Abstract. We prove that in a family of plurisubharmonic functions with Monge-Ampère measures bounded from above by such a measure of one function weak convergence is equivalent to convergence in capacity. We also show a very general statement on the existence of solutions of the complex Monge-Ampère type equation.

## 0. Introduction

It is well known (see e.g. [H]) that a sequence of plurisubharmonic (psh) functions which converges in the sense of distributions is also convergent in $L^p_{loc}$ for any finite $p \geq 1$. This rather strong type of convergence of functions does not, however, imply the weak* convergence of Monge-Ampère measures associated to those functions (see [L][C1]).

An example from [CK] shows that there exists a sequence of measures in a ball $d\mu_j$ converging in weak* topology to $d\mu$ and such that the solutions to the Dirichlet problem for the Monge-Ampère equation

$$(dd^c u_j)^n = d\mu_j,$$

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with zero boundary data, are uniformly bounded, but $u_j$ fail to converge to $u$ which solves

$$(dd^c u)^n = d\mu.$$

In this paper we show that on subsets of $PSH(\Omega)$ where the Monge-Ampère masses are under control the above phenomena: discontinuity of the Monge-Ampère operator and the lack of stability of the solutions of the Monge-Ampère equation no longer occur. Furthermore, the weak* convergence of functions within those subsets is equivalent to the convergence in capacity (see Preliminaries below for definitions). By a theorem of Xing [X] for a sequence of psh functions converging in capacity the Monge-Ampère measures corresponding to those functions tend to the Monge-Ampère measure of the limit function.

We shall consider families of psh functions on which the Monge-Ampère operator and its inverse restricted to the image are continuous. For a strictly pseudoconvex domain $\Omega$, a continuous function $\varphi$ on $\partial\Omega$, and a psh function $u$ from $\mathcal{F}(\varphi)$ (defined in [A], see Section 1) the family with the properties mentioned above consists of those psh functions $v$ in $\Omega$ which satisfy

$$v \in \mathcal{F}(\varphi), \ (dd^c v)^n \leq d\mu := (dd^c u)^n,$$

provided $d\mu$ is a finite measure which vanishes on pluripolar sets. The condition $v \in \mathcal{F}(\varphi)$ implies that $v$ has boundary data $\varphi$ in the sense that

$$\limsup_{\zeta \to z} v(\zeta) = \varphi(z), \quad z \in \partial\Omega.$$

The same holds for a hyperconvex $\Omega$ if $\mathcal{F}(\varphi)$ contains the maximal plurisubharmonic function which is continuous up to the boundary and equal $\varphi$ on $\partial\Omega$.

In the families described above the convergence $u_j \to u$ in the sense of distributions is the same as the convergence in capacity. Since they contain non continuous functions one cannot expect anything more. In [K4] it was shown that with suitable bounds on Monge-Ampère measures (for instance in $L^p$ norm , $p > 1$) we may get equicontinuity of families of psh functions.

In the second part of the paper we prove a very general existence theorem on the Dirichlet problem for the complex equation of the Monge-Ampère type

$$(dd^c u)^n = F(u(\cdot), \cdot)\, d\mu$$

with $d\mu$ finite and vanishing on pluripolar sets and $F \geq 0$ which is only continuous in the first variable and upper bounded by a function from $L^1(d\mu)$ . Our theorem strengthens the results from [BT2][C2] and [K3]. The case of smooth functions is done in [CKNS].

## 1. Preliminaries.

We call a domain in $\mathbb{C}^n$ hyperconvex if there exists nonzero $u \in PSH(\Omega) \cap C(\overline{\Omega})$ such that $u = 0$ on $\partial\Omega$. Throughout the paper we shall assume that $\Omega$ is hyperconvex.

**Definition [C3][C4][A].** *The set of bounded psh functions $u$ satisfying $\lim_{z\to\zeta} u(z) = 0$ for any $\zeta \in \partial\Omega$, and $\int_\Omega (dd^c u)^n < \infty$ we denote by $\mathcal{E}_0$. A psh function $u$ belongs to $\mathcal{F}$ if there exists a sequence $u_j \in \mathcal{E}_0$ with $u_j \downarrow u$ in $\Omega$ and*

$$\sup_j \int_\Omega (dd^c u_j)^n < \infty.$$

*A negative psh function $u$ belongs to $\mathcal{E}$ if for every $z_0 \in \Omega$ there is a neighborhood $\Omega' \subset \Omega$ of $z_0$ and a sequence $u_j \in \mathcal{E}_0$ with $u_j \downarrow u$ in $\Omega'$ and*

$$\sup_j \int_\Omega (dd^c u_j)^n < \infty.$$

*Furthermore, for $\varphi \in C(\partial\Omega)$ we denote by $\mathcal{E}_0(\varphi)$ (resp. $\mathcal{F}(\varphi)$ ) the set of those $u \in PSH(\Omega)$ for which*

$$h \geq u \geq h + v$$

*for some $v \in \mathcal{E}_0$ (resp. $v \in \mathcal{F}$) and $h := \sup\{w \in PSH \cap L^\infty(\Omega) : \limsup_{z\to\zeta} w(z) \leq \varphi(\zeta), \zeta \in \partial\Omega\}$ provided that $h \to \varphi$ on $\partial\Omega$ (by the Walsh theorem $h$ is then continuous on the closure of $\Omega$). The subclass $\mathcal{F}^a(\varphi)$ consists of those functions from $\mathcal{F}(\varphi)$ for which $(dd^c v)^n$ vanishes on all pluripolar sets.*

*Finally, by $\mathcal{F}_1$ we denote the set of those $u \in \mathcal{F}$ for which (with $u_j$ as above)*

$$\sup_j \int_\Omega -u_j (dd^c u_j)^n < \infty.$$

By [A] $\mathcal{F}(\varphi) \subset \mathcal{E}$ if $\varphi$ is non positive.

Let us observe that functions from $\mathcal{F}$ are not necessarily locally bounded and, according to [C3], we have

$$\limsup_{z\to\zeta} u(z) = 0$$

for any $\zeta \in \partial\Omega$.

Given a bounded plurisubharmonic function $u$ one can define the Monge-Ampère measure

$$(dd^c u)^n = dd^c u \wedge dd^c u \wedge ... \wedge dd^c u \quad (n \text{ terms})$$

as in [BT1]. This is a nonnegative Borel measure.

According to [C4] the Monge-Ampère operator is also well defined for $u \in \mathcal{F}(\varphi)$ and then $(dd^c u)^n = \lim (dd^c u_j)^n$, where $u_j \in \mathcal{E}_0(\varphi)$, $u_j \downarrow u$. The limit measure is independent of the choice of this sequence. Furthermore, by Theorem 3.8 from [C3] we have for $u \in \mathcal{F}_1$

$$\int_\Omega -u_j (dd^c u_j)^n \uparrow \int_\Omega -u (dd^c u)^n, \ j \to \infty. \tag{1.1}$$

The following capacity was introduced in [BT3] for Borel sets $E \subset \Omega$:

$$cap(E,\Omega) = \sup\{\int_E (dd^c u)^n : u \in PSH(\Omega), -1 \leq u < 0\}.$$

We call it the Bedford-Taylor capacity. It is closely related to the relative extremal function of the given compact set $K$:

$$u_K(z) = \sup\{u(z) : u \in PSH \cap L^\infty(\Omega), u < 0 \text{ in } \Omega, u \leq -1 \text{ on } K\}.$$

Its upper semicontinuous regularization $u_K^*(z) := \overline{\lim}_{\zeta\to z} u_K(\zeta)$ is a plurisubharmonic function and, by [BT3]

$$cap(K,\Omega) = \int_K (dd^c u_K^*)^n = \int_\Omega (dd^c u_K^*)^n. \tag{1.2}$$

It is a subadditive Choquet capacity and in bounded $\Omega$ it vanishes exactly on pluripolar sets (see [BT3]). Recall that a set $E$ in $\mathbb{C}^n$ is *pluripolar* if for any $z \in E$ there exists a neighbourhood $V$ of $z$ and $v \in PSH(V)$ such that $E \cap V \subset \{v = -\infty\}$.

**Definition.** *A sequence $u_j$ of functions defined in $\Omega$ is said to converge in capacity to $u$ if for any $t > 0$ and $K \subset\subset \Omega$*

$$\lim_{j\to\infty} cap(K \cap \{|u - u_j| > t\}, \Omega) = 0.$$

With respect to this type of convergence the Monge-Ampère operator is continuous.

**Theorem 1.1 (Convergence theorem)[X].** *Let $\{u_k^j\}_{j=1}^\infty$ be a locally uniformly bounded sequence of psh functions in $\Omega$ for $k = 1, 2, ..., n$; and let $u_k^j \to u_k \in PSH \cap L^\infty_{loc}(\Omega)$ in capacity as $j \to \infty$ for $k = 1, 2, ..., n$. Then*

$$dd^c u_1^j \wedge ... \wedge dd^c u_n^j \to dd^c u_1 \wedge ... \wedge dd^c u_n$$

*in the weak topology of currents.*

This result has been generalized in [C5] in the following sense.

**Theorem 1.2 [C5].** *Suppose that $u_j \in \mathcal{F}$, $j = 0, 1, ....$ and $u_0 \leq u_j$. If $u_j$ converges to $u$ in capacity then $(dd^c u_j)^n$ tends to $(dd^c u)^n$ as $j \to \infty$.*

One of the basic properties of plurisubharmonic functions is their *quasicontinuity.*

**Theorem 1.3 [BT3].** *For any psh function $u$ defined in $\Omega$ and any positive number $\epsilon$ one can find an open set $U \in \Omega$ with $cap(U, \Omega) < \epsilon$ and such that $u$ restricted to $\Omega \setminus U$ is continuous.*

We shall use the well known comparison principle from [BT3] as well as its generalization to the class $\mathcal{F}_1$.

**Comparison principle [BT3][C3].** *If $u, v \in PSH \cap L^\infty(\Omega)$ and $\liminf_{z\to\partial\Omega}(u(z) - v(z)) \geq 0$, then*

$$\int_{\{u<v\}} (dd^c v)^n \leq \int_{\{u<v\}} (dd^c u)^n.$$

*The inequality remains valid if $u$ and $v$ belong to $\mathcal{F}_1$.*

The following lemma has been shown in [C5] for functions from $\mathcal{F}$. The proof carries over to $\mathcal{E}$ since a function from $\mathcal{E}$ is locally in $\mathcal{F}$.

**Lemma 1.4.** *Assume that a finite positive measure $\mu$ in $\Omega$ vanishes on all pluripolar sets. If $u_0, u_j \in \mathcal{E}$, $u_0 \le u_j$, $\int -u_0 d\mu < \infty$, and $u_j \to u$ as distributions then*

$$\lim_{j\to\infty} \int_\Omega u_j d\mu = \int_\Omega u\, d\mu.$$

**Corollary.** *Any Borel measure $\mu$ in $\Omega$ which vanishes on pluripolar sets and such that $\mu(\Omega)$ is finite has the following property. Given a sequence $u_j \in \mathcal{E}$ converging weakly to $u$ one can extract a subsequence $u_{j_k}$ converging pointwise $d\mu$ almost everywhere.*

*Proof.* Indeed, for any positive integer $N$ one can find, using Lemma 1.4, a subsequence of $\max(u_j, -N)$ which tends to $\max(u, -N)$ $d\mu$ almost everywhere. Then using diagonal procedure we get the desired subsequence.

## 2. Stability.

First we shall prove stability results in a strictly pseudoconvex domain for the class of bounded functions with arbitrary continuous boundary data. Next we extend this result first to the case of hyperconvex domains and then to the bigger class of functions $\mathcal{F}(\varphi)$. Finally we show that the Monge-Ampère operator is continuous on $\mathcal{F}_1$.

**Lemma 2.1.** *Let $\Omega$ be a bounded strictly pseudoconvex domain in $\mathbb{C}^n$, $\varphi \in C(\partial\Omega)$, and let $dV$ denote the Lebesgue measure. Suppose $u_j, v_j$ are uniformly bounded sequences of plurisubharmonic functions in $\Omega$ with*

$$\lim_{\zeta\to z} u_j(\zeta) = \varphi(z) \text{ for } z \in \partial\Omega, \quad \varphi \in C^\infty(\partial\Omega),$$

*and such that $v_j \to v$ in capacity, $v, v_j$ are bounded and psh in a neighbourhood of the closure of $\Omega$, $v_j = v = 0$ on $\partial\Omega$, and $v_j = v$ outside $\Omega$. Furthermore, $u_j \to u$ in $L^1(\Omega)$ for some $u \in PSH \cap L^\infty(\Omega)$, with $\lim_{\zeta\to z} u(\zeta) = \varphi(z)$ for $z \in \partial\Omega$. If*

$$(dd^c u_j)^n \le (dd^c v_j)^n = g_j dV, \quad g_j \in L^\infty(\Omega)$$

*then*

*a)* $\int |u - u_j| (dd^c u_j)^n \to 0$ *as* $j \to \infty$.
*b)* $(dd^c u_j)^n \to (dd^c u)^n$.
*c)* $cap(\{|u - u_j| \ge a\}) \to 0$ *as* $j \to \infty$ *for any* $a > 0$.

*Proof.* Part a). It is proved in [K1, Lemma 2.7 ] or [K2, Lemma 3.4.5] that under our assumptions

$$\int_{\{u-u_j>a\}} (dd^c u_j)^n \to 0 \text{ for any } a > 0. \tag{2.1}$$

Those proofs are conducted for some specific sequences, so for the sake of completeness the proof of (2.1) is given in Appendix.

We use notation $f_+ = \max(f, 0)$, $f_- = -\min(f, 0)$. By a version of Hartogs' lemma [H] $(u - u_j)_-$ tends uniformly to zero on any set where $u$ is continuous.

Since $u$ is continuous off a set of arbitrarily small capacity (see Theorem 1.3) and the sequence $u_j$ is uniformly bounded we can conclude that

$$\int (u-u_j)_-(dd^c u_j)^n \to 0.$$

This combined with (2.1) and the fact that by the assumptions and Chern-Levine -Nirenberg inequalities [CLN] the integrals $\int (dd^c u_j)^n$ are uniformly bounded, give a).

Part b) follows from a) and [K1] (see also Proposition 3.2.1 in [K2]).

For the proof of part c) let us fix $a>0$ and denote by $U_j$ the set $\{u_j < u - a\}$. Since cap is an inner capacity (see [BT3]) one can choose for every $j$ a compact set $K_j \subset U_j$ with

$$cap(K_j, \Omega_j) \geq cap(U_j, \Omega) - 1/j.$$

We denote by $u_{K_j}$ the relative extremal function of $K_j$ . By (1.2)

$$cap(K_j, \Omega) = \int_{K_j} (dd^c u^*_{K_j})^n.$$

Observe that

$$K_j \subset U_j \subset \{u_j < u + \frac{a}{3}u^*_{K_j} - \frac{a}{3}\} =: V_j \subset \{u_j < u - \frac{a}{3}\} \subset\subset \Omega.$$

Using this and the comparison principle we get

$$\begin{aligned}(\frac{a}{3})^n(cap(U_j,\Omega) - 1/j) \leq (\frac{a}{3})^n cap(K_j, \Omega) &\leq (\frac{a}{3})^n \int_{V_j} (dd^c u^*_{K_j})^n \\ &\leq \int_{V_j} (dd^c u_j)^n \leq \int_{\{u_j < u - \frac{a}{3}\}} (dd^c u_j)^n.\end{aligned}$$

Since, by part a), the right hand side tends to 0 we conclude that

$$cap(U_j, \Omega) \to 0,$$

which completes the proof.

**Theorem 2.2 (Stability theorem).** *Let $\Omega$ be a strictly pseudoconvex domain. Suppose $d\mu = (dd^c v)^n$ for $v \in PSH \cap L^\infty(\Omega)$ with*

$$\lim_{\zeta \to z} v(\zeta) = 0 \text{ for } z \in \partial\Omega, \quad \int_\Omega d\mu < \infty.$$

*Let $f_j$ be a sequence of $d\mu$- measurable functions satisfying $0 \leq f_j \leq 1$ and such that $f_j d\mu \to f\, d\mu$ as measures. Suppose that $u_j$ solves the Dirichlet problem*

$$(*) \qquad \begin{aligned} &u_j \in PSH \cap L^\infty(\Omega), \\ &(dd^c u_j)^n = f_j d\mu, \\ &\lim_{\zeta\to z} u_j(\zeta) = \varphi(z) \text{ for } z \in \partial\Omega, \end{aligned}$$

*for some continuous $\varphi$. Then for $u = \lim u_j$ (in $L^1(\Omega)$) we have $(dd^c u)^n = f\, d\mu$. Furthermore $u_j$ tends to $u$ in capacity.*

*Proof.* We shall reduce the proof to the case of smooth $\varphi$, continuous $f_j$ and compactly supported $\mu$. Suppose first that we have shown the statement for smooth $\varphi$. Then we can approximate the given $\varphi \in C(\partial\Omega)$ by a decreasing sequence $\varphi_s \in C^\infty(\partial\Omega)$ with $||\varphi_s - \varphi|| < 1/s$ (where $||\cdot||$ is the $L^\infty$ norm) and denote by $u_j^s$ the solution of $(*)$ with $\varphi_s$ in place of $\varphi$. By the comparison principle $||u_j^s - u_j|| < 1/s$ and for $u^s = \lim_{j\to\infty} u_j^s$ we have $||u^s - u|| \leq 1/s$. Thus

$$\begin{aligned} ||u_j - u|| &\leq ||u_j^s - u_j|| + ||u_j^s - u^s|| + ||u^s - u|| \\ &\leq ||u_j^s - u^s|| + 2/s. \end{aligned}$$

Given $\delta > 0$ and $\epsilon > 0$ choose $s > 6/\delta$ and, using the convergence in capacity $u_j^s \to u^s$ , $j \to \infty$, find $j_0$ such that for $j > j_0$

$$cap(\{|u_j^s - u^s| > \delta/3\}, \Omega) < \epsilon.$$

Then, by the above inequalities

$$cap(\{|u_j - u| > \delta\}, \Omega) < \epsilon, \quad j > j_0.$$

Therefore in what follows we can assume that $\varphi$ is smooth.

The following proposition allows to reduce the proof to the case of continuous $f_j$'s.

**Proposition.** *Given $d\mu$ measurable $f$ with $0 \leq f \leq 1$ and $\delta > 0$ one can find continuous $g$ such that for $u, v$ the solutions of the Dirichlet problem $(*)$ corresponding to measures $f\, d\mu$ and $g\, d\mu$ respectively, we have*

$$cap(\{|u - v| \geq \delta\}, \Omega) < \delta.$$

*Proof.* Fix $\epsilon = \delta^{n+1}/2$ and use Egorov's theorem to find continuous $g$ with $0 \leq g \leq 1$ and $\mu(E) < \epsilon$, where $E = \{f \neq g\}$. Denote by $\rho$ the solution of $(*)$ with measure $2d\mu_{|E}$ and zero boundary data. Then by the comparison principle

$$|u - v| \leq -\rho.$$

Consider a regular compact $K \subset \{\rho < -\delta\}$ (regular means that the extremal function is continuous). Then $K \subset \{\rho < \delta u_K\}$ and by the comparison principle

$$\begin{aligned} \delta^n cap(K, \Omega) &= \delta^n \int_K (dd^c u_K)^n \\ &\leq \int_\Omega (dd^c \rho)^n < 2\epsilon. \end{aligned}$$

Thus from the choice of $\epsilon$ and $K$ the statement follows.

Using this proposition we can find for any $f_j$ continuous $\tilde{f}_j$ such that the solutions $u_j$, $\tilde{u}_j$ corresponding to those measures obey

$$cap(\{|u_j - \tilde{u}_j| \geq 1/j\}, \Omega) < 1/j.$$

Since $cap$ is subadditive the last inequality says that the convergence in capacity of $u_j$ is equivalent to the same type of convergence of $\tilde{u}_j$. So we may assume that $f_j$ are continuous.

In the next step we prove $(dd^c u)^n = f\,d\mu$ under extra assumption that $\mu$ is compactly supported. Then we can replace $v$ by $\max(v-1, \rho)$ with $\rho$ being the defining function of $\Omega$ such that $\rho < v-2$ on $K \subset\subset \Omega$, where $K$ contains $supp\,\mu$ in its interior. Then this new function is defined and smooth in a neighbourhood of $\partial\Omega$. Note that the Monge-Ampère measure of new $v$ is equal to $d\mu$ plus a measure supported on $\Omega \setminus int\,K$ and thus $f_j d\mu$ are not changed if we take $f_j = 0$ off $K$. For such $v$ one can define a sequence of smooth psh functions $\rho_j \downarrow v$ in the closure of $\Omega$. Set $\alpha_j = \sup_\Omega \rho_j$. Consider a regularizing sequence $m_j$ for the function $m(x,y) = \max(x,y)$ of two real variables such that $m_j$ are smooth, non decreasing in both variables and convex, with $m_j \downarrow m$, $0 \le m_j - m \le \alpha_j$, and $m_j = m$ for $|x-y| < \alpha_j/2$. Then the functions $v_j = m_j(\rho_j - 2\alpha_j, \rho)$ are smooth and plurisubharmonic in a neighbourhood of the closure of $\Omega$. Since

$$\rho_j - 2\alpha_j \le v_j \le \rho_j + \alpha_j$$

and $\alpha_j \to 0$ we conclude that $v_j \to v$ uniformly on each compact set where the restriction of $v$ is continuous. Thus, in view of Theorem 1.3, it is convergent in capacity.

Define $g_j$ by $(dd^c v_j)^n = g_j dV$. For fixed $j$ we apply [BT1] to find $\tilde{w}_s$ such that

$$(dd^c \tilde{w}_s)^n = f_j g_s dV,$$

$\lim_{\zeta\to z} \tilde{w}_s(\zeta) = \varphi(z)$ for $z \in \partial\Omega$. Then by Lemma 2.1 and the uniqueness of the solution of the Dirichlet problem we have the convergence in capacity $\tilde{w}_s \to u_j$ as $s \to \infty$. Thus for $s(j)$ large enough the function $w_j = \tilde{w}_{s(j)}$ satisfies

$$\begin{aligned} ||u_j - w_j||_{L^1(\Omega)} &\le 1/j, \\ cap(\{|w_j - u_j| \ge 1/j\}) &\le 1/j. \end{aligned} \tag{2.2}$$

By (2.2) and the convergence $u_j \to u$ in $L^1(\Omega)$ we get that $w_j \to u$ in $L^1(\Omega)$. Applying Lemma 2.1 once more with $w_j$ in place of $u_j$ we conclude that $w_j \to u$ in capacity. Therefore, passing to a subsequence, we can assume that

$$cap(\{|w_j - u| \ge 1/j\}) \le 1/j.$$

This estimate combined with (2.2) and the subadditivity of cap give

$$cap(\{|u_j - u| \ge 2/j\}) \le 2/j.$$

Thus $u_j \to u$ in capacity. (Precisely speaking, we have shown that one can extract from $u_j$ a subsequence converging in capacity but this is enough to get the statement.) Furthermore, applying Xing's result (Theorem 1.1) we conclude that $(dd^c u_j)^n \to (dd^c u)^n$ which gives the statement for compactly supported $\mu$. To get the general case take an exhaustion sequence $K_s \uparrow \Omega$ of compact sets and define $\mu_s$ as the restriction of $\mu$ to $K_s$. Denote by $u_j^s$ the solution of $(*)$ corresponding to the measure $f_j d\mu_s$. By the first part of the proof $u_s = \lim_{j\to\infty} u_j^s$ satisfies $(dd^c u_s)^n = f\,d\mu_s$. Since $u_s \downarrow u$ we get by the convergence theorem [BT3] $(dd^c u)^n = f\,d\mu$. The proof is complete.

We can show a version of this theorem in hyperconvex domains for suitable boundary data.

**Theorem 2.2'.** *Let $\Omega$ be hyperconvex, $\varphi \in \partial\Omega$, and $d\mu = (dd^c v)^n$ for $v \in PSH \cap L^\infty(\Omega)$ with*

$$\lim_{\zeta \to z} v(\zeta) = \varphi(z) \ \textit{for}\ z \in \partial\Omega, \quad \int_\Omega d\mu < \infty.$$

*Then the statement of Theorem 2.2 remains true.*

*Proof.* Let $h \in PSH \cap C(\bar{\Omega})$ be such that $(dd^c h)^n = 0$ and $h = \varphi$ on $\partial\Omega$ (it exists by the assumptions). Take an exhaustion sequence of strictly pseudoconvex sets $\Omega_k \uparrow \Omega$. Fix $E \subset\subset \Omega$ and $\delta, \epsilon > 0$. Choose $k_0$ so large that $E \subset \Omega_{k_0}$ and

$$h - v < \delta/2 \quad \text{on } \partial\Omega_k,\ k \geq k_0. \tag{2.3}$$

Applying [K1] solve

$$\begin{aligned}
&u_{jk} \in PSH \cap L^\infty(\Omega_k),\\
&(dd^c u_{jk})^n = f_j d\mu_{|\Omega_k},\\
&\lim_{\zeta \to z} u_{jk}(\zeta) = h(z) \text{ for } z \in \partial\Omega_k,
\end{aligned}$$

$$\begin{aligned}
&u \in PSH \cap L^\infty(\Omega),\\
&(dd^c u)^n = f d\mu,\\
&\lim_{\zeta \to z} u(\zeta) = h(z) \text{ for } z \in \partial\Omega.
\end{aligned}$$

and

$$\begin{aligned}
&v_k \in PSH \cap L^\infty(\Omega_k),\\
&(dd^c v_k)^n = f d\mu_{|\Omega_k},\\
&\lim_{\zeta \to z} v_k(\zeta) = h(z) \text{ for } z \in \partial\Omega_k.
\end{aligned}$$

By the comparison principle and (2.3) we have

$$u \leq v_k < u + \delta/2, \quad u_j \leq u_{jk} < u_j + \delta/2 \ \text{ in } \Omega_k.$$

Therefore

$$\begin{aligned}
u_j - u &\leq u_{jk} - u = (u_{jk} - v_k) + (v_k - u) < (u_{jk} - v_k) + \delta/2,\\
u - u_j &\leq (v_k - u_{jk}) + (u_{jk} - u_j) < v_k - u_{jk} + \delta/2.
\end{aligned} \tag{2.4}$$

Fix $k > k_0$ and use Theorem 2.2 to find $j_0$ with

$$cap(\{|u_{jk} - v_k| > \delta/2\} \cap E) < \epsilon/2 \tag{2.5}$$

for $j > j_0$. Using subadditivity of $cap$ we infer from (2.4) and (2.5) that

$$cap(\{|u_j - u| > \delta\} \cap E) \leq 2cap(\{|u_{jk} - v_k| > \delta/2\} \cap E) < \epsilon.$$

This proves the statement.

In the most general version of the stability theorem we consider the family of functions from $\mathcal{F}(\varphi)$ with a bound on their Monge-Ampère masses.

**Theorem 2.2" (stability in $\mathcal{F}^a(\varphi)$).** *Let $\Omega$ be hyperconvex and let $d\mu$ be a non-negative Borel measure which vanishes on pluripolar sets, such that $d\mu = (dd^c v)^n$ for $v \in \mathcal{F}(\varphi)$, $\varphi \in C(\partial\Omega)$, and satisfying $\int_\Omega d\mu < \infty$. Then for $u_j, u \in \mathcal{F}(\varphi)$ which solve*

$$(dd^c u_j)^n = f_j d\mu, \quad (dd^c u)^n = f d\mu$$

*for $f, f_j$ measurable with values in $[0,1]$, and satisfying $f_j d\mu \to f\, d\mu$, we have $u_j \to u$ in capacity.*

*Proof.* By [C3] there exists $\psi \in \mathcal{E}_0$ and $g \in L^1((dd^c\psi)^n)$ with $d\mu = g(dd^c\psi)^n$. Set

$$d\mu_k = \min(g,k)(dd^c\psi)^n.$$

Then, by [K1][C3][C4] there exist $v_k, u_{jk} \in \mathcal{F}(\varphi) \cap L^\infty(\Omega)$ and $\eta_k \in \mathcal{F}$ solving

$$\begin{aligned}&(dd^c v_k)^n = f d\mu_k, \quad (dd^c u_{jk})^n = f_j d\mu_k\\ &(dd^c \eta_k)^n = [g - \min(g,k)](dd^c\psi)^n.\end{aligned}$$

By the comparison principle $u_{jk} \downarrow u_j, v_k \downarrow u$. Hence

$$\begin{aligned}&u_j - u \le u_{jk} - u = (u_{jk} - v_k) + (v_k - u)\\ &u - u_j \le (v_k - u_{jk}) + (u_{jk} - u_j) \le v_k - u_{jk} - \eta_k,\end{aligned} \tag{2.6}$$

where the last inequality follows from the comparison principle.

Fix $E \subset\subset \Omega$ and $\delta, \epsilon > 0$. Using the fact that $\eta_k \uparrow 0$ (in capacity) choose $k$ so large that

$$cap(\{\eta_k < -\delta/4\} \cap E) < \epsilon/4, \quad cap(\{|v_k - u| > \delta/4\} \cap E) < \epsilon/4.$$

By Theorem 2.2' $u_{jk} \to v_k$ in capacity as $j \to \infty$. Find $j_0$ with

$$cap(\{|u_{jk} - v_k| > \delta/4\} \cap E) < \epsilon/4$$

for $j > j_0$. Those inequalities and (2.6) yield

$$\begin{aligned}cap(\{|u_j - u| > \delta\} \cap E) &\le 2cap(\{|u_{jk} - v_k| > \delta/4\} \cap E)\\ +cap(\{|v_k - u| > \delta/4\} \cap E) &+ cap(\{\eta_k < -\delta/4\} \cap E) < \epsilon.\end{aligned}$$

The proof is completed.

**Corollary.** *Let $\mu$ be a measure and $\varphi \in C(\partial\Omega)$ a function satisfying the hypothesis of Theorem 2.2". Denote by $A(\varphi,\mu)$ the set of all solutions of the Dirichlet problem*

$$\begin{aligned}&u \in \mathcal{F}^a(\varphi),\\ &(dd^c u)^n = f d\mu,\end{aligned}$$

*where $f$ varies over all $d\mu$ measurable functions with $0 \le f \le 1$. Then in $A(\varphi,\mu)$ the weak* convergence is equivalent to the convergence in capacity.*

*Proof.* Let $u_j \to u$ in the sense of distributions and let $(dd^c u_j)^n = f_j d\mu$. Choose a subsequence $f_{j_k}$ converging in weak* topology to some $f$. By Theorem 2.2" $(dd^c u)^n = f\, d\mu$ and $u_{j_k} \to u$ in capacity. Since this argument works for any subsequence extracted from the original one we get $u_j \to u$ in capacity. Thus the corollary follows.

We end this section proving the continuity of the Monge-Ampère operator in the class $\mathcal{F}_1$.

**Theorem 2.3.** *Let $u, u_j \in \mathcal{F}_1$ with $u_j \to u$ as distributions. If*

$$\lim_{j\to\infty} \int_\Omega u_j (dd^c u_j)^n = \int_\Omega u(dd^c u)^n$$

*then*

*a)* $\int |u - u_j|(dd^c u_j)^n \to 0$ *as* $j \to \infty$.

*b)* $(dd^c u_j)^n \to (dd^c u)^n$.

*c)* $u_j \to u$ *in capacity.*

*Proof.* Denote by $v_j$ the function which is the upper semicontinuous regularization of $\sup_{k\geq j} u_k$. Then $v_j \geq u_j$ and $v_j \downarrow u$. So

$$\int_\Omega |u - u_j|(dd^c u_j)^n \leq \int_\Omega [(v_j - u_j) + (v_j - u)](dd^c u_j)^n. \tag{2.7}$$

The inequality $v_j \geq u_j$ and integration by parts, applied repeatedly, give

$$\begin{aligned}\int_\Omega -v_j(dd^c v_j)^n &\leq \int_\Omega -u_j(dd^c v_j)^n \\ &= \int_\Omega -v_j dd^c u_j \wedge (dd^c v_j)^{n-1} \leq ... \leq \int_\Omega -v_j (dd^c u_j)^n.\end{aligned}$$

Hence by (1.1) we have

$$\int_\Omega u_j(dd^c u_j)^n \leq \int_\Omega v_j (dd^c u_j)^n \leq \int_\Omega v_j(dd^c v_j)^n \to \int_\Omega u(dd^c u)^n.$$

From [C3, Theorem 3.2] and the assumptions one obtains

$$\int_\Omega -u(dd^c u_j)^n \leq (\int_\Omega -u(dd^c u)^n)^{1/n+1} (\int_\Omega -u_j(dd^c u_j)^n)^{n/n+1} \to \int_\Omega -u(dd^c u)^n.$$

Therefore, in view of the last two formulae, the right hand side of (2.7) goes to zero.

Part b) of the theorem is a consequence of the first part, [C3, Lemma 5.3] and (1.1). The convergence in capacity we prove in the same way as in Lemma 2.1 using the fact that the comparison principle works in $\mathcal{F}_1$.

## 3. The equation of complex Monge-Ampère type.

In this section we show a very general existence theorem for the complex equation of Monge-Ampère type. We shall need the following two results from [C6].

**Theorem 3.1.** *Suppose $\Omega$ is a bounded hyperconvex domain and $\varphi$ is a continuous function on $\partial\Omega$ such that $\varphi = u$ for some function $u$ which is psh in $\Omega$ and continuous along $\partial\Omega$. Let $\mu$ be a finite positive measure in $\Omega$ which vanishes on all pluripolar sets. Then there is a unique $v \in \mathcal{F}^a(\varphi)$ with*

$$(dd^c v)^n = d\mu.$$

**Theorem 3.2.** *The comparison principle is valid in $\mathcal{F}^a(\varphi)$.*

Now we are in a position to prove the following existence result.

**Theorem 3.3.** *Let $\Omega$ be a bounded hyperconvex domain and let $\mu$ and $\varphi$ satisfy the assumptions of Theorem 3.1. Suppose further that $F(t, z)$ is a $dV \times d\mu$- measurable function on $(-\infty, 0] \times \Omega$ which is continuous in $t$ and which satisfies*

$$0 \leq F(t, z) \leq g(z)$$

*for some $g \in L^1(d\mu)$. Then there exists $u \in \mathcal{F}^a(\varphi)$ such that*

$$(dd^c u)^n = F(u(\cdot), \cdot)\, d\mu.$$

*Proof.* It is no loss of generality to assume that $\varphi$ is negative. Denote by $A$ the set of $u \in \mathcal{F}^a(\varphi)$ such that $u \geq u_0$ where $u_0 \in \mathcal{F}^a(\varphi)$ satisfies

$$(dd^c u_0)^n = g\, d\mu.$$

The set is convex and compact in $L^1(dV)$ topology. The solution $u_0$ exists and is uniquely determined by Theorem 3.1. We define a map $\mathcal{G} : A \to A$ so that $\mathcal{G}(u) = v$ if

$$(dd^c v)^n = F(u(\cdot), \cdot)\, d\mu.$$

Then $(dd^c v)^n \leq g\, d\mu$ and so, by Theorem 3.2, $v \geq u_0$ and $v \in A$. By Schauder's fixed point theorem [S] we are done as soon as we show that $\mathcal{G}$ is continuous. So suppose $u_j \in A$ with $u_j \to u$. Applying the corollary to Lemma 1.4 one can extract a subsequence (still denoted by $u_j$) converging to $u$ $d\mu$ almost everywhere. Let $w_k \in \mathcal{F}^a(\varphi)$ be the solution of

$$(dd^c w_k)^n = \sup_{j \geq k} F(u_j(\cdot), \cdot) d\mu \geq F(u_k(\cdot), \cdot) d\mu.$$

Then, by the comparison principle (Theorem 3.2), $w_k \leq v_k$, where $v_k = \mathcal{G}(u_k)$. Furthermore, let $\tilde{w}_k$ solve

$$(dd^c \tilde{w}_k)^n = \inf_{j \geq k} F(u_j(\cdot), \cdot) d\mu \leq F(u_k(\cdot), \cdot) d\mu.$$

Then $\tilde{w}_k \geq v_k$. Since $\lim_{k\to\infty} w_k$ and $\lim_{k\to\infty} \tilde{w}_k$ have the same Monge-Ampère mass both limits define the same function $v = \mathcal{G}(u)$ (see Theorem 3.1). From the above inequalities we get

$$v = \lim_{k\to\infty} w_k \leq \lim_{k\to\infty} v_k \leq \lim_{k\to\infty} \tilde{w}_k = v.$$

Thus $\mathcal{G}(u_j) \to \mathcal{G}(u)$ and the theorem follows.

## Appendix

*Proof of formula* (2.1).

**Lemma A.** *Suppose that, after passing to a subsequence (which does not change $u$ since $u_j$ converge in $L^1$) we have*

$$\int_{E_j(a_0)} (dd^c u_j)^n > A_0, \quad A_0 > 0, a_0 > 0,$$

*with the notation $E_j(a_0) = \{u_j < u - a_0\}$. Then there exist $a_m > 0, A_m > 0, k_1 > 0$ such that*

$$\int_{E_j(a_m)} (dd^c v_j)^{n-m} \wedge (dd^c v_k)^m > A_m, \; j > k > k_1. \tag{A.1}$$

*Proof.* We shall proceed by induction over $m$. For $m = 0$ the statement holds by the hypothesis. We assume that (A.1) is true for some fixed $m < n$ and now we shall prove it for $m + 1$.

Let us observe that by the Chern-Levine-Nirenberg inequalities [CLN] there exists $C > 0$ such that

$$\int_\Omega T \le C, \tag{A.2}$$

for currents $T$ which are wedge products of $dd^c v_j$ or $dd^c u_j$. Indeed, $v_j$ are bounded and plurisubharmonic in a fixed neighbourhood of $\bar\Omega$ and each $u_j$ can be extended to the same set as a plurisubharmonic function by $h + v_j$, where $h$ is a smooth psh function in a neighbourhood of $\bar\Omega$ which is equal $\varphi$ on $\partial\Omega$. On this larger domain we apply the Chern-Levine-Nirenberg inequalities. We also assume that all the functions involved have $L^\infty$ norm bounded by one.

Denote by $T = T(j, k, m)$ the current $(dd^c v_j)^{n-m-1} \wedge (dd^c v_k)^m$. For fixed $\epsilon \in (0, \frac{a_m A_m}{4C+4})$ we choose an open set $U$ such that

$$cap(U, \Omega) < \epsilon/2^{n+1},$$

and both $u$ and $v$ are continuous on $\Omega \setminus U$. Note that, assuming for simplicity, that $-1 < v_j, u_j < 0$, we have

$$\int_U (dd^c(v_j + v_k))^n \le 2^n cap(U, \Omega) < \epsilon/2. \tag{A.3}$$

Moreover one can replace $v_j$ by $u_j$ in this inequality. Then for $k > k_0$ and we have

$$\begin{aligned} v_k &\le v + \epsilon, \\ u_k &\le u + \epsilon, \end{aligned} \tag{A.4}$$

on $\Omega \setminus U$. Indeed, the inequalities are valid in a neighbourhood of $\partial\Omega$ because all $u_j$ (resp. $v_j$) are bounded from above by the maximal function in $\Omega$ with boundary data $\varphi$ (resp. 0). On the remaining part of $\Omega \setminus U$ one obtains (A.4) by the Hartogs lemma. Set

$$\begin{aligned} J'(j,k) &:= \int_\Omega (u - u_j) dd^c v_j \wedge T, \\ J(j,k) &:= \int_\Omega (u - u_j) dd^c v_k \wedge T, \quad j > k > k_0. \end{aligned}$$

Integrating by parts we get

$$J'(j,k) - J(j,k) = \int_\Omega (v_j - v_k) dd^c(u - u_j) \wedge T = \int_{\Omega \setminus U} \dots + \int_U \dots .$$

Since $v_j \to v$ uniformly away from $U$ one can find $k_1 > k_0$ such that $||v_j - v_k|| < \epsilon/2C$ on $\Omega \setminus U$ for $j > k > k_1$. Thus, using (A.2) and (A.3), we conclude that each integral on the right hand side does not exceed $\epsilon/2$ for such $j, k$. So

$$J'(j,k) - J(j,k) \leq \epsilon \leq \frac{a_m A_m}{4}, j > k > k_1. \tag{A.5}$$

Using the induction hypothesis, (A.2), (A.3), and (A.4) we have

$$\begin{aligned} J'(j,k) &\geq a_m \int_{E_j(a_m)} dd^c v_j \wedge T - \epsilon \int_{\Omega \setminus U} dd^c v_j \wedge T - \int_U dd^c v_j \wedge T \\ &\geq a_m \int_{E_j(a_m)} dd^c v_j \wedge T - \epsilon(C+1) \geq a_m A_m - \epsilon(C+1) \geq \frac{3 a_m A_m}{4}, \end{aligned}$$

for $j > k > k_2 > k_1$. Combined with (A.5) this gives

$$J(j,k) \geq \frac{a_m A_m}{2}, \quad j > k > k_2. \tag{A.6}$$

Fixing $d > 0$ one can estimate $J(j,k)$ from above as follows

$$\begin{aligned} J(j,k) &\leq \int_{\{u_j < u - d\}} dd^c v_k \wedge T + d \int_\Omega dd^c v_k \wedge T \\ &\leq \int_{\{u_j < u - d\}} dd^c v_k \wedge T + dC. \end{aligned}$$

Setting $a_{m+1} := d = \frac{a_m A_m}{4C}$ in the last formula and combining it with (A.6) we finally arrive at

$$\int_{E_j(a_{m+1})} dd^c v_k \wedge T \geq \frac{a_m A_m}{4} := A_{m+1}, \quad j > k > k_2,$$

which concludes the proof of the inductive step. Thus Lemma A follows.

Now we shall prove formula (2.1) reasoning by contradiction. So, suppose the hypothesis of Lemma A is valid. Then using its statement for $m = n$ we can fix $k > k_1$ such that

$$\int_{E_j(a_n)} (dd^c v_k)^n > A_n \quad \text{if } j > k.$$

Since, by the construction, $(dd^c v_k)^n \leq M_k dV$ for some $M_k > 0$ one infers from the last inequality that

$$V(E_j(a_n)) \geq M_k^{-1} \int_{E_j(a_n)} (dd^c v_k)^n > \frac{A_n}{M_k}, \quad j > k,$$

which contradicts the fact that $u_j \to u$ in $L^1_{loc}$. Thus (2.1) follows.

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