Local approximation of analytic sets by Nash sets

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IMUJ PREPRINT 2004/04
21 April 2004

Abstract

We show that every analytic subset of $\mathbf{C}^m$ of pure dimension n can be locally approximated by a sequence of Nash sets of pure dimension n.

1 Introduction

The problem of algebraic approximations of analytic objects appears in a natural way in many areas of research. In complex geometry various results have been obtained, especially in the case where the considered objects are manifolds or mappings (see [7], [9], [12], [13], [14]).

This paper is devoted to approximating analytic sets by Nash sets. The approximation is expressed by means of the notion of the *local uniform convergence*. More precisely, let U be an open subset of $\mathbf{C}^n$ and let Y, Y_ν, for $\nu \in \mathbf{N}$, be closed subsets of U. We say that $\{Y_\nu\}$ converges to Y *locally uniformly* if and only if:

(1) for every $a \in Y$ there exists a sequence $\{a_\nu\}$ such that $a_\nu \in Y_\nu$ and $a_\nu \longrightarrow a$ in the standard topology of $\mathbf{C}^n$,

(2) for every compact subset K of U such that $K \cap Y = \emptyset$ there exists $\nu_0 \in \mathbf{N}$ such that $Y_\nu \cap K = \emptyset$ for all $\nu > \nu_0$.

Then we write $Y_\nu \longrightarrow Y$.

For details concerning the local uniform convergence see [16].

[1] Research partially supported by KBN Grant no. 2P03A 015 22

It has been well known that if the set X of pure dimension k that we want to approximate is a set-theoretic complete intersection in $\mathbf{C}^n$, that is there are entire functions $f_1, \ldots, f_{n-k}$ describing it, we obtain a sequence $\{X_\nu\}$ of algebraic subsets of $\mathbf{C}^n$ converging to X by approximating f_j by polynomials (see [16] Corollary 4, p. 392). Due to the Sard theorem, we may assume that X_ν are non-singular of pure dimension k (if not, we add small generic constants to the polynomials). Unfortunately, this method does not work in the case where the number of the functions describing X is greater than its codimension.

The problem of approximation of analytic sets which are not complete intersections is discussed in [2]. The main result of that paper concerning the local approximation is the following. Let D be an open connected subset of $\mathbf{C}^n$ and let X be an analytic subset of $D \times \mathbf{C}^k$ of pure dimension n with proper projection onto D such that the set Σ_X is a Nash subset of D. Here $\Sigma_X = \{x \in D : \sharp((\{x\} \times \mathbf{C}^k) \cap X) < \max_{x' \in D} \sharp((\{x'\} \times \mathbf{C}^k) \cap X)\}$. Then for every $a \in D$ there exists a neighborhood V such that $(V \times \mathbf{C}^k) \cap X$ can be approximated by Nash sets of pure dimension n. Additionally, these sets have the same generic cardinality of the fibers of the projection onto D as X. The fact that every analytic surface can be locally approximated by two-dimensional Nash sets has been obtained as a corollary.

Let us mention that the question whether it is possible to approximate a given analytic set by Nash ones is related to the problem of the existence of a Nash variety biholomorphically equivalent to this set. In [9] (Theorem 1.2, p. 336) Lempert proved that for every compact subset of a reduced Stein space with isolated singularities there exists a neighborhood which is biholomorphic to an open subset of some algebraic variety. In view of this result and the fact that the space obtained by normalization of a two-dimensional analytic set has isolated singularities, one can easily prove that every analytic set of dimension two can be locally approximated by two-dimensional Nash sets. (For works on biholomorphic equivalence see also [1], [3], [11]).

In this paper we generalize the result for analytic surfaces showing that every analytic set can be locally approximated by Nash sets independently of its dimension and the dimension of the space in which it is contained. The main result is the following

Theorem 1.1 *Let Ω be an open subset of $\mathbf{C}^m, m \in \mathbf{N}$ and let X be an analytic subset of Ω of pure dimension n. Then for every $a \in X$ there exists an open neighborhood U of a in Ω and a sequence $\{X_\nu\}$ of Nash subsets of U of pure dimension n such that $X_\nu \longrightarrow X \cap U$.*

The organization of this paper is as follows. Section 2 is of preparatory nature. There we fix the notation and collect facts and definitions concerning Nash functions, Nash sets and the structure of analytic sets. In Section 3 we prove our main result.

2 Preliminaries

2.1 Analytic sets with proper projection

In this subsection we collect facts concerning analytic sets with proper projection that will be useful in the proof of the main theorem.

The following mappings will denote natural projections.
$\pi : \mathbf{C}^{n+1} \times \mathbf{C}^k \longrightarrow \mathbf{C}^{n+1}$,
$\pi_i : \mathbf{C}^{n+1} \times \mathbf{C}_1 \times \ldots \times \mathbf{C}_k \longrightarrow \mathbf{C}^{n+1} \times \mathbf{C}_i$
$\widetilde{\pi}_i : \mathbf{C}^{n+1} \times \mathbf{C}_i \longrightarrow \mathbf{C}^{n+1}$.
(The subscripts will be written, if necessary, to distinguish coordinates.)

In the sequel the restrictions of those mappings to open subsets of their domains will be denoted by the same letters.

Let D be an open connected subset of $\mathbf{C}^{n+1}$. For an analytic subset X of $D \times \mathbf{C}^k$ of pure dimension $n+1$ with proper projection onto D we denote $X_i = \pi_i(X)$. The sets $\mathcal{S}(X), \mathcal{S}(X_i)$ are by definition the sets of singular points of the projections $\pi|_X, \widetilde{\pi}_i|_{X_i}$ respectively:

$$\mathcal{S}(X) = Sing(X) \cup \{x \in Reg(X) : (\pi|_X)'(x) \text{ is not an isomorphism}\},$$
$$\mathcal{S}(X_i) = Sing(X_i) \cup \{x \in Reg(X_i) : (\widetilde{\pi}_i|_{X_i})'(x) \text{ is not an isomorphism}\}.$$

These sets are analytic subsets of $D \times \mathbf{C}^k, D \times \mathbf{C}_i$ respectively (cp.[5], p. 50) hence by the Remmert theorem $\pi(\mathcal{S}(X))$ and $\widetilde{\pi}_i(\mathcal{S}(X_i))$ are also analytic. Moreover, the following hold (cp. [15] Theorem 2.1, p. 233):

The mapping $\pi|_X : X \longrightarrow D$ is surjective and open and there exists an integer s which will be denoted by $s(\pi|_X)$ such that:
(1) $\sharp(\pi|_X)^{-1}(\{a\}) = s$ for $a \in D - \pi(\mathcal{S}(X))$,
$\sharp(\pi|_X)^{-1}(\{a\}) < s$ for $a \in \pi(\mathcal{S}(X))$,
(2) for every $a \in D - \pi(\mathcal{S}(X))$ there exists a neighborhood W of a and holomorphic mappings $f_1, \ldots, f_s : W \longrightarrow \mathbf{C}^k$ such that $f_i \cap f_j = \emptyset$ for $i \neq j$ and $f_1 \cup \ldots \cup f_s = (W \times \mathbf{C}^k) \cap X$.

For an analytic subset X of $D \times \mathbf{C}^k$ as above we define the sets:
$\sigma_i(X) = \widetilde{\pi}_i(\mathcal{S}(X_i))$,
$\sigma(X) = \bigcup\{\sigma_i(X) : i \in \{1, \ldots, k\}\}$
which by the Remmert theorem are analytic. Note that
$\sigma(X) = \{x \in D \mid \exists_i : \sharp((\widetilde{\pi}_i|_{\pi_i(X)})^{-1}(x)) < s(\widetilde{\pi}_i|_{\pi_i(X)})\}$.
In the paper D is often taken to be $(\Delta r)^n \times \Delta t$, for some $r > 0, t > 0$, where
$\Delta r = \{x \in \mathbf{C} : |x| < r\}$.
Here we also use the notation Δ_i, for $i \in \{1, \ldots, k\}$, to distinguish the coordinates.

For any analytic subset Y of an open subset of $\mathbf{C}^n$ by $Y_{(k)}$ we denote the union of the k-dimensional irreducible components of Y.

The notion introduced below will be useful in Section 3. Let Y be an analytic subset of $D \times \mathbf{C}_z$ of pure dimension $n+1$ with proper projection onto D, where D is an open connected subset of $\mathbf{C}^{n+1}$. It is a well known fact that for such a set Y there exists a unitary polynomial $p \in \mathcal{O}(D)[z]$ such that

$$Y = \{(x_1, \ldots, x_{n+1}, z) \in D \times \mathbf{C}_z : p(x_1, \ldots, x_{n+1}, z) = 0\};$$

moreover, the discriminant of p is not identically zero. (p is a Weierstrass polynomial such that $\max\{\sharp((\{x\} \times \mathbf{C}_z) \cap Y) : x \in D\}$ is equal to the degree of p in z.) The polynomial p will be called *the optimal polynomial for* Y. Any polynomial $q \in \mathcal{O}(D)[z]$ which is optimal for $\{q(x_1, \ldots, x_{n+1}, z) = 0\}$ will be called an *optimal polynomial*.

2.2 Nash sets

Let Ω be an open subset of $\mathbf{C}^n$ and let f be a holomorphic function on Ω. We say that f is a *Nash function* at $x_0 \in \Omega$ if there exist an open neighborhood U of x_0 and a polynomial $P : \mathbf{C}^n \times \mathbf{C} \longrightarrow \mathbf{C}, P \neq 0$, such that $P(x, f(x)) = 0$ for $x \in U$. A holomorphic function defined on Ω is said to be a *Nash function* if it is a Nash function at every point of Ω.

Let us recall some basic properties of Nash functions (see [15], p. 232):

Theorem 2.1 *Let Ω be an open subset of $\mathbf{C}^n$. The family of Nash functions on Ω is a subring of the ring $\mathcal{O}(\Omega)$ of holomorphic functions on Ω.*

A holomorphic mapping $f = (f_1, \ldots, f_m) : \Omega \longrightarrow \mathbf{C}^m$ is said to be a *Nash mapping* if each of its components is a Nash function on Ω.

Theorem 2.2 *Let Ω, U be open subsets of $\mathbf{C}^n, \mathbf{C}^m$ respectively.
Let $f = (f_1, \ldots, f_m) : \Omega \longrightarrow U$ be a Nash mapping. Then for every Nash function $g : U \longrightarrow \mathbf{C}$ the composition $g \circ f : \Omega \longrightarrow \mathbf{C}$ is also a Nash function.*

Let us turn to Nash sets. A subset Y of an open set $\Omega \subset \mathbf{C}^n$ is said to be a *Nash subset* of Ω if and only if for every $y_0 \in \Omega$ there exist a neighborhood $U \subset \Omega$ of y_0 and Nash functions $f_1, \ldots, f_r$ on U such that

$$Y \cap U = \{y \in U : f_1(y) = \ldots = f_r(y) = 0\}.$$

We present two facts from [15] concerning the structure of Nash sets (Proposition 2.6, Theorem 2.11, pp. 235, 237):

Theorem 2.3 *The union of a locally finite family of Nash subsets of an open subset $\Omega \subset \mathbf{C}^n$ is a Nash subset of Ω.*

Theorem 2.4 *An irreducible analytic component of a Nash subset of an open $\Omega \subset \mathbf{C}^n$ is a Nash subset of Ω.*

3 Proof of the main result

Let X be as in Theorem 1.1 and let $a \in X$ be the point in a neighborhood of which we want to approximate X. We may assume that X is of codimension at least 2. Otherwise it is locally principle and the problem is trivial. The first step in the proof will be to find a neighborhood U of a and an analytic subset $\widetilde{Y}$ of U of pure dimension $n+1$ such that $X \subset Sing(\widetilde{Y})$. Then X is a union of some of the irreducible components of $Sing(\widetilde{Y})$. Next we shall approximate $\widetilde{Y}$ by Nash sets $\widetilde{Y}^\nu$ of pure dimension $n+1$ in such a way that some n-dimensional Nash subsets of $\widetilde{Y}^\nu$ will be converging to X.

At this place we mention that the problem of the construction of an analytic set with a prescribed set of singular points have already been discussed in [6]. One of the results of that paper is that for a given analytic subset X of $\mathbf{C}^n$ of codimension at least two there exists an irreducible hypersurface H such that $Sing(H) = X$.

We shall start with the following Proposition 3.1. The version of its proof that we present below is due to P. Tworzewski.

Proposition 3.1 *Let Ω be an open subset of $\mathbf{C}^m$ and let X be an analytic subset of Ω of pure dimension n, $m \geq n+2$. Then for every $a \in X$ there exists a neighborhood U of a in Ω and an analytic subset Y of U of pure dimension $n+1$ such that $X \subset Sing(Y)$.*

Proof. Since the problem is local we may assume that X is an analytic subset of $\Delta^n \times \Delta_1 \times \ldots \times \Delta_k$ with proper projection onto Δ^n, where $n + k = m$.

Let

$$\begin{aligned} \varrho_1 &: \Delta^n \times \Delta_1 \times \ldots \times \Delta_k \longrightarrow \Delta^n \times \Delta_2 \times \ldots \times \Delta_k, \\ \varrho_k &: \Delta^n \times \Delta_1 \times \ldots \times \Delta_k \longrightarrow \Delta^n \times \Delta_1 \times \ldots \times \Delta_{k-1} \end{aligned}$$

be natural projections and let A_1, A_k denote $\varrho_1^{-1}(\varrho_1(X)), \varrho_k^{-1}(\varrho_k(X))$, respectively.

Now, observe that A_1, A_k are analytic subsets of $\Delta^n \times \Delta_1 \times \ldots \Delta_k$ of pure dimension $n+1$ and moreover, $A_1 \cap A_k$ is of pure dimension n and $X \subset A_1 \cap A_k$ thus $X \subset Sing(A_1 \cup A_k)$.■

We shall require that $\widetilde{Y}$ we want to obtain will additionally satisfy some extra conditions that will be stated precisely in the sequel. First we shall prove the following technical lemma that will be useful to us.

Lemma 3.2 *Let W be an open connected subset of $\mathbf{C}^{n+1}$ and let Y be an analytic subset of $W \times \Delta^k$ of pure dimension $n+1$ with proper projection onto W. Let $\overline{y}$ be an arbitrary point from Y. Then there exist an open connected subset U of W and a polynomial biholomorphism $\Phi : \mathbf{C}^{n+1} \times \mathbf{C}^k \longrightarrow \mathbf{C}^{n+1} \times \mathbf{C}^k$ for which $\overline{y} \in U \times \mathbf{C}^k$, $\Phi(\overline{y}) = \overline{y}$ and $Y' = \Phi(Y) \cap (U \times \mathbf{C}^k)$ has the following properties:*

(a) Y' is an analytic subset of $U \times \Delta^k$ with proper projection onto U,

(b) $s(\pi|_Y) = s(\pi|_{Y'}) = s(\widetilde{\pi}_i|_{Y'_i})$ for every $i \in \{1, \ldots, k\}$,

(c) there exists an analytic subset T of U such that $dim T < n$ and $\widetilde{\pi}_i|_{\mathcal{S}(Y'_i)-(T\times \mathbf{C}_i)} : \mathcal{S}(Y'_i) - (T \times \mathbf{C}_i) \longrightarrow \widetilde{\pi}_i(\mathcal{S}(Y'_i)) - T$ is injective for every $i \in \{1, \ldots, k\}$.

Proof. Note that passing to the image of Y by $(u,v)\mapsto(u,L(v))$, where $u\in\mathbf{C}^{n+1}, v\in\mathbf{C}^k$ and $L:\mathbf{C}^k\longrightarrow\mathbf{C}^k$ is an appropriately selected linear isomorphism we obtain $s(\pi|_Y)=s(\widetilde{\pi}_i|_{Y_i})$ for every $i\in\{1,\ldots,k\}$.

Now we prove the lemma for $W=(\Delta r)^n\times\Delta r$, where $r>1$ and for $\overline{y}=0$. (Recall that $\Delta r=\{z\in\mathbf{C}:|z|<r\}$.) We also assume that $\sigma(Y)$ is contained in $(\Delta r)^n\times\Delta t$ for some $0<t<1$ (which implies that $\sigma(Y)$ is with proper projection onto $(\Delta r)^n$).

One easily checks that the image of Y by the biholomorphism

$$\Phi_{w,i}:\mathbf{C}^n\times\mathbf{C}\times\mathbf{C}^k\longrightarrow\mathbf{C}^n\times\mathbf{C}\times\mathbf{C}^k$$

given by the formula $\Phi_{w,i}(x,y,z_1,\ldots,z_k)=(x,y+w(z_i),z_1,\ldots,z_k)$, where $w(z_i)$ is a polynomial such that $\sup_{z\in\Delta 2}|w(z)|<\epsilon$, has the following properties. If ϵ is small enough then

$$Y_w=\Phi_{w,i}(Y)\cap((\Delta r')^n\times\Delta r'\times\Delta^k)$$

is with proper projection onto $(\Delta r')^n\times\Delta r'$ where $r'=\frac{r+1}{2}$ and

$$s(\pi|_{Y_w})=s(\pi|_Y)=s(\widetilde{\pi}_j|_{(Y_w)_j})$$

for every $j\in\{1,\ldots,k\}$. Moreover, $\sigma(Y_w)$ is contained in $(\Delta r')^n\times\Delta t'$, where $t'=\frac{t+1}{2}$, and

$$\begin{array}{c}l_{w,j}=\max\{\sharp((\{x\}\times\Delta r')\cap\sigma_j(Y_w)):x\in(\Delta r')^n\}\geq\\ \max\{\sharp((\{x\}\times\Delta r)\cap\sigma_j(Y)):x\in(\Delta r)^n\}\end{array}$$

for every $j\in\{1,\ldots,k\}$. The inequality follows, by the Rouche theorem, from the fact that $\sigma_j(Y_w),\sigma_j(Y)$ are the zero-sets of the discriminants of the optimal polynomials for $(Y_w)_j,Y_j$ respectively so it is sufficient to keep ϵ small enough (then by $s(\widetilde{\pi}_j|_{(Y_w)_j})=s(\widetilde{\pi}_j|_{Y_j})$ the optimal polynomials are of the same degree).

On the other hand, observe that if ϵ is small then $l_{w,j}$ is bounded from above by the degree of the Weierstrass polynomial in y obtained by applying the Weierstrass preparation theorem to the discriminant of the optimal polynomial for Y_j.

Now, denote $T_i=$

$$\begin{array}{c}\{x\in(\Delta r)^n:\sharp((\{x\}\times\Delta r)\cap\sigma_i(Y))<\\ \max_{x\in(\Delta r)^n}(\sharp((\{x\}\times\Delta r)\cap\sigma_i(Y)))\}\end{array}$$

and put $T=(\bigcup_{i=1}^k(T_i\times\Delta r))\cap\sigma(Y)$.

We show that if there exists $i\in\{1,\ldots,k\}$ such that

$$\widetilde{\pi}_i|_{\mathcal{S}(Y_i)-(T\times\mathbf{C}_i)}:\mathcal{S}(Y_i)-(T\times\mathbf{C}_i)\longrightarrow\widetilde{\pi}_i(\mathcal{S}(Y_i))-T$$

is not injective then for every $\epsilon>0$ there exists a polynomial w such that $\sup_{z\in\Delta 2}|w(z)|<\epsilon$ and the following hold. The set $Y_w=\Phi_{w,i}(Y)\cap((\Delta r')^n\times\Delta r'\times\Delta^k)$, for $r'=\frac{r+1}{2}$ and $\Phi_{w,i}$ as before, has all the properties described above and, furthermore,

$$l_{w,i}\geq 1+\max\{\sharp((\{x\}\times\Delta r)\cap\sigma_i(Y)):x\in(\Delta r)^n\}.$$

(w will be such that $w(0) = 0$. Then $\Phi_{w,i}(\overline{y}) = \overline{y}$.)

Next we repeat the procedure starting with Y_w. Since $l_{w,i}$ are bounded from above this process must stop after finite number of steps (denoted by m) and the obtained set $Y' = (\ldots((Y_{w_1})_{w_2})\ldots)_{w_m}$ satisfies (a), (b) and (c).

One observes that if at every step we take $\epsilon \ll 1$ then Φ defined as the superposition of the mappings $\Phi_{w,i}$ is a biholomorphism satisfying the assertion of the lemma.

The polynomial w is chosen as follows. Let $(x,y) \in \widetilde{\pi}_i(\mathcal{S}(Y_i)) - T$ be such that $(x,y,z),(x,y,z') \in \mathcal{S}(Y_i), z \neq z'$. Then

$$\sharp((\{x\} \times \Delta r) \cap \sigma_i(Y)) = \max_{x \in (\Delta r)^n} (\sharp((\{x\} \times \Delta r) \cap \sigma_i(Y)))$$

thus it is sufficient to take w with very small norm and such that in a small neighborhood of (x,y) there are two points $(x,y_1),(x,y_2), y_1 \neq y_2$ such that $(x,y_1),(x,y_2) \in \widetilde{\pi}_i(\mathcal{S}((Y_w)_i))$. One easily checks that for every $\epsilon > 0$ there exists a polynomial w such that $\sup_{z \in \Delta 2} |w(z)| < \epsilon$ and $w'(z) = w'(z') = w(z) = w(0) = 0$ and $w(z') \neq 0$ (if $z' = 0$ we transpose z and z'). We complete the proof by observing that $(x, y + w(z)), (x, y + w(z')) \in \widetilde{\pi}_i(\mathcal{S}((Y_w)_i))$.

The proof in the case where W is an arbitrary open connected subset of $\mathbf{C}^{n+1}$ is an easy consequence of the proof presented above.■

Now, let Y be an analytic subset of $\Delta^n \times \Delta \times \mathbf{C}^k$, of pure dimension $n+1$ with proper projection onto $\Delta^n \times \Delta$ such that $\sigma(Y) \subset \Delta^n \times \Delta t$, $0 < t < 1$. Moreover we assume that:

(i) $s(\pi|_Y) = s(\widetilde{\pi}_i|_{Y_i})$, for $i = 1, \ldots, k$,

(ii) there exists an analytic subset T of $\Delta^n \times \Delta$ such that $dim T < n$ and $\widetilde{\pi}_i|_{\mathcal{S}(Y_i) - (T \times \mathbf{C}_i)} : \mathcal{S}(Y_i) - (T \times \mathbf{C}_i) \longrightarrow \widetilde{\pi}_i(\mathcal{S}(Y_i)) - T$ is injective for $i = 1, \ldots, k$.

If we show that every X of pure dimension n for which there exists Y satisfying the conditions above and such that $X \subset Sing(Y)$, can be locally approximated then by Proposition 3.1 and Lemma 3.2 every X of pure dimension n can be locally approximated.

By $\widetilde{Y}$ we denote the set

$$\{(x,y,z_1,\ldots,z_k) \in \Delta^n \times \Delta \times \mathbf{C}^k : p_1(x,y,z_1) = \ldots = p_k(x,y,z_k) = 0\},$$

where $p_i(x,y,z_i) = z_i^s + z_i^{s-1} a_{i,1}(x,y) + \ldots + a_{i,s}(x,y) \in \mathcal{O}(\Delta^n \times \Delta)[z_i]$ is the optimal polynomial for $Y_i = \pi_i(Y)$, $i = 1, \ldots, k$.

Now define $\mathcal{E}(\widetilde{Y}) =$

$$\{(x,y,z_1,\ldots,z_k) \in \widetilde{Y} : \frac{\partial p_1}{\partial z_1}(x,y,z_1) = \ldots = \frac{\partial p_k}{\partial z_k}(x,y,z_k) = 0\}$$

and observe that $\mathcal{E}(\widetilde{Y}) =$

$$\{(x,y,z_1,\ldots,z_k) \in \widetilde{Y} : \pi_i(x,y,z_1,\ldots,z_k) \in \mathcal{S}(\widetilde{Y}_i), i = 1, \ldots, k\}$$

and that, by (i), $\mathcal{S}(Y) \subset \mathcal{E}(\widetilde{Y})$.

Assume that $dim\, \mathcal{S}(Y) = n$ and put $\mathcal{L} = (\pi(\mathcal{S}(Y)))_{(n)}$. The following lemma explains the purpose of the condition (ii).

Lemma 3.3 *In the situation described above assume that $\mathcal{L}_0$ is an irreducible component of $\mathcal{L}$. Then*

$$((\mathcal{L}_0 \times \mathbf{C}^k) \cap \mathcal{E}(\widetilde{Y}))_{(n)}$$

is an irreducible analytic subset of $\Delta^n \times \Delta \times \mathbf{C}^k$.

Proof. $(\mathcal{L}_0 \times \mathbf{C}^k) \cap \mathcal{E}(\widetilde{Y})$ is n-dimensional as $\mathcal{S}(Y)$ is n-dimensional and $\mathcal{S}(Y) \subset \mathcal{E}(\widetilde{Y})$. Suppose that there exist $\mathcal{E}_1, \mathcal{E}_2$ which are distinct irreducible analytic subsets of $\mathcal{E}(\widetilde{Y})$ of dimension n such that $\pi(\mathcal{E}_1) = \pi(\mathcal{E}_2) = \mathcal{L}_0$. Then there exists a set A such that $int_{\mathcal{L}_0} A \neq \emptyset$ and such that for every $(x, y) \in A$ there are $\overline{z}, \overline{z}' \in \mathbf{C}^k, \overline{z} \neq \overline{z}'$ and $(x, y, \overline{z}), (x, y, \overline{z}') \in \mathcal{E}(\widetilde{Y})$. But then there are $i \in \{1, \ldots, k\}$ and $A' \subset \mathcal{L}_0, int_{\mathcal{L}_0} A' \neq \emptyset$ such that for every $(x, y) \in A'$ the fiber of $\widetilde{\pi}_i|_{(A' \times \mathbf{C}_i) \cap \pi_i(\mathcal{E}(\widetilde{Y}))}$ above (x, y) contains at least two elements which contradicts the condition (ii) as $\pi_i(\mathcal{E}(\widetilde{Y})) \subset \mathcal{S}(\widetilde{Y}_i) = \mathcal{S}(Y_i)$. ■

Note that by the fact that $\mathcal{S}(Y) \subset \mathcal{E}(\widetilde{Y})$ and by Lemma 3.3 for every n-dimensional irreducible component A of $\mathcal{S}(Y)$ there exists $\mathcal{L}_0$ which is an irreducible component of $\mathcal{L}$ such that $A = ((\mathcal{L}_0 \times \mathbf{C}^k) \cap \mathcal{E}(\widetilde{Y}))_{(n)}$.

We now turn to approximating the coefficients of the polynomials p_i. Let $dc_s \in \mathbf{C}[u_1, \ldots, u_s]$ denote the polynomial such that for every $b_1, \ldots, b_s$ from $\mathbf{C}$ the complex number $dc_s(b_1, \ldots, b_s)$ is a discriminant of $z^s + z^{s-1}b_1 + \ldots + b_s$.

By the Weierstrass preparation theorem we have

$$dc_s(a_{i,1}(x, y), \ldots, a_{i,s}(x, y)) = H_i(x, y) W_{i,1}^{\alpha_{i,1}}(x, y) \ldots W_{i,r_i}^{\alpha_{i,r_i}}(x, y)$$

on $\Delta^n \times \Delta$ where $W_{i,j} \in \mathcal{O}(\Delta^n)[y]$ are irreducible optimal polynomials pairwise distinct for a fixed i, whereas H_i is a holomorphic function non-vanishing on $\Delta^n \times \Delta$.

Next consider the system of polynomial equations

$$dc_s(u_{i,1}, \ldots, u_{i,s}) - t_i v_{i,1}^{\alpha_{i,1}} \ldots v_{i,r_i}^{\alpha_{i,r_i}} = 0,$$

for $i = 1, \ldots, k$ in variables $u_{i,j}, v_{i,l}, t_i$. Moreover the equations

$$v_{i,j} = v_{k,l}$$

are included in the system if $W_{i,j} = W_{k,l}$ for any i, j, k, l.

In the process of approximation of the coefficients of the optimal polynomials we shall use the following theorem which is due to Lempert (see [9] Theorem 3.2, pp. 338, 339).

Theorem 3.4 *Let K be a holomorphically convex compact subset of $\mathbf{C}^n$ and $f : K \longrightarrow \mathbf{C}^k$ a holomorphic mapping that satisfies a system of equations $Q(z, f(z)) = 0$ for $z \in K$. Here Q is a Nash mapping from a neighborhood $U \subset \mathbf{C}^n \times \mathbf{C}^k$ of the graph of f into some $\mathbf{C}^q$. Then f can be uniformly approximated by Nash $F : K \longrightarrow \mathbf{C}^k$ satisfying $Q(z, F(z)) = 0$.*

Since we consider the approximation locally we may assume that the optimal polynomials p_i extend to slightly bigger sets in such a way that by Theorem 3.4 we obtain Nash functions $a^{\nu}_{i,j}, H^{\nu}_i, W_{\nu,i,j}$ defined on $\Delta^n \times \Delta$, converging uniformly to $a_{i,j}, H_i, W_{i,j}$ respectively and satisfying the conditions

$$dc_s(a^{\nu}_{i,1}, \ldots, a^{\nu}_{i,s}) = H^{\nu}_i W^{\alpha_{i,1}}_{\nu,i,1} \ldots W^{\alpha_{i,r_i}}_{\nu,i,r_i}$$

on $\Delta^n \times \Delta$ for every $i \in \{1, \ldots, k\}$. Moreover, H^{ν}_i are non-vanishing on $\Delta^n \times \Delta$, $\{(x, y) \in \Delta^n \times \Delta : dc_s(a^{\nu}_{i,1}(x, y), \ldots, a^{\nu}_{i,s}(x, y)) = 0\} \subset \Delta^n \times \Delta t'$, for some $t' < 1$ and $W_{\nu,i,j} = W_{\nu,k,l}$ for all ν if $W_{i,j} = W_{k,l}$.

Now, we put

$$p^{\nu}_i(x, y, z_i) = z^s_i + z^{s-1}_i a^{\nu}_{i,1} + \ldots + a^{\nu}_{i,s}$$

and next define $\widetilde{Y}^{\nu} =$

$$\{(x, y, z_1, \ldots, z_k) \in \Delta^n \times \Delta \times \mathbf{C}^k : p^{\nu}_1(x, y, z_1) = \ldots = p^{\nu}_k(x, y, z_k) = 0\}.$$

Proposition 3.5 *Keeping the notation established above we have*

$$\mathcal{E}(\widetilde{Y}^{\nu})_{(n)} \longrightarrow \mathcal{E}(\widetilde{Y})_{(n)}$$

Proof. We start with the remark that we do not need to assume that Y satisfies (i) and (ii) to obtain the assertion of the proposition.

By $\overline{z}$ we denote $(z_1, \ldots, z_k)$. Assume that $(x, y, \overline{z}) \notin \mathcal{E}(\widetilde{Y})_{(n)}$. First we show that there is an open neighborhood U of $(x, y, \overline{z})$ such that $U \cap \mathcal{E}(\widetilde{Y}^{\nu})_{(n)} = \emptyset$ for sufficiently large ν. (This is equivalent to the second condition of the definition of the local uniform convergence.)

Since $\widetilde{Y}$ is with proper projection onto $\Delta^n \times \Delta$ and $\sigma(\widetilde{Y})$ is with proper projection onto Δ^n there exist open connected relatively compact $\widetilde{U} \subset \Delta^n, \widetilde{V} \subset \Delta, \widetilde{W} \subset \mathbf{C}^k$ such that $(x, y, \overline{z}) \in \widetilde{U} \times \widetilde{V} \times \widetilde{W}$, $(\overline{\widetilde{U} \times \widetilde{V}} \times \partial\widetilde{W}) \cap \widetilde{Y} = \emptyset$, $(\overline{\widetilde{U}} \times \partial\widetilde{V}) \cap \sigma(\widetilde{Y}) = \emptyset$ (then, in particular if $\widetilde{Y} \cap (\widetilde{U} \times \widetilde{V} \times \widetilde{W})$ and $\sigma(\widetilde{Y}) \cap (\widetilde{U} \times \widetilde{V})$ are not empty then they are with proper projection onto $\widetilde{U} \times \widetilde{V}$, $\widetilde{U}$ respectively). Moreover, we may assume that $dim(\mathcal{E}(\widetilde{Y}) \cap (\widetilde{U} \times \widetilde{V} \times \widetilde{W})) \leq n - 1$.

We observe that there exists an open ball $U' \subset \widetilde{U}$ such that $(U' \times \widetilde{V} \times \widetilde{W}) \cap \mathcal{E}(\widetilde{Y}) = \emptyset$. Indeed, if $E \neq \emptyset$ then $\pi(E)$ is with proper projection onto $\widetilde{U}$ where $E = (\widetilde{U} \times \widetilde{V} \times \widetilde{W}) \cap \mathcal{E}(\widetilde{Y})$. But then $\varrho(\pi(E))$ is an s-dimensional analytic subset of $\widetilde{U}$ where $\varrho : \widetilde{U} \times \widetilde{V} \longrightarrow \widetilde{U}$ is a natural projection and $s \leq n-1$ so there exists an open ball $U' \subset \widetilde{U}$ such that $U' \cap \varrho(\pi(E)) = \emptyset$.

Consequently, we may assume that there are $\varepsilon > 0, \delta > 0$ such that

$$\sum_{j=1}^{k} |\frac{\partial p_j}{\partial z_j}|_{|\widetilde{Y}_{\varepsilon} \cap (U' \times \widetilde{V} \times \widetilde{W})} > \delta,$$

where $\widetilde{Y}_{\varepsilon} = \{a \in \Delta^n \times \Delta \times \mathbf{C}^k : \exists_{b \in \widetilde{Y}} ||a - b|| < \varepsilon\}$.

If for large ν we had $(\mathcal{E}(\widetilde{Y}^\nu))_{(n)}\cap(\widetilde{U}\times\widetilde{V}\times\widetilde{W})\neq\emptyset$ then the set $(\mathcal{E}(\widetilde{Y}^\nu))_{(n)}\cap(U'\times\widetilde{V}\times\widetilde{W})\cap\widetilde{Y}_\varepsilon$ would also be nonempty. Indeed, we have assumed that $(\overline{\widetilde{U}\times\widetilde{V}}\times\partial\widetilde{W})\cap\widetilde{Y}=\emptyset$, $(\overline{\widetilde{U}}\times\partial\widetilde{V})\cap\sigma(\widetilde{Y})=\emptyset$, it holds $\widetilde{Y}^\nu\longrightarrow\widetilde{Y}$ and clearly $\sigma(\widetilde{Y}^\nu)\longrightarrow\sigma(\widetilde{Y})$. Hence for large ν the set $(\widetilde{U}\times\widetilde{V}\times\widetilde{W})\cap(\mathcal{E}(\widetilde{Y}^\nu))_{(n)}$ is with proper projection onto $\widetilde{U}$, which, because of equity of the dimensions of $(\mathcal{E}(\widetilde{Y}^\nu))_{(n)}$ and $\widetilde{U}$, is surjective (recall that $\widetilde{U}$ is connected) so $((\mathcal{E}(\widetilde{Y}^\nu))_{(n)})\cap(U'\times\widetilde{V}\times\widetilde{W})\neq\emptyset$. The latter set is obviously contained in $\widetilde{Y}_\varepsilon$ for large ν. Here, however, we obtain a contradiction as on one hand $\sum_j|\frac{\partial p_j^\nu}{\partial z_j}|_{|\mathcal{E}(\widetilde{Y}^\nu)}=0$ whereas, on the other hand $\sum_j|\frac{\partial p_j^\nu}{\partial z_j}|\longrightarrow\sum_j|\frac{\partial p_j}{\partial z_j}|$.

To complete the proof of the proposition we need to show that for every $(x_0,y_0,\overline{z}_0)\in\mathcal{E}(\widetilde{Y})_{(n)}$ there exists a sequence $(x_\nu,y_\nu,\overline{z}_\nu)\in\mathcal{E}(\widetilde{Y}^\nu)_{(n)}$ such that $(x_\nu,y_\nu,\overline{z}_\nu)\longrightarrow(x_0,y_0,\overline{z}_0)$.

Let B be any open neighborhood of $(x_0,y_0,\overline{z}_0)$. By the fact that $dim(\mathcal{E}(\widetilde{Y}))_{(n)}$ $=dim(\sigma(\widetilde{Y}))$, we may assume that $(x_0,y_0)\in\sigma(\widetilde{Y})$ is such that there is an open ball $\widetilde{U}\ni x_0$ contained in Δ^n such that $(\widetilde{U}\times\Delta)\cap\sigma(\widetilde{Y})$ is a union of pairwise disjoint graphs of holomorphic mappings on $\widetilde{U}$. We may also assume that $(\widetilde{U}\times\widetilde{V})\cap\sigma(\widetilde{Y})$ is one such graph, where $\widetilde{V}$ is an open ball in $\mathbf{C}, y_0\in\widetilde{V}$ and $(\overline{\widetilde{U}}\times\partial\widetilde{V})\cap\sigma(\widetilde{Y})=\emptyset$.

Then for large ν the set $(\widetilde{U}\times\widetilde{V})\cap\sigma(\widetilde{Y}^\nu)$ is also a graph of a single holomorphic function on $\widetilde{U}$. (This follows from the way the polynomials $W_{j,i}$ were approximated by the functions $W_{\nu,j,i}$, (in particular from the presence of the equations of the form $v_{i,j}=v_{k,l}$ in the considered system).)

Finally, we assume that there is an open polydisc $\widetilde{W}$ contained in $\mathbf{C}^k$ such that $(x_0,y_0,\overline{z}_0)\in\widetilde{U}\times\widetilde{V}\times\widetilde{W}\subset B$ and $(\overline{\widetilde{U}\times\widetilde{V}}\times\partial\widetilde{W})\cap\widetilde{Y}=\emptyset$. (Then $\widetilde{Y}\cap(\widetilde{U}\times\widetilde{V}\times\widetilde{W})$ is with proper projection onto $\widetilde{U}\times\widetilde{V}$.)

We show that for sufficiently large ν, $dim((\widetilde{U}\times\widetilde{V}\times\widetilde{W})\cap\mathcal{E}(\widetilde{Y}^\nu))=n$ which will complete the proof.

Fix $i\in\{1,\ldots,k\}$, put

$$\begin{aligned}A_i&=\{(x,y,z_i)\in\widetilde{U}\times\widetilde{V}\times\pi_i(\widetilde{W}):p_i(x,y,z_i)=0\},\\A_i'&=\{(x,y,z_i)\in\widetilde{U}\times\widetilde{V}\times\pi_i(\widetilde{W}):\tfrac{\partial p_i}{\partial z_i}(x,y,z_i)=0\},\\A_{i,\nu}&=\{(x,y,z_i)\in\widetilde{U}\times\widetilde{V}\times\pi_i(\widetilde{W}):p_i^\nu(x,y,z_i)=0\},\\A_{i,\nu}'&=\{(x,y,z_i)\in\widetilde{U}\times\widetilde{V}\times\pi_i(\widetilde{W}):\tfrac{\partial p_i^\nu}{\partial z_i}(x,y,z_i)=0\},\end{aligned}$$

and observe that $dim(A_i\cap A_i')=n$ so these two sets intersect properly. On the other hand one easily checks that $A_{i,\nu}\longrightarrow A_i$ and $A_{i,\nu}'\longrightarrow A_i'$. Thus it follows, by Theorem 3 from [16], that $dim(A_{i,\nu}'\cap A_{i,\nu})=n$. As a result, in view of the fact that $(\widetilde{U}\times\widetilde{V})\cap\sigma(\widetilde{Y}^\nu)$ is irreducible we obtain that for every i and for large ν it holds

$$\widetilde{\pi}_i(A_{i,\nu}'\cap A_{i,\nu})=(\widetilde{U}\times\widetilde{V})\cap\sigma(\widetilde{Y}^\nu).$$

Now define

$$Z_\nu = \{(x, y, z_1, \ldots, z_k) \in \widetilde{U} \times \widetilde{V} \times \widetilde{W} : (x, y, z_i) \in (A_{i,\nu} \cap A'_{i,\nu})\}.$$

By what we have shown so far we see that for large ν, $dim Z_\nu = n$ and $Z_\nu \subset \mathcal{E}(\widetilde{Y}^\nu)$. Thus the proof of the proposition has been completed. ■

Now we are ready to prove the main result of this paper.

Proof of Theorem 1.1. Since every irreducible component of the considered set may be approximated separately we assume, without loss of generality, that X is locally irreducible. Furthermore, we assume that X is of codimension at least two (otherwise it is locally principle). Then, by Proposition 3.1, Lemma 3.2 and the fact that the problem is local we also may assume that X is an irreducible component of the set $\mathcal{S}(Y)$ where Y is an analytic subset of $\Delta^n \times \Delta \times \mathbf{C}^k$ of pure dimension $n+1$ with proper projection onto $\Delta^n \times \Delta$ such that $\sigma(Y) \subset \Delta^n \times \Delta t$, $0 < t < 1$. Moreover it holds:
(i) $s(\pi|_Y) = s(\widetilde{\pi}_i|_{Y_i})$, for $i = 1, \ldots, k$,
(ii) there exists an analytic subset T of $\Delta^n \times \Delta$ such that $dim T < n$ and $\widetilde{\pi}_i|_{\mathcal{S}(Y_i)-(T\times\mathbf{C}_i)} : \mathcal{S}(Y_i) - (T \times \mathbf{C}_i) \longrightarrow \widetilde{\pi}_i(\mathcal{S}(Y_i)) - T$ is injective for $i = 1, \ldots, k$.

Let $\mathcal{L}_0$ denote $\pi(X)$. Since $dim X = dim \mathcal{S}(Y) = n$ and (by the condition (i)) $\mathcal{S}(Y) \subset \mathcal{E}(\widetilde{Y})$, the set X is an irreducible component of $\mathcal{E}(\widetilde{Y})$. Then, by Lemma 3.3 it holds

$$((\mathcal{L}_0 \times \mathbf{C}^k) \cap \mathcal{E}(\widetilde{Y}))_{(n)} = X.$$

Now we approximate the coefficients of the optimal polynomials for $\widetilde{Y}_i$ exactly in the same way as described in the setup preceding Proposition 3.5. Then $\mathcal{L}_0$ is described by a polynomial $W = W_{j,i}$ for some j, i and the sequence of the sets $\{\mathcal{L}_0^\nu\}$ described by $W_{\nu,j,i}$ converges to $\mathcal{L}_0$.

We show that

$$((\mathcal{L}_0^\nu \times \mathbf{C}^k) \cap \mathcal{E}(\widetilde{Y}^\nu))_{(n)} \longrightarrow ((\mathcal{L}_0 \times \mathbf{C}^k) \cap \mathcal{E}(\widetilde{Y}))_{(n)}.$$

Fix $(x, y, \overline{z})$ from $((\mathcal{L}_0 \times \mathbf{C}^k) \cap \mathcal{E}(\widetilde{Y}))_{(n)}$. First we check that there are $(x_\nu, y_\nu, \overline{z}_\nu) \in ((\mathcal{L}_0^\nu \times \mathbf{C}^k) \cap \mathcal{E}(\widetilde{Y}^\nu))_{(n)}$ such that $(x_\nu, y_\nu, \overline{z}_\nu) \longrightarrow (x, y, \overline{z})$. Without loss of generality we restrict our attention to the case where there exist open balls U, V contained in Δ^n, Δ respectively such that $(x, y) \in U \times V$, $(\overline{U} \times \partial V) \cap \sigma(\widetilde{Y}) = \emptyset$ and $(U \times V) \cap \sigma(\widetilde{Y})$ is a graph of a holomorphic mapping on U. Then the way the polynomials $W_{j,i}$ are approximated by the functions $W_{\nu,j,i}$ implies that there is ν_0 such that for every $\nu > \nu_0$ it holds $(U \times V) \cap \sigma(\widetilde{Y}^\nu) = (U \times V) \cap \mathcal{L}_0^\nu$.

Now, by Proposition 3.5 it follows that $\mathcal{E}(\widetilde{Y}^\nu)_{(n)} \longrightarrow \mathcal{E}(\widetilde{Y})_{(n)}$. Moreover, by the definitions of $\mathcal{E}(\widetilde{Y}^\nu), \mathcal{E}(\widetilde{Y})$ we have $\pi(\mathcal{E}(\widetilde{Y}^\nu)) \subset \sigma(\widetilde{Y}^\nu)$ and $\pi(\mathcal{E}(\widetilde{Y})) \subset \sigma(\widetilde{Y})$, hence, in view of the previous paragraph, one easily obtains a sequence $(x_\nu, y_\nu, \overline{z}_\nu)$ satisfying the required conditions.

To complete the proof of the claim it is sufficient to show that for every $(x, y, \overline{z}) \notin ((\mathcal{L}_0 \times \mathbf{C}^k) \cap \mathcal{E}(\widetilde{Y}))_{(n)}$ there exists an open neighborhood E such that $E \cap ((\mathcal{L}_0^\nu \times \mathbf{C}^k) \cap \mathcal{E}(\widetilde{Y}^\nu))_{(n)} = \emptyset$ for ν large enough.

Since Y is with proper projection onto $\Delta^n \times \Delta$ and $\sigma(Y)$ is with proper projection onto Δ^n we can select open balls $U \ni x, V \ni y, W \ni \overline{z}$ which are relatively compact subsets of $\Delta^n, \Delta, \mathbf{C}^k$ respectively and satisfy the following conditions: $(\overline{U \times V} \times \partial W) \cap \widetilde{Y} = \emptyset$, $(\overline{U} \times \partial V) \cap \sigma(\widetilde{Y}) = \emptyset$, $(U \times V \times W) \cap ((\mathcal{L}_0 \times \mathbf{C}^k) \cap \mathcal{E}(\widetilde{Y}))_{(n)} = \emptyset$. Then there are an open ball U' contained in U and $\varepsilon > 0$ such that

$$((\overline{U' \times V} \cap \overline{(\mathcal{L}_0)_\varepsilon}) \times \overline{W}) \cap \mathcal{E}(\widetilde{Y})_{(n)} = \emptyset.$$

(Recall that $(\mathcal{L}_0)_\varepsilon = \{a \in \Delta^n \times \Delta : \exists_{b \in \mathcal{L}_0} ||a - b|| < \varepsilon\}$, where $|| \cdot ||$ is a standard norm in $\mathbf{C}^{n+1}$.)

Now, if $(U \times V \times W) \cap ((\mathcal{L}_0^\nu \times \mathbf{C}^k) \cap \mathcal{E}(\widetilde{Y}^\nu))_{(n)}$ is non-empty for large ν then $((\overline{U' \times V} \cap \overline{(\mathcal{L}_0)_\varepsilon}) \times \overline{W}) \cap \mathcal{E}(\widetilde{Y}^\nu)_{(n)}$ is also non-empty which contradicts the fact that $\mathcal{E}(\widetilde{Y}^\nu)_{(n)} \longrightarrow \mathcal{E}(\widetilde{Y})_{(n)}$.

Obviously $((\mathcal{L}_0^\nu \times \mathbf{C}^k) \cap \mathcal{E}(\widetilde{Y}^\nu))_{(n)}$ is a Nash set as $\widetilde{Y}^\nu$ is Nash. Thus the proof of the theorem has been completed. ■

We shall finish with an observation that if X in Theorem 1.1 is locally irreducible then the sets in the approximating sequence can also be taken irreducible. This is a simple consequence of the following

Remark 3.6 *Let Ω be an open subset of $\mathbf{C}^{n+1} \times \mathbf{C}^k$ such that $\Delta^n \times \Delta \times \Delta^k \subset\subset \Omega$ and let X, Y^ν, $\nu \in \mathbf{N}$, be analytic subsets of Ω of pure dimension $n+1$ such that $Y^\nu \longrightarrow X$. Moreover, assume that $(\overline{\Delta^n \times \Delta} \times \partial\Delta^k) \cap X = \emptyset$ and that $(\Delta^n \times \Delta \times \Delta^k) \cap X$ is non-empty and irreducible. Let X^ν be an arbitrary irreducible component of $(\Delta^n \times \Delta \times \Delta^k) \cap Y^\nu$. Then $X^\nu \longrightarrow (\Delta^n \times \Delta \times \Delta^k) \cap X$.*

Proof. It is sufficient to show that for every $(x, y) \in (\Delta^{n+1} \times \Delta^k) \cap X$ there are $(x_\nu, y_\nu) \in X^\nu$ such that $(x_\nu, y_\nu) \longrightarrow (x, y)$.

Denote $X_r = ((\Delta r)^n \times \Delta r \times \Delta^k) \cap X$ and fix $r > 1$ such that $(\overline{(\Delta r)^n \times \Delta r} \times \partial\Delta^k) \cap X = \emptyset$ which is possible by the hypotheses. Next define

$$X_{r,\varepsilon} = \{(x, y) \in \mathbf{C}^{n+1} \times \mathbf{C}^k : \exists_{y'}((x, y') \in X_r \wedge ||y - y'||_k < \varepsilon)\},$$

where $|| \cdot ||_k$ is the standard norm in $\mathbf{C}^k$. Now observe that for every $\varepsilon > 0$ there exists ν_0 such that for every $\nu > \nu_0$ it holds $(Y^\nu \cap (\Delta^n \times \Delta \times \Delta^k)) \subset X_{r,\varepsilon}$. Indeed, we have $X \cap (\overline{\Delta^n \times \Delta \times \Delta^k} - X_{r,\varepsilon}) = \emptyset$ and, since $Y^\nu \longrightarrow X$ and $\overline{\Delta^n \times \Delta \times \Delta^k} - X_{r,\varepsilon}$ is compact, it holds $Y^\nu \cap (\overline{\Delta^n \times \Delta \times \Delta^k} - X_{r,\varepsilon}) = \emptyset$, for large ν, which gives our claim.

Choose $\varepsilon > 0$. There exists (cp. [2], Theorem 4.2) an open connected relatively compact subset E of

$$(\Delta^n \times \Delta) - \pi(\mathcal{S}(X \cap (\Delta^n \times \Delta \times \Delta^k)))$$

with the following properties:

(1) for every $a \in (((\Delta^n \times \Delta) - E) \times \Delta^k) \cap X$ there exists $b \in (E \times \Delta^k) \cap X$ such that $||a - b||_{n+k+1} < \frac{\varepsilon}{3}$,

(2) $(E \times \Delta^k) \cap X$ is irreducible (i.e. a connected manifold).

Now, put $\theta = \inf_{x \in E} d((\{x\} \times \Delta^k) \cap X)$, where

$$d(B) = \inf\{||x - y||_{n+k+1} : x, y \in B, x \neq y\}$$

for any finite set $B \subset \mathbf{C}^{n+k+1}$, and observe that $\theta > 0$. Let ν_0 be so large that $Y^\nu \cap (\Delta^n \times \Delta \times \Delta^k) \subset X_{r,\alpha}$ for every $\nu > \nu_0$, where $\alpha = \min\{\frac{\theta}{3}, \frac{\varepsilon}{3}\}$. Fix $\nu > \nu_0$ and define

$$A = \{(u, v) \in X \cap (E \times \Delta^k) : \exists_{\widetilde{v}}((u, \widetilde{v}) \in X^\nu \wedge ||v - \widetilde{v}||_k < \alpha)\}.$$

Obviously, $A \neq \emptyset$. Since X^ν is with proper projection onto E and $Y^\nu \cap (\Delta^n \times \Delta \times \Delta^k) \subset X_{r,\alpha}$ it follows that A is an open and closed subset of $X \cap (E \times \Delta^k)$ which immediately implies the assertion of the remark.■

Acknowledgements. I am grateful to P. Tworzewski for suggesting the problem and for helpful discussions.

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Kraków, 21.04.2004

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