Relative Tangent Cone of Analytic Sets

by

Danuta Ciesielska

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Summary. The purpose of this paper is to give a characterization of the relative tangent cone of an analytic curve and an analytic set with an isolated intersection. Moreover, we present an effective computation of the multiplicity intersection of a curve and a set with an $s-$parametrization.

1. Introduction. We consider an analytic curve X and an analytic set Y in a neighbourhood Ω of a in $\mathbb{C}^m$ such that $X \cap Y = \{a\}$ and study the relative tangent cone of them at a (see Section 2 in [ATW]). By an analytic curve we mean an analytic set of pure dimension 1. The relative tangent cone and the intersection multiplicity of analytic curves are additive, so we restrict our attention, without lose of generality, to an analytic curves with irreducible germs at 0.

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The main result of this paper, which is an improvement of the result from the paper [C], is the equality $C_a(X,Y) + C_a(X) = C_a(X,Y)$. This theorem gives a strong geometric characterization of the relative tangent cone of an analytic curve and an analytic set.

In the last section we give an effective calculation of the intersection multiplicity of an analytic sets with $s-$parametrization and an analytic curve.

2. Main result. Let Ω be an open neighbourhood of $a \in \mathbb{C}^m$ ($m \geqslant 2$), X analytic curve and Y analytic set, both of which in Ω, such that $X \cap Y = \{a\}$.

By the relative tangent cone $C_a(X,Y)$ of the analytic sets X, Y at the point a we mean (see [ATW]) the set:
$\{v \in \mathbb{C}^m : \exists \ \{x \ \} \in X, \ \{y \ \} \in Y, \ \{ \quad \} \in \mathbb{C}$:
$x \quad \to a, \ y \quad \to a, \qquad (x \quad - y \ ) \to v\}$.

The main goal of this paper is to prove the following theorem.

Theorem 2.1. *If the curve X have irreducible germ at the point a and $X \cap Y = \{a\}$, then $C_a(X,Y) + C_a(X) = C_a(X,Y)$.*

P r o o f. The germ of the curve X at a is irreducible, so the relative tangent cone $C_a(X)$ is a complex line (see [Ł], [Ch]). If this line is transitive to the set Y, which means $C_a(X) \cap C_a(Y) = \{a\}$, then by Property 2.9 in [ATW] we obtain the equality $C_a(X,Y) = C_a(X) + C_a(Y)$.

In the second case we have $C_a(X) \qquad C_a(Y)$. Without loose of generality, we may assume that $a = 0$. Now, after a suitable biholomorphic change of coordinates, we have $C_0(X) = \mathbb{C}_1 := \{x \in \mathbb{C}^m : x_2 = x_3 = ... = x_m = 0\}$.

By the Puiseux Theorem ([Ł], II. 6.2) we can parametrize X (shrinking Ω, if necessary) in the following way:

$$U \ni t \mapsto (t^p, \varphi(t)) \in X, \ \operatorname{ord}_0 \varphi > p$$

by $\operatorname{ord}_0 \varphi$ we denote $\min\{\operatorname{ord}_0 \varphi_i : i = 2, \dots, m\}$.

Fix $v = (v_1, v_2, \dots, v_m) \in C_a(X,Y)$ and $(c, 0, \dots, 0) \in \mathbb{C}_1 = C_a(X)$. By the definition of the relative tangent cone the are sequences: $\{t_n\} \in \mathbb{C}$

and $\{(y_n^1, y_n^2, \ldots, y_n^m)\} \in Y$ such that $t_n \to 0$, $(y_n^i) \to 0$, $n^p(t_n^p - y_n^1) \to v_1$ and $n^p(\varphi_i(t_n) - y_n^i) \to v_i$ for $i \in \{2, 3, \ldots, m\}$.

Applying Lemma 2.1 from [C] to the sequence $\{t_n\}$ yields the sequence $\{h_n\}$ which has the following properties:

(1) $h_n \to 0$,

(2) $n^d((t_n + h_n)^d - t_n^d) \to c$,

(3) for any holomorphic function $\varphi : \Omega \to \mathbb{C}$ defined in an open neighbourhood Ω of $0 \in \mathbb{C}$ with $\operatorname{ord}_0 \varphi > d$ we have
$n^d(\varphi(t_n + h_n) - \varphi(t_n)) \to 0$.

Substituting $t_n + h_n$ for t_n in the parametrization we move a little the points on the curve X and

i) for the first coordinate we have:

$$n^p((t_n + h_n)^p - y_n^1) = n^p((t_n + h_n)^p - t_n) + n^p(t_n^p - y_n^1)$$

and observe that the left side of the equality converges to the first coordinate of a vector in the relative tangent cone of $C_a(X, Y)$, wheras components of the sum converges respectively to c and v_1.

ii) for the $i-$th coordinate ($i \in \{2, \ldots, m\}$) we have

$$n^p(\varphi_i(t_n + h_n) - y_n^i) = n^p(\varphi_i(t_n + h_n) - \varphi_i(t_n)) + n^p(\varphi_i(t_n) - y_n^i).$$

and observe that the left side of the $i-$th equality converges to $i-$th coordinate of a vector in the relative tangent cone $C_a(X, Y)$, wheras the first component of the sum converges to 0 and the second converges to respective v_i.

Consequently, $v + (c, 0, \ldots, 0) \in C_0(X, Y)$. Since $C_0(X) = \mathbb{C}_1$ we conclude that $C_0(X, Y) + C_0(X) = C_0(X, Y)$ and the theorem follows.

The following corollary, which follows from Theorem 2.1, is an strong geometric descriptions of the relative tangent cone of an curve and a set:

Corollary 2.2. *Let Ω be an open neighbourhood of $0 \in \mathbb{C}^m$ $(m \geqslant 2)$ and let X be an analytic curve with irreducible germs at the origin and Y be an analytic set both in Ω such that $X \cap Y = \{0\}$. If the relative tangent cone of X is the axis $\mathbb{C}_1$, then there exists an algebraic cone $\mathbb{S}$ $\mathbb{C}^{m-1}$ such that $C_0(X, Y) = \mathbb{C} \times \mathbb{S}$.*

3. Multiplicity of the intersection of analytic sets. We present an effective computation of multiplicity intersection of an analytic curve and the an analytic set with $s-$parametrization (for definition see [TW]). For the effective computation for two analytic curves see ([K], [KN]). Intersection multiplicity is additive, so without lose of generality we may assume that the analytic curve X has irreducible germ in a point in which we compute multiplicity. Now we state a simpler version of Theorem 4.2 from [TW], which is more suitable for our purpose.

Theorem 3.1. *Let $\Omega \quad \mathbb{C}^m$ be an open set, Y be a pure dimensional analytic set of Ω, a set $D \quad \mathbb{C}^d$ be a domain $(m \quad d)$ and $\varphi : D \longrightarrow X$ be an $s-$parametrization of Y. Moreover, let mapping $f : \Omega \longrightarrow \mathbb{C}^d$ be holomorphic. If $\dim f^{-1}(0) = m - d$ and $f^{-1}(0) \cap Y = \{a\}$, then*

$$(f^* \cdot Y; a) = \frac{1}{s} \sum_{b \in \varphi^{-1}(a)} \quad (f \quad \varphi; b).$$

Let Ω be an open neigbourhood of $0 \in \mathbb{C}^m$, Y an irreducible pure dimensional analytic set and X analytic curve of Ω with irreducible germ in origin. The curve X have Puiseux's parametrization:

$U \ni t \mapsto \ \Phi(t) := \ (t^p, \varphi(t)) \in X, \quad \operatorname{ord}_0 \varphi > p$, which is 1-parametrization of the cure X. We assume that the set Y have $s-$parametrization:

$\Psi : D \ni \quad \mapsto \quad (\quad) \in Y$, where D is a domain in $\mathbb{C}^k$

and $X \cap Y = \{0\}$. Moreover, we have $\Psi^{-1}(0) = \{0\}$. To makes computations easier we will describe the relative tangent cone in a simpler way. By Corollary 2.2 there exists a $k-$dimensional algebraic cone $\mathbb{S} \quad \mathbb{C}^m$ such that $C_0(X, Y) = \mathbb{C} \times S = \mathbb{C}_1 + (\{0\} \times S) \quad \mathbb{C} \times \mathbb{C}^{m-1} = \mathbb{C}^m$.
For the linear surjection $l : \mathbb{C}^{m-1} \longrightarrow \mathbb{C}^k$ we denote

$f_l : \mathbb{C}^m \times \mathbb{C}^m \ni (x, y) = ((x_1, x'), (y_1, y')) \mapsto (x_1 - y_1, l(x' - y')) \in \mathbb{C}^{k+1}$.

Using the above notation and multiplicity of the holomorphic map we have

Theorem 3.2. *If $\Psi^{-1}(0) = \{0\}$ and $\operatorname{Ker} l \cap S = \{0\}$, then*

$$i(X, Y; 0) = \frac{1}{s} \ _0(f_l \quad (\Phi \times \Psi)).$$

P r o o f. Let $T := X \times Y$, $: \mathbb{C}^m \times \mathbb{C}^m \ni (x,y) \mapsto x - y \in \mathbb{C}^m$ and $\Delta := \mathrm{Ker}$. By the theory developed in [ATW], to compute the multiplicity of the isolated intersection of X and Y we should calculate the multiplicity of the isolated intersection of T with the subspace Δ at the point $0 \in \mathbb{C}^m \times \mathbb{C}^m$ (see [ATW], Def. 5.1). Observe that $\mathrm{Ker}\, f_l$ is a linear subspace of $\mathbb{C}^m \times \mathbb{C}^m$ of codimension $k+1$ and $C_0(T,\Delta) = {}^{-1}(\mathbb{C} \times \mathbb{S}) = \{(x,y) \in \mathbb{C}^m \times \mathbb{C}^m : x' - y' \in \mathbb{S}\}$ (see [ATW], Lemma 2.4). Therefore, our assumption implies $\mathrm{Ker}\, f_l \cap C_0(T,\Delta) = \Delta$ and by Theorem 4.4 in [ATW] we get $i(X \cdot Y; 0) = i(T \cdot \mathrm{Ker}\, f_l; 0)$. According to Definition 4.1 in [TW], the mapping $\Phi \times \Psi : U \times D \longrightarrow T = X \times Y$ is an s–parametrization of T and $\mathrm{Ker}\, f_l = Z_{f_l}$ is a cycle of zeros of the mapping f_l. By Theorem 3.1 we have $(T \cdot Z_{f_l}; 0) = \frac{1}{s} (Z_{f_l \, (\Phi\times\Psi)}; 0)$ and $(T \cdot Z_{f_l}; 0) = (T \cdot \mathrm{Ker}\, f_l; 0) = i(X,Y;0)$, so $(Z_{f_l \, (\Phi\times\Psi)}; 0) = {}_0(f_l \, (\Phi \times \Psi))$ which completes the proof.

Example 3.3. Consider the analytic curve:
$X = \{(t^3, t^5, 0, 0) \in \mathbb{C}^4 \mid t \in \mathbb{C}\}$
and the pure dimensional analytic set:
$Y = \{(\ ^3, \ ^2\varrho, \ ^2\varrho\, \varrho^3) \in \mathbb{C}^4 | \ , \varrho \in \mathbb{C}\}$ of dimension 2.

The curve X have 1–parametrization and Y have 3–parametrization
$\Psi : \mathbb{C}^3 \ni (\ , \varrho) \mapsto (\ ^3, \ ^2\varrho, \ ^2\varrho\, \varrho^3) \in Y$
and $X \cap Y = \{0\}$. Moreover, the axis $\mathbb{C}_1$ lays in $C_0(X)$ and $C_0(Y)$. The cone $C_0(X,Y)$ have the form $\mathbb{C} \times \mathbb{S}$, for an algebraic 2–dimensional cone $\mathbb{S}$. Observe that the vector $(0,1,1) \notin \mathbb{S}$. By Theorem 3.1, intersection multiplicity $i(X,Y;0) = \frac{1}{3} \ {}_0(f_l)$ where the mapping is following

$$f_l : \mathbb{C}^3 \ni (t, \ , \varrho) \mapsto (t^3 - \ ^3, l(t^5 - \ ^2\varrho, \ ^2\varrho\, \varrho^3)) \in \mathbb{C}^3$$

and a linear surjection $l : \mathbb{C}^3 \longrightarrow \mathbb{C}^2$ is such that $\mathrm{Ker}\, l \cap \mathbb{S} = \{0\}$. Consider $l : \mathbb{C}^3 \ni (x_2, x_3, x_4) \mapsto (x_2, x_3 - x_4) \in \mathbb{C}^2$, so $\mathrm{Ker}\, l = \{(0,t,t) : t \in \mathbb{C}\}$ and $\mathrm{Ker}\, l \cap \mathbb{S} = \{0\}$. We will compute the multiplicity ${}_0(f_l)$ using Theorem 4.3 from [TW]. Denote

$g : \mathbb{C}^3 \ni (t, \ , \varrho) \mapsto t^5 - \ ^2\varrho \in \mathbb{C}$,

$h : \mathbb{C}^3 \ni (t, \ , \varrho) \mapsto (t^3 - \ ^3, \ ^2\varrho - \varrho^3) \in \mathbb{C}^2$.

To compute the multiplicity of $f_l = (g,h)$ at 0, we observe that $Z_h = 2A_1 + A_2$ where A_1 and A_2 are the sets of three lines with respective equations $t^3 - \quad {}^3 = 0, \varrho = 0$ and $t^3 - \quad {}^3 = 0, \qquad - \varrho = 0$. Now, we have ${}_0(f_l) = 3 \cdot 2 \cdot 5 + 3 \cdot 3 = 39$, so $i(X, Y; 0) = 13$.

Note that in the presented method we do not need to know the form of the relative tangent cone. It is enough to know a suitable linear subspace which allowed to chose a linear surjection.

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