CONDITIONALLY MINIMAL PROJECTIONS

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Abstract. Sets of projections which do not contain orthogonal projection are considered. In such sets projections of minimal norm are found.
The research presented in this paper has been motivated by earlier results concerning oblique projections (see [14]).
Keywords: projection, operator norm, extremal points

1. Introduction.

Let $(X, \|\cdot\|)$ denote a normed space and let S be a linear subspace of X.A linear, bounded operator $P : X \longrightarrow S$ is called a projection if $P(s) = s$ for any $s \in S$. Projections play an important role in approximation, optimization, spectral theory and orthogonal decomposition.

Recently, applications of projections to signal processing [4], [7], [18], sampling [5], [30], information theory [30], wavelets [1] and least square approximation [16], [17], [31] have been established.

Results connected with projections can be found in the following papers [8], [9], [28], [29], [33].

Survey on some results on so called oblique projections can be found in [14].

Among all projections $P : X \longrightarrow S$ we will distinguish a minimal projection. A projection $P_0 : X \longrightarrow S$ is called minimal if

$$\|P_0\| = inf\{\|P\| : P : X \longrightarrow S\}.$$

A lot of papers have been devoted to minimal projections. Let us mention the following [2], [3], [6], [10], [11], [12], [13], [15], [19], [20], [21], [22], [23], [24], [25], [26], [32].

If the space X is a Hilbert space then orthogonal projection is minimal. So seeking for minimal projection we will focus our attention on sets of projections which do not contain orthogonal projection. In this paper we will assume that $X = \mathbb{R}^n$.

Let us consider the space $\mathbb{R}^n$ with the standard scalar product given by the following formula:

$$\langle x, y\rangle = x_1y_1 + x_2y_2 + ... + x_ny_n,$$

where $x = (x_1, x_2, ..., x_n),\ y = (y_1, y_2, ..., y_n) \in \mathbb{R}^n$.
By $\mathbb{L}(\mathbb{R}^n)$ we will denote a set of all bounded, linear operators from $\mathbb{R}^n$ to $\mathbb{R}^n$.
Let $S \subset \mathbb{R}^n$ be a closed, linear subspace of $\mathbb{R}^n$. By $P(S)$ we will denote a set of all projections on $\mathbb{R}^n$ with the range S, so:

$$P(S) = \{Q \in \mathbb{L}(\mathbb{R}^n) :\ Q^2 = Q,\ Q(\mathbb{R}^n) = S\}.$$

Let P_S denote the orthogonal projection on $\mathbb{R}^n$. Of course $P_S \in P(S)$.
For a given operator $A \in \mathbb{L}(\mathbb{R}^n)$ and a closed subspace $S \subset \mathbb{R}^n$ set

$$P(A, S) = \{Q \in P(S) :\ AQ = Q^*A\}.$$

If we take under consideration only non-emptiness of the set $P(A, S)$; we obtain that an operator A need not be symmetric. In this paper we will assume that in the basis of subspaces $S, S^\perp$ an operator A has the matrix representation such that:
$A = [a_{ij}]^n_{i,j=1}$, where $a_{ij} = a_{ji},\ i = 1, 2, ..., k,\ j = 1, 2, ..., n$, and $k = dimS$.
This is much more general situation in comparison with that considered in [14].

2. Preliminary results.

In this paper we present three types of results:
- for a fixed $S \subset \mathbb{R}^n$; establish the set of bounded operators on $\mathbb{R}^n$ such that, for any operator A from this set, holds $P(A, S) \neq \emptyset$,
-describe nonempty sets $P(A, S)$ for which $P_S \notin P(A, S)$,
-for some special $P(A, S)$ such that $P_S \notin P(A, S)$ and for some norms on $\mathbb{R}^n$ we will find Q_0 such that:

$$\|Q_0\| = \inf_{Q \in P(A,S)} \|Q\|.$$

These results have their origin in earlier research concerning oblique projections (see [14]).
For a closed, k-dimensional subspace $S \subset \mathbb{R}^n$ we consider:

$$S^\perp = \{y \in \mathbb{R}^n :\ \langle x, y\rangle = 0,\ x \in S\}.$$

Let $v_1, v_2, ..., v_k$ be a basis of the subspace S and $v_{k+1}, v_{k+2}, ..., v_n$ be a basis of $S^\perp$.
For any $Q \in P(S)$ we obtain in such bases the following matrix representation:

$$
Q = \begin{bmatrix}
1 & 0 & . & . & 0 & r_{1k+1} & r_{1k+2} & . & . & r_{1n} \\
0 & 1 & . & . & 0 & r_{2k+1} & r_{2k+2} & . & . & r_{2n} \\
. & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . \\
0 & 0 & . & . & 1 & r_{kk+1} & r_{kk+2} & . & . & r_{kn} \\
0 & 0 & . & . & 0 & 0 & 0 & 0 & 0 & 0 \\
. & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . \\
0 & 0 & . & . & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} \tag{1}
$$

where

$$r_{1j}, r_{2j}, ..., r_{kj} \in \mathbb{R},\ \ j = k+1, k+2, ..., n.$$

Let us consider a sequence of variables:
$r = (r_{1k+1}, ..., r_{kk+1}, r_{1k+2}, ..., r_{kk+2}, ..., r_{1n}, ..., r_{kn})$ and let $A = [a_{ij}]_{i,j=1}^{n}$. The equation $AQ = Q^* A$ leads to the system of linear equations which can be written as follows:

$$Cr = b, \tag{2}$$

where
$b = (a_{1k+1}, ..., a_{1n}, a_{2k+1}, ..., a_{2n}, ..., a_{kk+1}, ..., a_{kn}, 0, ..., 0)^T \in \mathbb{R}^{\frac{n(n-1)-k(k-1)}{2} \times 1}$
and

$$C^T = [\, C_1^T \quad C_2^T \quad . \quad . \quad . \quad C_k^T \quad C_{k+1}^T \quad C_{k+2}^T \quad . \quad . \quad . \quad C_{n-1}^T \,], \tag{3}$$

where

$$C_1 = \begin{bmatrix} a_{11} & 0 & . & . & 0 \\ a_{12} & 0 & . & . & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ a_{1k} & 0 & . & . & 0 \\ 0 & a_{11} & 0 & . & 0 \\ 0 & a_{12} & 0 & . & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & a_{1k} & 0 & . & 0 \\ 0 & 0 & . & . & 0 \\ . & . & . & . & 0 \\ . & . & . & . & 0 \\ 0 & 0 & . & . & 0 \\ 0 & 0 & . & . & a_{11} \\ 0 & 0 & . & . & a_{12} \\ 0 & 0 & . & . & . \\ 0 & 0 & . & . & . \\ 0 & 0 & . & . & a_{1k} \end{bmatrix}, \quad C_2 = \begin{bmatrix} a_{12} & 0 & . & . & 0 \\ a_{22} & 0 & . & . & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ a_{2k} & 0 & . & . & 0 \\ 0 & a_{12} & 0 & . & 0 \\ 0 & a_{22} & 0 & . & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & a_{2k} & 0 & . & 0 \\ 0 & 0 & . & . & 0 \\ . & . & . & . & 0 \\ . & . & . & . & 0 \\ 0 & 0 & . & . & 0 \\ 0 & 0 & . & . & a_{12} \\ 0 & 0 & . & . & a_{22} \\ 0 & 0 & . & . & . \\ 0 & 0 & . & . & . \\ 0 & 0 & . & . & a_{2k} \end{bmatrix}, \quad .... \quad ,$$

$$C_k = \begin{bmatrix} a_{1k} & 0 & . & . & 0 \\ a_{2k} & 0 & . & . & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ a_{kk} & 0 & . & . & 0 \\ 0 & a_{1k} & 0 & . & 0 \\ 0 & a_{2k} & 0 & . & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & a_{kk} & 0 & . & 0 \\ 0 & 0 & . & . & 0 \\ . & . & . & . & 0 \\ . & . & . & . & 0 \\ 0 & 0 & . & . & 0 \\ 0 & 0 & . & . & a_{1k} \\ 0 & 0 & . & . & a_{2k} \\ 0 & 0 & . & . & . \\ 0 & 0 & . & . & . \\ 0 & 0 & . & . & a_{kk} \end{bmatrix}$$

$$C_{k+1}=\begin{bmatrix}
-a_{1k+2} & -a_{1k+3} & . & . & -a_{1n}\\
. & . & . & . & .\\
. & . & . & . & .\\
-a_{kk+2} & -a_{kk+3} & . & . & -a_{kn}\\
a_{1k+1} & 0 & . & . & 0\\
. & . & . & . & .\\
. & . & . & . & .\\
a_{kk+1} & 0 & . & . & 0\\
0 & a_{1k+1} & 0 & . & 0\\
. & . & . & . & .\\
. & . & . & . & .\\
0 & a_{kk+1} & 0 & . & 0\\
. & . & . & . & .\\
. & . & . & . & .\\
. & . & . & . & .\\
. & . & . & . & .\\
. & . & . & . & .\\
. & . & . & . & .\\
. & . & . & . & .\\
. & . & . & . & .\\
0 & . & . & 0 & a_{1k+1}\\
. & . & . & . & .\\
. & . & . & . & .\\
0 & . & . & 0 & a_{kk+1}
\end{bmatrix},\quad
C_{k+2}=\begin{bmatrix}
0 & . & . & . & 0\\
0 & . & . & . & 0\\
0 & . & . & . & 0\\
0 & . & . & . & 0\\
-a_{1k+3} & -a_{1k+4} & . & . & -a_{1n}\\
. & . & . & . & .\\
. & . & . & . & .\\
-a_{kk+3} & -a_{kk+4} & . & . & -a_{kn}\\
a_{1k+2} & 0 & . & . & 0\\
. & . & . & . & .\\
. & . & . & . & .\\
a_{kk+2} & 0 & . & . & 0\\
0 & a_{1k+2} & 0 & . & 0\\
. & . & . & . & .\\
. & . & . & . & .\\
0 & a_{kk+2} & 0 & . & 0\\
. & . & . & . & .\\
. & . & . & . & .\\
. & . & . & . & .\\
. & . & . & . & .\\
0 & . & . & . & a_{1k+2}\\
. & . & . & . & .\\
. & . & . & . & .\\
0 & . & . & . & a_{kk+2}
\end{bmatrix},$$

$$\ldots.\quad ,C_{n-1}=\begin{bmatrix}
0\\ .\\ .\\ .\\ .\\ .\\ .\\ .\\ .\\ .\\ 0\\ -a_{1n}\\ .\\ .\\ -a_{kn}\\ a_{1n-1}\\ .\\ .\\ a_{kn-1}
\end{bmatrix}$$

Matrices $C_1, C_2, ..., C_k \in \mathbb{R}^{k(n-k)\times(n-k)}$, $C_m \in \mathbb{R}^{k(n-k)\times(n-m)}$, $m \in \{k+1, k+2, ..., n-1\}$.

Matrix $C \in \mathbb{R}^{\frac{n(n-1)-k(k-1)}{2}\times k(n-k)}$ where $k = dimS$ and $k(n-k)$ is the number of variables.

Example 1.

Let $n=3,\ k=dimS=1$ *. Let* v_1 *and* v_2, v_3 *be the basis for subspaces* S *and* $S^{\perp}$ *respectively. Suppose that the matrix representation of projection* Q *and an operator* A *is:*

$$Q=\begin{bmatrix}1 & r_{12} & r_{13}\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix},\quad A=\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{12} & a_{22} & a_{23}\\ a_{13} & a_{32} & a_{33}\end{bmatrix}.$$

Then in the equation (2) we have:

$$C=\begin{bmatrix}a_{11} & 0\\ 0 & a_{11}\\ -a_{13} & a_{12}\end{bmatrix}\quad and\quad r=(r_{12},r_{13})^T,\quad b=(a_{12},a_{13},0)^T.$$

3. Non-emptiness of the set $P(A,S)$.

Theorem 1.

Let $A=[a_{ij}]_{i,i=1}^n,\ a_{ij}=a_{ji},\ i=1,2,...,k,\ j=1,2,...,n,$ *and let* $S\subset\mathbb{R}^n$ *be the subspace such that* $dimS=k$*.*
Then

$$P(A,S)\neq\emptyset\quad if\ and\ only\ if\quad r(C)=r(C_d),$$

where

$$C_d=\begin{bmatrix}C_1^T & A_1\\ C_2^T & A_2\\ . & .\\ . & .\\ . & .\\ C_k^T & A_k\\ C_{k+1}^T & \mathbb{O}_{k+1}\\ C_{k+2}^T & \mathbb{O}_{k+2}\\ . & .\\ . & .\\ . & .\\ C_{n-1}^T & \mathbb{O}_{n-1}\end{bmatrix},$$

and

$$A_1=\begin{bmatrix}a_{1k+1}\\ .\\ .\\ .\\ a_{1n}\end{bmatrix},\quad A_2=\begin{bmatrix}a_{2k+1}\\ .\\ .\\ .\\ a_{2n}\end{bmatrix},\quad ....\quad ,A_k=\begin{bmatrix}a_{kk+1}\\ .\\ .\\ .\\ a_{kn}\end{bmatrix},$$

$$\mathbb{O}_m=\begin{bmatrix}0\\ .\\ .\\ .\\ 0\end{bmatrix}\in\mathbb{R}^{(n-m)\times 1},\ m\in\{k+1,...,n-1\}.$$

Proof.

The obvious consequence of linear equations theory.

Corollary.

Let matrix representation of an operator A be such that $a_{ij} = 0,\ i = 1, 2, ..., k$, $j = k+1, k+2, ..., n$ and let $dimS = k$.
Then $P(A, S) \neq \emptyset$.

Note that, each operator A considered in this paper has a representation:

$$A = \begin{bmatrix} a & b \\ b^* & c \end{bmatrix} \tag{4}$$

where $a \in \mathbb{L}(S),\ a = a^*,\ b \in \mathbb{L}(S^\perp, S),\ c \in \mathbb{L}(S^\perp),\ S \subset \mathbb{R}^n,\ dimS = k$.

Then, for $A \in \mathbb{L}(\mathbb{R}^n)$ with matrix representation $A = [a_{ij}]_{i,j=1}^n$, we have:

$$a = \begin{bmatrix} a_{11} & . & . & a_{1k} \\ . & . & . & . \\ . & . & . & . \\ a_{1k} & . & . & a_{kk} \end{bmatrix},\quad b = \begin{bmatrix} a_{1k+1} & . & . & a_{1n} \\ . & . & . & . \\ . & . & . & . \\ a_{kk+1} & . & . & a_{kn} \end{bmatrix},$$

$$c = \begin{bmatrix} a_{k+1k+1} & . & . & a_{k+1n} \\ . & . & . & . \\ . & . & . & . \\ a_{nk+1} & . & . & a_{nn} \end{bmatrix}. \tag{5}$$

Theorem 2.

Let an operator A has the matrix representation given by the formula (4) where operators a, b, c are given by (5). Let $S \subset \mathbb{R}^n$ be a k-dimensional subspace.
Then

$$P(A, S) \neq \emptyset \quad \textit{if and only if} \quad R(b) \subset R(a).$$

Proof.

Let $v_1, v_2, ..., v_k$ be a basis of the subspace S and let $v_{k+1}, v_{k+2}, ..., v_n$ be a basis of the subspace $S^\perp$.

Let us assume that $R(b) \subset R(a)$. Hence:

$$bv_{k+1}, bv_{k+2}, ..., bv_n \in R(a),$$

which means that there exist $x_{k+1}, x_{k+2}, ..., x_n \in S$ such that

$$bv_{k+1} = ax_{k+1},\ bv_{k+2} = ax_{k+2}, ...,\ bv_n = ax_n.$$

From, given by (5), the matrix representation of the operator b we have:

$$bv_{k+1} = \begin{bmatrix} a_{1k+1} \\ a_{2k+1} \\ . \\ . \\ . \\ a_{kk+1} \end{bmatrix},\ bv_{k+2} = \begin{bmatrix} a_{1k+2} \\ a_{2k+2} \\ . \\ . \\ . \\ a_{kk+2} \end{bmatrix}, ...,\ bv_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ . \\ . \\ . \\ a_{kn} \end{bmatrix}.$$

Let $x_{k+1}, x_{k+2}, ..., x_n$ have, in the basis of S, the following representation:

$$x_{k+1} = (x_{1k+1}, ..., x_{kk+1}), ..., x_n = (x_{1n}, ..., x_{kn}).$$

From the above we have:

$$ax_{k+1} = \begin{bmatrix} a_{11}x_{1k+1} + ... + a_{1k}x_{kk+1} \\ . \\ . \\ . \\ a_{1k}x_{1k+1} + ... + a_{kk}x_{kk+1} \end{bmatrix}, ..., ax_n = \begin{bmatrix} a_{11}x_{1n} + ... + a_{1k}x_{kn} \\ . \\ . \\ . \\ a_{1k}x_{1n} + ... + a_{kk}x_{kn} \end{bmatrix}.$$

Hence we obtain

$$\begin{gathered} a_{11}x_{1k+1} + ... + a_{1k}x_{kk+1} = a_{1k+1}, \\ . \\ . \\ a_{1k}x_{1k+1} + ... + a_{kk}x_{kk+1} = a_{kk+1}, \\ . \\ . \\ . \\ a_{11}x_{1n} + ... + a_{1k}x_{kn} = a_{1n}, \\ . \\ . \\ a_{1k}x_{1n} + ... + a_{kk}x_{kn} = a_{kn}. \end{gathered}$$

Consequently the system of linear equations given by the matrix C (see formula (3)) has a solution. This means that $P(A, S) \neq \emptyset$.

Now, let us assume that $P(A, S) \neq \emptyset$. Hence there exist $x_{k+1}, x_{k+2}, ..., x_n \in S$ such that:

$$bv_{k+1} = ax_{k+1},\ bv_{k+2} = ax_{k+2}, ...,\ bv_n = ax_n.$$

This means that $bv_{k+1}, ..., bv_n \in R(a)$. Since $R(a)$ is a linear space,

$$\gamma_{k+1}bv_{k+1} + ... + \gamma_n bv_n = b(\gamma_{k+1}v_{k+1} + ... + \gamma_n v_n) \in R(a)$$

for any $\gamma_{k+1}, ..., \gamma_n \in \mathbb{R}$.

From the above we obtain that $b(S^{\perp}) = R(b) \subset R(a)$.

The proof is complete.

Remark. *Note that, for the space $\mathbb{R}^n$, the above Theorem 2 is a generalization of the result obtained in [14].*
In [14]; positive operator A has been considered whereas an operator A considered in Theorem 2 is not even symmetric.

Corollary.

Let the assumptions of Theorem 2 hold.
If $b \equiv 0$ then $P(A, S) \neq \emptyset$.

4. The orthogonal projection.

In what follows, we will assume that $P(A,S) \neq \emptyset$. If the orthogonal projection P_S belongs to $P(A,S)$ then P_S is an element which norm is minimal.
The theorem below characterizes operators $A \in \mathbb{L}(\mathbb{R}^n)$ such that $P_S \in P(A,S)$.

Theorem 3.
Let an operator A has the matrix representation $A = [a_{ij}]_{i,j=1}^n$, $a_{ij} = a_{ji}$, $i = 1,2,...,k$, $j = 1,2,...,n$ and let $dim S = k$.

$$P_S \in P(A,S) \quad \textit{if and only if}$$
$$a_{ij} = a_{ji} = 0, \ i = 1,2,...,k, \ j = k+1, k+2, ..., n.$$

Proof.
Application of the theory of linear, homogenous equations.

Theorem 4. *Let the assumptions of Theorem 3 hold. Let $P(A,S) \neq \emptyset$. Then $P_S \notin P(A,S)$ if and only if there exists $l \in \{1,2,...,k\}$, $m \in \{k+1, k+2, ..., n\}$ such that $a_{lm} \neq 0$.*
Additionally:
if $r(C) = k(n-k)$ then $\sharp P(A,S) = 1$;
if $r(C) < k(n-k)$ then $P(A,S)$ is not finite,
where the matrix C is given by (2) and $r(C)$ denotes the range of C.

Proof. Application of the theory of linear equations.

Example 2.
Let us consider the situation from Example 1. This situation leads to the following system of linear equations:

$$\begin{cases} a_{11}r_{12} = a_{12}, \\ a_{11}r_{13} = a_{13}, \\ -a_{13}r_{12} + a_{12}r_{13} = 0. \end{cases}$$

If we assume that the solution of such linear system exists; we obtain that there is only one solution if and only if $a_{11} \neq 0$ and then $P(A,S)$ has only one element. The set $P(A,S)$ is infinite if and only if $a_{11} = a_{12} = a_{13} = 0$. In the second situation r_{12}, r_{13} can be chosen arbitrary so $P_S \in P(A,S)$.

Example 3.

Let us consider the following situation:
$n = 3,\ dimS = 2,$ *and*

$$A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 5 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 & r_{13} \\ 0 & 1 & r_{23} \\ 0 & 0 & 0 \end{bmatrix}.$$

From (3) we have:

$$\begin{cases} 4r_{13} + 2r_{23} = 2, \\ 2r_{13} + r_{23} = 1. \end{cases}$$

From the above we obtain that $r_{23} = 1 - 2r_{13},\ r_{13} \in \mathbb{R}$ *and hence we have:*

$$P(A,S) = \{Q = \begin{bmatrix} 1 & 0 & r_{13} \\ 0 & 1 & 1-2r_{13} \\ 0 & 0 & 0 \end{bmatrix},\ r_{13} \in \mathbb{R}\}.$$

So $P(A,S)$ *is not finite and* $P_S \notin P(A,S)$.

5. Projections of minimal norms.

Now we focus our attention to the case when $P(A,S)$ has more than one element and $P_S \notin P(A,S)$. In such a case, for the special class of operators A, we will find an element which norm is minimal. We will consider different norms in $\mathbb{R}^n$.

Let $S \subset \mathbb{R}^n$ be k-dimensional subspace and let the operator A has the following matrix representation:

$$A = \begin{bmatrix} a_{11} & 0 & . & . & a_{1k} & 0 & . & . & a_{1n} \\ 0 & a_{22} & 0 & . & 0 & 0 & . & . & 0 \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ a_{1k} & 0 & . & 0 & a_{kk} & 0 & . & . & 0 \\ 0 & 0 & . & . & 0 & a_{k+1k+1} & . & . & a_{k+1n} \\ . & . & . & . & . & . & . & . & . \\ a_{1n} & 0 & . & . & 0 & a_{nk+1} & . & . & a_{nn} \end{bmatrix}$$

where $a_{1k}, a_{1n} \neq 0$.

Any projection Q has the representation given by (1).
From the equation $AQ = Q^*A$ we obtain:

$$\begin{cases} a_{11}r_{1n} + a_{1k}r_{kn} = a_{1n} \\ a_{1k}r_{1n} + a_{kk}r_{kn} = 0 \\ a_{22}r_{2k+1} = ... = a_{22}r_{2n} = 0 \\ ... \\ ... \\ a_{k-1k-1}r_{k-1k+1} = ... = a_{k-1k-1}r_{k-1n} = 0 \\ r_{kk+1} = ... = r_{kn-1} = 0 \end{cases}$$

Hence:
- if there exists $l \in \{2, 3, ..., k-1\}$ such that $a_{ll} = 0$ then we have that $r_{lk+1}, ..., r_{ln} \in \mathbb{R}$;
- if $a_{22}, ..., a_{k-1k-1} \neq 0$ then we have that $r_{2k+1} = ... = r_{k-1n} = 0$.

Additionally from the equations

$$\begin{cases} a_{11}r_{1n} + a_{1k}r_{kn} = a_{1n} \\ a_{1k}r_{1n} + a_{kk}r_{kn} = 0, \end{cases}$$

we obtain:

$$r_{kn} = \frac{a_{1n}a_{1k}}{a_{1k}^2 - a_{11}a_{kk}}, \quad r_{1n} = \frac{-a_{kk}a_{1n}}{a_{1k}^2 - a_{11}a_{kk}}.$$

So $r_{kn} \neq 0$ and $r_{1n} = 0$ if and only if $a_{kk} = 0$.
Summarizing the above, we obtain that $P(A, S)$ looks like as follows:
- if there exists $l \in \{2, 3, ..., k-1\}$ such that $a_{ll} = 0$ then

$$P(A,S) = \{Q = \begin{bmatrix} 1 & 0 & . & 0 & 0 & . & . & 0 & \frac{-a_{kk}a_{1n}}{a_{1k}^2 - a_{11}a_{kk}} \\ 0 & 1 & . & . & . & . & . & . & . \\ . & . & . & . & r_{lk+1} & . & . & . & r_{ln} \\ . & . & . & 0 & . & . & . & . & . \\ 0 & . & . & 1 & 0 & . & . & 0 & \frac{a_{1n}a_{1k}}{a_{1k}^2 - a_{11}a_{kk}} \\ 0 & . & . & . & . & . & . & . & 0 \\ . & . & . & . & . & . & . & . & . \\ 0 & . & . & . & . & . & . & . & 0 \end{bmatrix},$$

$$r_{lk+1}, ..., r_{ln} \in \mathbb{R}\}, \tag{6}$$

- if $a_{22}, ..., a_{kk} \neq 0$ then

$$P(A,S) = \{Q = \begin{bmatrix} 1 & 0 & . & 0 & 0 & . & . & 0 & \frac{-a_{kk}a_{1n}}{a_{1k}^2 - a_{11}a_{kk}} \\ 0 & 1 & . & . & . & . & . & . & 0 \\ 0 & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & 0 \\ 0 & . & . & 1 & 0 & . & . & 0 & \frac{a_{1n}a_{1k}}{a_{1k}^2 - a_{11}a_{kk}} \\ 0 & . & . & . & . & . & . & . & 0 \\ . & . & . & . & . & . & . & . & . \\ 0 & . & . & . & . & . & . & . & 0 \end{bmatrix}\}. \tag{7}$$

In (7); $P(A, S)$ has only one element and from the point of seeking for an element of minimal norm such a situation is not interesting.

From now on, we focus our attention on the case given by (6). In such a case the set $P(A, S)$ is not finite.

Example 4. *Let us assume that an operator Q has the matrix representation: $Q = [q_{ij}]_{i,j=1}^n$ and let us consider the following norm:*

$$\|Q\| = \sqrt{\sum_{i,j=1}^{n} | q_{ij} |^2}.$$

For any $Q \in P(A,S)$, where $P(A,S)$ is given by (6), we have:

$$\|Q\|^2 = k + r_{lk+1}^2 + ... + r_{ln}^2 + (\frac{-a_{kk}a_{1n}}{a_{1k}^2 - a_{11}a_{kk}})^2 + (\frac{a_{1n}a_{1k}}{a_{1k}^2 - a_{11}a_{kk}})^2.$$

The above formula has minimal value if $r_{lk+1} = ... = r_{ln} = 0$.

Now, for projection $Q \in P(A,S)$, we will consider different types of operator norms. These norms will be given by different norms in the space $\mathbb{R}^n$.
For an operator $Q : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ the operator norm is defined by the formula:

$$\|Q\|_{op} = \sup_{\|x\|=1} \|Qx\|. \tag{8}$$

Let D be a set in $\mathbb{R}^n$. An element $d \in D$ will be called an extremal point of D if: for any $d_1, d_2 \in D$ if there exists $\alpha \in (0,1)$ such that $d = \alpha d_1 + (1-\alpha)d_2$ then $d = d_1 = d_2$.

Let us denote by $E \subset \mathbb{R}^n$ the set of extremal points of the unit sphere. It is known [27] that:

$$\|Q\|_{op} = \sup_{y \in E} \|Qy\|.$$

Example 5. *Consider $\mathbb{R}^n$ with the l_1-norm meaning that:*

$$\|x\|_1 = \sum_{i=1}^{n} \mid x_i \mid, \quad x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n. \tag{9}$$

Then

$$E = \{(\pm 1, 0, ..., 0), ..., (0, ..., 0, \pm 1)\}, \quad \sharp E = 2n.$$

Because of the definition of $\|\cdot\|_1$, it is enough to consider only norms $\|Qe_i\|_1$, $i = 1, 2, ..., n$, $e_i = (0, 0, .., 1, .., 0)$ (1 on $i-th$ position).

Because of the matrix representation of any $Q \in P(A,S)$, where $P(A,S)$ given by (6), we have:

$$Qe_1 = (1, 0, ..., 0), ..., Qe_k = (0, ..., 0, 1, 0...0), (\textit{1 on k-th position}),$$
$$Qe_{k+1} = (r_{1k+1}, ..., r_{kk+1}, 0, ..., 0), ..., Qe_n = (r_{1n}, ..., r_{kn}, 0, ..., 0),$$

where

$$r_{1n} = \frac{-a_{kk}a_{1n}}{a_{1k}^2 - a_{11}a_{kk}}, \; r_{kn} = \frac{a_{1n}a_{1k}}{a_{1k}^2 - a_{11}a_{kk}}.$$

From the above, for any $Q \in P(A,S)$, we obtain:

$$\|Q\|_{op} = max\{1, \sum_{i=1}^{k} \mid r_{ik+1} \mid, ..., \sum_{i=1}^{k} \mid r_{in-1} \mid,$$
$$\mid \frac{-a_{kk}a_{1n}}{a_{1k}^2 - a_{11}a_{kk}} \mid + \mid r_{2n} \mid + ... + \mid r_{k-1n} \mid + \mid \frac{a_{1n}a_{1k}}{a_{1k}^2 - a_{11}a_{kk}} \mid \}.$$

So, for any $Q \in P(A,S)$, holds that $\|Q\|_{op} \geq 1$ and $\|Q\|_{op} = 1$ if, in its matrix representation, Q has elements such that:

$$\sum_{i=1}^{k} \mid r_{ik+1} \mid \leq 1, ..., \sum_{i=1}^{k} \mid r_{in-1} \mid \leq 1,$$

$$\mid \frac{-a_{kk}a_{1n}}{a_{1k}^2 - a_{11}a_{kk}} \mid + \mid r_{2n} \mid + ... + \mid r_{k-1n} \mid + \mid \frac{a_{1n}a_{1k}}{a_{1k}^2 - a_{11}a_{kk}} \mid \leq 1.$$

Example 6. *Consider $\mathbb{R}^n$ with the norm $\|\cdot\|_{max}$ where*

$$\|x\|_{max} = max\{\mid x_1 \mid, \mid x_2 \mid, ..., \mid x_n \mid\}, \ x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n. \tag{10}$$

Then

$$E = \{e_i = (e_{i1}, ..., e_{in}), \ e_{il} \in \{1, -1\}, l = 1, 2, ..., n, \ i = 1, 2, .., 2^n\}.$$

So we have

$$\|Q\|_{op} = max\{\mid 1 + \sum_{i=k+1}^{n} r_{1i} \mid, \mid 1 + \sum_{i=k+1}^{n} r_{2i} \mid, ..., \mid 1 + \sum_{i=k+1}^{n} r_{ki} \mid\}$$

where

$$r_{ml} \geq 0, \ m = 1, 2, ..., k, \ l = k+1, ..., n.$$

So minimal norm has such an operator Q which, in its matrix representation, has:

$$r_{1k+1} = ... = r_{1n-1} = r_{2k+1} = ... = r_{2n} = ... = r_{kk+1} = ... = r_{kn-1} = 0.$$

Example 7. *Consider*

$$Q : (\mathbb{R}^n, \|\cdot\|_1) \longrightarrow (\mathbb{R}^n, \|\cdot\|_{max}),$$

where $\|\cdot\|_1, \ \|\cdot\|_{max}$ are given by (9) and (10) respectively.

Then

$$E = \{(\pm 1, 0, ..., 0), ..., (0, ..., 0, \pm 1)\}, \quad \sharp E = 2n.$$

and

$$\|Q\|_{op} = max\{1, \mid r_{1k+1} \mid, ..., \mid r_{kk+1} \mid, ..., \mid r_{1n} \mid, ..., \mid r_{kn} \mid\}.$$

So $\|Q\| \geq 1$ and $\|Q\| = 1$ if $\mid r_{lm} \mid \leq 1, \ l = 1, 2, ..., k, \ m = k+1, k+2, ..., n.$

Example 8. *We consider*

$$Q : (\mathbb{R}^n, \| \cdot \|_{max}) \longrightarrow (\mathbb{R}^n, \| \cdot \|_1),$$

where $\| \cdot \|_1, \ \| \cdot \|_{max}$ *are given by (9) and (10) respectively.*
Then

$$E = \{e_i = (e_{i1}, ..., e_{in}), \ e_{il} \in \{1, -1\}, l = 1, 2, ..., n, \ i = 1, 2, .., 2^n\},$$

and

$$\|Q\|_{op} = | \ 1 + r_{1k+1} + ... + r_{1n} \ | + ... + | \ 1 + r_{kk+1} + ... + r_{kn} \ |,$$

where

$$r_{lm} \geq 0, \ l = 1, 2, ..., k, \ m = k + 1, k + 2, ..., n.$$

So the above norm is minimal for an operator Q which, in its matrix representation, has:

$$r_{1k+1} = ... = r_{1n-1} = ... = r_{kk+1} = ... = r_{kn-1} = 0.$$

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