A geometrical version of the Moore theorem in the case of infinite dimensional Banach spaces

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Abstract. In this paper the Author shows that if one defines the triods in a suitable way, then it it possible to prove the theorem of Moore in the infinite dimensional case.

1. Introduction.

The classical theorem of Moore is a certain refinement of the Suslin property of separable spaces (each family of pairwise disjoint open sets is countable). In (4) Moore has formulated the following property:

each family of triods in R^2 *is countable.*

A triode is a set homeomorphic with $(-1,1)\times\{0\}\cup\{0\}\times[0,1)$. The generalization of this theorem for ^{n} was proved by Young in (5). By a "triode" in R^n one means a set which is homeomorphic to "an umbrella" (by an n - *dimensional umbrella* we understand the union of an n - ball Q and of a simple arc L such that the set $Q \cup L$ consists of only one point a lying in the set $Q \setminus intQ$ and being an end point of L). Another version of such properties was proved by Bing and Borsuk in (1).

A direct generalization to the case of infinite dimensional Banach spaces is not true. Indeed, let us consider the space l_2. Let

$$B = \{x \in l_2 : x_1 = 0 \wedge \|x\| \leq 1\} \cup \{x \in l_2 : x_1 \in [0,1] \wedge \forall k \geq 2 \ x_k = 0\}.$$

If one understand a triode as a set, which is homeomorphic (or even isometric) to B, then the property from the theorem of Moore does not hold. Indeed let us consider the hyperplanes $H_c = \{x \in l_2 : x_1 = c\}$ and $c \in R$. It follows from the Riesz theorem, that H_0 is isometric to l_2. Let $v = (c, 0, ...)$, then $H_c = H_0 + v$ and thus $T_v(H_0) = H_c$ where $T_v : l_2 \to l_2$ and $T_v(x) = x + v$. Hence we have a triode in each hyperplane H_c. But these hyperplanes form an uncountable family of pairwise sets.

However it is possible to prove a kind of the theorem of Moore in infinite dimensional case if one consider more "rigid" notion of the triode.

2. The main theorem.

Let $(E, ||\cdot||)$ a Banach space, let E^* be the conjugate of E and let $x, z \in E, r > 0, f \in E^*$ such that $f(x) \neq f(z)$ and $\|x - z\| = r$.

Definition 2.1 The hyperplane defined by a functional f and a constant c is the set $\{y \in E : f(y) = c\}$. We will denote it by $H_{f,c}$ (clearly $H_{f,0} = \ker f$).

Definition 2.2 A triode given by the parameters x, r, f and z is a set

$$\left(B(x, r) \cap H_{f,f(x)}\right) \cup \{\alpha x + (1 - \alpha)z : \alpha \in [0, 1]\} .$$

It will be denoted by $T(x, r, f, z)$. The point x will be said the emanation point, the number r will be said the radius of the triode and the segment joining the points x and z will be called a handle.

Clearly if $\lambda \neq 0$, then $H_{f,f(x)} = H_{\lambda f,\lambda f(x)}$, i.e. without loss of generality we may assume that the norm of f equals 1.

We will use below the following simple lemma:

Lemma 2.3 If A is an uncountable set and $h : A \to (0, +\infty)$, then there exists a real positive number d such that, $\text{card}\{a \in A : h(a) \geq d\} > \aleph_0$.

Proof. Since $A = \bigcup A_n$ where

$$A_n = \{a \in A : h(a) \geq \frac{1}{n}\}$$

then one of A_n must be uncountable.

We will also use the following theorem: (its proof is to be found in (2))

Theorem 2.4 If X is a topological space satisfying the second countability axiom then for each set $A \subset X$ the set of points in A, which are not its condensation points is countable.

Theorem 2.5 If E is a separable Banach space, then each family of pairwise disjoint triods in E is countable.

Proof. Let us suppose that E is a separable Banach space and let $\Im$ be an uncountable family of pairwise disjoint triods.

It follows from the lemma 2.3 that there exists $d > 0$ and an uncountable subset $\Im_1$ of $\Im$ such that all triods in $\Im_1$ have the radius at d.

Without loss of generality we may assume that all triods in $\Im_1$ have the radius equal d, (the family of "smaller" triods are still pairwise disjoint.)

Let us observe that the set $\Im_1$ can be written as the union of two sets $\{T(x,d,f,z) \in \Im_1 : f(x) < f(z)\}$ and $\{T(x,d,f,z) \in \Im_1 : f(x) > f(z)\}$. Hence at least one of them (without loss of generality we assume that the first) is uncountable. It follows from the lemma 2.3 that there exists $\delta > 0$ such that the set $\Im_2 = \{T(x,d,f,z) \in \Im_1 : f(z-x) \geq \delta\}$ is uncountable. Since the triods are pairwise disjoint then the set of their emanation points $G = \{x \in E : T(x,d,f,z) \in \Im_2\}$ is uncountable. By the theorem 2.4 there exists in G an emanation point, which is its condensation point. Without loss of generality we may assume that it is the origin θ. The triod corresponding to the origin will be denoted by $T(\theta,d,g,w)$. Hence $g(w) \geq \delta$.

Let us consider the ball with the center at the origin and the radius $\frac{\delta}{4}$. Let the triode $T(x,d,f,z)$ be from $\Im_2$ and let $0 < \|x\| \leq \frac{\delta}{4}$.

Since $\|g\| = 1, g(w) \geq \delta$ and from the definition of the radius it follows that

$$\delta \leq g(w) \leq \|w\| = d \tag{1}$$

$$g(x) \leq \|x\| \leq \frac{\delta}{4} \tag{2}$$

$$\frac{g(x)}{g(w)} \leq \frac{1}{4} \tag{3}$$

Let us consider the following cases:

1. $g(x) > 0$.

   Let us observe, that $x \notin Rw$.

   (a) Rw and $H_{f,f(x)}$ have exactly one common point.
   We denote this point by $\overline{w}$. Then there exists $\lambda \in R$ such that

   $$\overline{w} = \lambda w \tag{4}$$

   Hence $\overline{w} \in H_{f,f(x)}$ and

   $$\|\overline{w}\| = |\lambda|\, d \tag{5}$$
   $$g(\overline{w}) = \lambda g(w) \tag{6}$$

   i. $0 < \lambda < \frac{1}{2}$.
   In consequence, using (2), (5) and (1) we obtain

   $$\|x - \overline{w}\| \leq \|x\| + \|\overline{w}\| \leq \frac{\delta}{4} + \frac{d}{2} \leq d$$

   But $\overline{w} \in H_{f,f(x)}$ hence $\overline{w} \in T(x,d,f,z)$. This is impossible since the triods are pairwise disjoint (clearly $\overline{w} \in T(\theta,d,g,w)$).

ii. $\frac{1}{2} \leq |\lambda|$.
Since $g(\overline{w}) \neq g(x)$ hence $R(\overline{w} - x)$ and $\ker g$ have exactly one common point, and let denote it by a. Let $s \in R$ such that $\overline{w} = a + s(x - a)$. Hence $\|\overline{w} - a\| = |s| \, \|x - a\|$ and $g(\overline{w}) = sg(x)$. In consequence

$$|g(\overline{w})| = \frac{\|\overline{w} - a\|}{\|x - a\|} g(x)$$

In consequence, using (4), (6) and (3) we obtain

$$\|x - a\| = \frac{g(x) \, \|\overline{w} - a\|}{|g(\overline{w})|} = \frac{g(x) \, \|\lambda w - a\|}{|\lambda| \, g(w)} \leq \frac{\|\lambda w - a\|}{4 \, |\lambda|} \leq \frac{\|w\|}{4} + \frac{\|a\|}{4 \, |\lambda|} \leq \frac{d}{4} + \frac{\|a\|}{2}$$

In consequence, using (2) and (1) we obtain

$$\|a\| \leq \|x\| + \|x - a\| \leq \frac{\delta}{4} + \frac{d}{4} + \frac{\|a\|}{2} \leq \frac{d}{2} + \frac{\|a\|}{2}$$
$$\|a\| \leq d$$

In consequence, using $a \in \ker g$ we obtain $a \in T(\theta, d, g, w)$. Moreover

$$\|x - a\| \leq \frac{d}{4} + \frac{\|a\|}{2} \leq d$$

Then, since $a \in H_{f,f(x)}$ we obtain $a \in T(x, d, f, z)$. This is impossible since the triods are pairwise disjoint.

iii. $-\frac{1}{2} < \lambda < 0$.
Hence $\|\overline{w}\| \leq \frac{d}{2}$ and $g(\overline{w}) < 0$. Since $x, \overline{w} \in B(\theta, \frac{d}{2})$ then the segment joining these points is still in the ball. In consequence this segment intersects $\ker g$ - and this intersection point is a common point of the triods $T(x, d, f, z)$ and $T(\theta, d, g, w)$.
($\lambda = 0$ will be considered in c))

(b) Rw and $H_{f,f(x)}$ are disjoint.
Let us denote $\widehat{w} = \frac{g(x)}{g(w)} w$, then $g(\widehat{w}) = g(x)$. Let us observe, that

$$x - \widehat{w} \in (Rw + x) \cap \ker g \wedge Rw + x \subset H_{f,f(x)} \tag{7}$$

Moreover

$$\|\widehat{w}\| = \frac{g(x)}{g(w)} d \leq \frac{d}{4} \tag{8}$$

$$\|x - \widehat{w}\| \leq \frac{\delta}{4} + \frac{d}{4} \leq \frac{d}{2} \tag{9}$$

It follows from (7) and (8) that $x - \widehat{w} \in T(x, d, f, z)$ but from (7) and (9) we have $x - \widehat{w} \in T(\theta, d, g, w)$. This is impossible since the triods are pairwise disjoint.

(c) Rw is contained in $H_{f,f(x)}$ or $\lambda = 0$.

In this situation $\theta \in H_{f,f(x)}$. Because $\mathrm{dist}(x,\theta) \leq \frac{\delta}{4} \leq \frac{d}{4}$ then $\theta \in T(x,d,f,z)$. This is impossible since the triods are pairwise disjoint.

2. $g(x) < 0$.

We use the translation given by $-x$ and we proceed in 1. (we change the roles of triods and thanks to the choice of δ the same argument is possible.)

3. $g(x) = 0$.

This is impossible since $x \in \ker g$ and $\|x\| \leq \frac{d}{4}$.

Remark. In the proof of the first case "the handle" is necessary only in the situation when Rw and $H_{f,f(x)}$ have exactly one point and this point is of the form λw for $\lambda \in (0, \frac{1}{2})$.

We can generalize a little the definition of the triode.

Let $(E, ||\cdot||)$ a Banach space, let E^* be the conjugate of E and let $x, z \in E, r > 0, f \in E^*, \varphi \in E^{[a,b]}$ such that $f(x) \neq f(z), \|x - z\| = r, \varphi$ - continuous and $\varphi(a) = x, \varphi(b) = z,\ f(\varphi(t)) \neq f(x)\ (\varphi(t) \notin H_{f,f(x)})$ for $t \in (a,b]$.

Definition 2.8 A generalized triode given by the parameters x, r, f, z and φ is a set

$$\left(B(x,r) \cap H_{f,f(x)}\right) \cup \{\varphi(t) : t \in [a,b]\}.$$

It will be denoted by $T(x,r,f,z,\varphi)$.

The main theorem in this paper is:

Theorem 2.9 If E is a separable Banach space, then each family of pairwise disjoint generalized triods in E is countable.

Proof. Let us suppose that E is a separable Banach space and let $\Im$ be an uncountable family of pairwise disjoint generalized triods.

Without loss of generality we may assume that all generalized triods in $\Im$ have the radius at $d > 0$. Let us fix an arbitrary triode $T(x,r,f,z,\varphi)$, and let us consider the sphere $S(x, \frac{d}{2})$. Then

$$\exists c \in (a,b) \left\{ \|x - \varphi(c)\| = \frac{d}{2} \wedge \forall t \in (a,c)\ \|x - \varphi(t)\| < \frac{d}{2} \right\}$$

($\varphi(c)$ - the first common point of the curve and the sphere).

Let us consider

$$\Im' = \{T(x,r',f,z',\varphi) : T(x,r,f,z,\varphi) \in \Im \wedge r' = \tfrac{d}{2} \wedge z' = \varphi(c)\}.$$

This is a family of pairwise disjoint generalized triods.

Without loss of generality we may assume that for all generalized triods in $\Im'$ the following inequality holds $f(z'-x) \geq \delta > 0$ and that the origin is a condensation point of the emanation points of the generalized triods from $\Im'$ and the corresponding generalized triod is $T(\theta, \frac{d}{2}, g, w', \psi)$.

Let us consider the ball with the center at the origin and the radius $\frac{\delta}{4}$. Let the triode $T(x, \frac{d}{2}, f, z', \varphi)$ be from $\Im'$ and let $0 < \|x\| \leq \frac{\delta}{4}$.

Similarly as in the proof of the the theorem 2.5 it is sufficient to consider the case when $g(x) > 0$ and $Rw \cap H_{f,f(x)} = \{\lambda w'\}$ for $\lambda \in (0, \frac{1}{2})$. Since θ and w' lie on opposite sides of the hyperplane $H_{f,f(x)}$ hence the curve joining θ and w' has the common point - say b - with this hyperplane. Since all the curve is contained in the ball $S(\theta, \frac{d}{2})$ hence

$$\|x-b\| \leq \frac{\delta}{4} + \frac{d}{2} \leq d$$

which is impossible since the generalized triods are pairwise disjoint.

References:

[1] R. H. Bing, K. Borsuk, *Some remarks concerning topologically homogeneous space*, Annals of Mathematics, 81 (1965), 100 - 111.

[2] K. Kuratowski, *Wstep do teorii mnogoci i topologii*, Warszawa 1972.

[3] A. Lelek, *On the Moore triodic theorem*, Bulletin of the Polish Academy of Sciences Mathematics, 8 (1960), 271 - 276.

[4] R. L. Moore, *Concerning triods in the plane and the junction points of plane continua*, Proceedings of the National Academy of Sciences USA, 14 (1928), 85 - 88.

[5] G. S. Young, *A generalization of Moore's theorem on simple triods*, Bulletin of the American Mathematical Society, 50 (1944), 714.

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