A geometrical version of the Moore theorem in the case of infinite dimensional Banach spaces

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Abstract. In this paper the Author shows that if one defines the triods in a suitable way, then it is possible to prove the theorem of Moore in the infinite dimensional case.

1. Introduction.

The classical theorem of Moore is a certain refinement of the Suslin property of separable spaces (each family of pairwise disjoint open sets is countable). In (4) Moore has formulated the following property:

\[
\text{each family of triods in } \mathbb{R}^2 \text{ is countable.}
\]

A triode is a set homeomorphic with \((-1, 1) \times \{0\} \cup \{0\} \times [0, 1]\). The generalization of this theorem for \(n\) was proved by Young in (5). By a "triode" in \(\mathbb{R}^n\) one means a set which is homeomorphic to "an umbrella" (by an \(n\)-dimensional umbrella we understand the union of an \(n\)-ball \(Q\) and of a simple arc \(L\) such that the set \(Q \cup L\) consists of only one point \(a\) lying in the set \(Q \setminus \text{int}\,Q\) and being an end point of \(L\)). Another version of such properties was proved by Bing and Borsuk in (1).

A direct generalization to the case of infinite dimensional Banach spaces is not true. Indeed, let us consider the space \(l_2\). Let

\[
B = \{x \in l_2 : x_1 = 0 \land \|x\| \leq 1\} \cup \{x \in l_2 : x_1 \in [0, 1] \land \forall k \geq 2 \; x_k = 0\}.
\]
If one understand a triode as a set, which is homeomorphic (or even isometric) to $B$, then the property from the theorem of Moore does not hold. Indeed let us consider the hyperplanes $H_c = \{ x \in l_2 : x_1 = c \}$ and $c \in R$. It follows from the Riesz theorem, that $H_0$ is isometric to $l_2$. Let $v = (c, 0, ...,)$, then $H_c = H_0 + v$ and thus $T_v(H_0) = H_c$ where $T_v : l_2 \to l_2$ and $T_v(x) = x + v$. Hence we have a triode in each hyperplane $H_c$. But these hyperplanes form an uncountable family of pairwise sets.

However it is possible to prove a kind of the theorem of Moore in infinite dimensional case if one consider more ”rigid” notion of the triode.

2. The main theorem.

Let $(E, || \cdot ||)$ a Banach space, let $E^*$ be the conjugate of $E$ and let $x, z \in E, r > 0, f \in E^*$ such that $f(x) \neq f(z)$ and $||x - z|| = r$.

**Definition 2.1** The hyperplane defined by a functional $f$ and a constant $c$ is the set $\{ y \in E : f(y) = c \}$. We will denote it by $H_{f,c}$ (clearly $H_{f,0} = \text{ker } f$).

**Definition 2.2** A triode given by the parameters $x, r, f$ and $z$ is a set
\[
(B(x, r) \cap H_{f,f(x)}) \cup \{ \alpha x + (1 - \alpha) z : \alpha \in [0, 1] \}.
\]
It will be denoted by $T(x, r, f, z)$. The point $x$ will be said the emanation point, the number $r$ will be said the radius of the triode and the segment joining the points $x$ and $z$ will be called a handle.

Clearly if $\lambda \neq 0$, then $H_{f,f(x)} = H_{\lambda f, \lambda f(x)}$, i.e. without loss of generality we may assume that the norm of $f$ equals 1.

We will use below the following simple lemma:

**Lemma 2.3** If $A$ is an uncountable set and $h : A \to (0, +\infty)$, then there exists a real positive number $d$ such that, $\text{card}\{ a \in A : h(a) \geq d \} > \aleph_0$.

**Proof.** Since $A = \bigcup A_n$ where
\[
A_n = \{ a \in A : h(a) \geq \frac{1}{n} \}
\]
than one of $A_n$ must be uncountable.

We will also use the following theorem: (its proof is to be found in (2))

**Theorem 2.4** If $X$ is a topological space satisfying the second countability axiom then for each set $A \subset X$ the set of points in $A$, which are not its condensation points is countable.

**Theorem 2.5** If $E$ is a separable Banach space, then each family of pairwise disjoint triods in $E$ is countable.

**Proof.** Let us suppose that $E$ is a separable Banach space and let $3$ be an uncountable family of pairwise disjoint triods.
It follows from the lemma 2.3 that there exists \( d > 0 \) and an uncountable subset \( \mathcal{I}_1 \) of \( \mathcal{I} \) such that all triods in \( \mathcal{I}_1 \) have the radius at \( d \).

Without loss of generality we may assume that all triods in \( \mathcal{I}_1 \) have the radius equal \( d \), (the family of "smaller" triods are still pairwise disjoint.)

Let us observe that the set \( \mathcal{I}_1 \) can be written as the union of two sets
\[
\{ T(x, d, f, z) \in \mathcal{I}_1 : f(x) < f(z) \} \quad \text{and} \quad \{ T(x, d, f, z) \in \mathcal{I}_1 : f(x) > f(z) \}.
\]
Hence at least one of them (without loss of generality we assume that the first) is uncountable. It follows from the lemma 2.3 that there exists \( \delta > 0 \) such that the set \( \mathcal{I}_2 = \{ T(x, d, f, z) \in \mathcal{I}_1 : f(z - x) \geq \delta \} \) is uncountable. Since the triods are pairwise disjoint then the set of their emanation points \( G = \{ x \in E : T(x, d, f, z) \in \mathcal{I}_2 \} \) is uncountable. By the theorem 2.4 there exists in \( G \) an emanation point, which is its condensation point. Without loss of generality we may assume that it is the origin \( \theta \). The triod corresponding to the origin will be denoted by \( T(\theta, d, g, w) \). Hence \( g(w) \geq \delta \).

Let us consider the ball with the center at the origin and the radius \( \frac{\delta}{4} \). Let the triode \( T(x, d, f, z) \) be from \( \mathcal{I}_2 \) and let \( 0 < \|x\| \leq \frac{\delta}{4} \).

Since \( \|g\| = 1 \), \( g(w) \geq \delta \) and from the definition of the radius it follows that
\[
\delta \leq g(w) \leq \|w\| = d
\]

\[
g(x) \leq \|x\| \leq \frac{\delta}{4}
\]

\[
g(x) \leq g(w) \leq \frac{1}{4}
\]

Let us consider the following cases:

1. \( g(x) > 0 \).

Let us observe, that \( x \not\in R_w \).

(a) \( R_w \) and \( H_{f, f(x)} \) have exactly one common point.

We denote this point by \( \overline{w} \). Then there exists \( \lambda \in R \) such that
\[
\overline{w} = \lambda w
\]

Hence \( \overline{w} \in H_{f, f(x)} \) and
\[
\|\overline{w}\| = |\lambda| \cdot d
\]
\[
g(\overline{w}) = \lambda g(w)
\]

i. \( 0 < \lambda < \frac{1}{4} \).

In consequence, using (2), (5) and (1) we obtain
\[
\|x - \overline{w}\| \leq \|x\| + \|\overline{w}\| \leq \frac{\delta}{4} + \frac{d}{2} \leq d
\]

But \( \overline{w} \in H_{f, f(x)} \) hence \( \overline{w} \in T(x, d, f, z) \). This is impossible since the triods are pairwise disjoint (clearly \( \overline{w} \in T(\theta, d, g, w) \)).
ii. \( \frac{1}{2} \leq |\lambda| \).

Since \( g(\omega) \neq g(x) \) hence \( R(\omega - x) \) and \( \ker g \) have exactly one common point, and let denote it by \( a \). Let \( s \in R \) such that \( \omega = a + s(x - a) \). Hence \( \|\omega - a\| = |s|\|x - a\| \) and \( g(\omega) = sg(x) \).

In consequence

\[
|g(\omega)| = \frac{\|\omega - a\|}{\|x - a\|} g(x)
\]

In consequence, using (4), (6) and (3) we obtain

\[
\|x - a\| = \frac{g(x) \|\omega - a\|}{|g(\omega)|} \leq \frac{\|\lambda w - a\|}{\|g(w)\|} \leq \frac{\|w\| + \|a\|}{\|\lambda\|} \leq \frac{d}{4} + \frac{\|a\|}{2}
\]

In consequence, using (2) and (1) we obtain

\[
\|a\| \leq \|x\| + \|x - a\| \leq \frac{\delta}{4} + \frac{d}{4} + \frac{\|a\|}{2} \leq \frac{d}{2} + \frac{\|a\|}{2}
\]

\[
\|a\| \leq d
\]

In consequence, using \( a \in \ker g \) we obtain \( a \in T(\theta, d, g, w) \).

Moreover

\[
\|x - a\| \leq \frac{d}{4} + \frac{\|a\|}{2} \leq d
\]

Then, since \( a \in H_{f,f}(x) \) we obtain \( a \in T(x, d, f, z) \). This is impossible since the triods are pairwise disjoint.

iii. \(-\frac{1}{2} < \lambda < 0\).

Hence \( \|\omega\| \leq \frac{d}{4} \) and \( g(\omega) < 0 \). Since \( x, \omega \in B(\theta, \frac{d}{4}) \) then the segment joining these points is still in the ball. In consequence this segment intersects \( \ker g \) - and this intersection point is a common point of the triods \( T(x, d, f, z) \) and \( T(\theta, d, g, w) \).

(\( \lambda = 0 \) will be considered in c))

(b) \( Rw \) and \( H_{f,f}(x) \) are disjoint.

Let us denote \( \hat{w} = \frac{g(x)}{g(\omega)} \omega \), then \( g(\hat{w}) = g(x) \). Let us observe, that

\[
x - \hat{w} \in (Rw + x) \cap \ker g \land Rw + x \subset H_{f,f}(x) \tag{7}
\]

Moreover

\[
\|\hat{w}\| = \frac{g(x)}{g(\omega)} d \leq \frac{d}{4} \tag{8}
\]

\[
\|x - \hat{w}\| \leq \frac{\delta}{4} + \frac{d}{4} \leq \frac{d}{2} \tag{9}
\]

It follows from (7) and (8) that \( x - \hat{w} \in T(x, d, f, z) \) but from (7) and (9) we have \( x - \hat{w} \in T(\theta, d, g, w) \). This is impossible since the triods are pairwise disjoint.
(c) $Rw$ is contained in $H_{f,f(x)}$ or $\lambda = 0$.

In this situation $\theta \in H_{f,f(x)}$. Because $\text{dist}(x,\theta) \leq \frac{\delta}{4} \leq \frac{d}{4}$ then $\theta \in T(x,d,f,z)$. This is impossible since the triods are pairwise disjoint.

2. $g(x) < 0$.

We use the translation given by $-x$ and we proceed in 1. (we change the roles of triods and thanks to the choice of $\delta$ the same argument is possible.)

3. $g(x) = 0$.

This is impossible since $x \in \ker g$ and $\|x\| \leq \frac{d}{4}$.

Remark. In the proof of the first case "the handle" is necessary only in the situation when $Rw$ and $H_{f,f(x)}$ have exactly one point and this point is of the form $\lambda w$ for $\lambda \in (0,\frac{1}{2})$.

We can generalize a little the definition of the triode.

Let $(E,||.||)$ a Banach space, let $E^*$ be the conjugate of $E$ and let $x,z \in E, r > 0, f \in E^*, \varphi \in E^{[a,b]}$ such that $f(x) \neq f(z), \|x-z\| = r, \varphi$ - continuous and $\varphi(a) = x, \varphi(b) = z, f(\varphi(t)) \neq f(x)$ ($\varphi(t) \notin H_{f,f(x)}$) for $t \in (a,b]$.

Definition 2.8 A generalized triode given by the parameters $x,r,f,z$ and $\varphi$ is a set

$$ \{B(x,r) \cap H_{f,f(x)}\} \cup \{\varphi(t) : t \in [a,b]\}.$$ 

It will be denoted by $T(x,r,f,z,\varphi)$.

The main theorem in this paper is:

**Theorem 2.9** If $E$ is a separable Banach space, then each family of pairwise disjoint generalized triods in $E$ is countable.

**Proof.** Let us suppose that $E$ is a separable Banach space and let $\mathcal{G}$ be an uncountable family of pairwise disjoint generalized triods.

Without loss of generality we may assume that all generalized triods in $\mathcal{G}$ have the radius at $d > 0$. Let us fix an arbitrary triode $T(x,r,f,z,\varphi)$, and let us consider the sphere $S(x,\frac{d}{2})$. Then

$$\exists c \in (a,b) \left\{\|x - \varphi(c)\| = \frac{d}{2} \land \forall t \in (a,c) \|x - \varphi(t)\| < \frac{d}{2}\right\}$$

($\varphi(c)$ - the first common point of the curve and the sphere).

Let us consider

$$\mathcal{G}' = \{T(x,r',f,z',\varphi) : T(x,r,f,z,\varphi) \in \mathcal{G} \land r' = \frac{d}{2} \land z' = \varphi(c)\}.$$
This is a family of pairwise disjoint generalized triods.

Without loss of generality we may assume that for all generalized triods in $\mathcal{Y}'$ the following inequality holds $f(z'-x) \geq \delta > 0$ and that the origin is a condensation point of the emanation points of the generalized triods from $\mathcal{Y}'$ and the corresponding generalized triod is $T(\theta, \frac{d}{2}, g, w', \psi)$.

Let us consider the ball with the center at the origin and the radius $\frac{d}{4}$. Let the triode $T(x, \frac{d}{2}, f, z', \varphi)$ be from $\mathcal{Y}'$ and let $0 < \|x\| \leq \delta.

Similarly as in the proof of the theorem 2.5 it is sufficient to consider the case when $g(x) > 0$ and $Rw \cap H_{f,f(x)} = \{\lambda w'\}$ for $\lambda \in (0, \frac{1}{2})$. Since $\theta$ and $w'$ lie on opposite sides of the hyperplane $H_{f,f(x)}$ hence the curve joining $\theta$ and $w'$ has the common point - say $b$ - with this hyperplane. Since all the curve is contained in the ball $S(\theta, \frac{d}{2})$ hence

$$\|x - b\| \leq \frac{\delta}{4} + \frac{d}{2} \leq d$$

which is impossible since the generalized triods are pairwise disjoint.

References:


