Minimal multi-convex projections

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Abstract

We say that a function from $X = C^L[a,b]$ is k-convex (for $k \leq L$) if the kth derivative of the function is nonnegative. Let P denote a projection from X onto $V = \Pi_n \subset X$, where Π_n denotes the space of algebraic polynomials of degree less than equal to n. If we want P to leave invariant the cone of k-convex functions ($k \leq n$), we find that such a demand is impossible to fulfill for nearly every k. Indeed only for $k = n-1$ and $k = n$ does such a projection exist. So let us consider instead a more general 'shape' to preserve. Let $\boldsymbol{\sigma} = (\sigma_0, \sigma_1, \ldots, \sigma_n)$ be an $(n+1)$-tuple with $\sigma_i \in \{0,1\}$; we say $f \in X$ is *multi-convex* if $f^{(i)} \geq 0$ for i such that $\sigma_i = 1$. In this paper we characterize those $\boldsymbol{\sigma}$ for which there exists a projection onto V preserving the multi-convex shape. For those shapes able to be preserved via a projection, we construct (in all but one case) a minimal norm multi-convex preserving projection. Out of necessity, we include some results concerning the geometrical structure of $C^L[a,b]$

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1 Introduction

When X is a Banach space and $V \subset X$ a subspace, we denote by $\mathcal{P}(X,V)$ the set of all projections from X onto V; in the cases where there no ambiguity, we will simply write $\mathcal{P}$. We say that a projection P_0 is a *minimal projection* if $\|P_0\| \leq \|P\|$ for all $P \in \mathcal{P}(X,V)$.

It is worth noting that there exists a large number of papers concerning minimal projections. Mainly the problems concern existence ([15], [18]), uniqueness ([14], [16], [27], [39], [40]), characterization of one-complemented subspaces ([1], [2], [29], [36], [37], citemu) concrete formulas for minimal projections ([3], [4], [5], [6], [7], [13], [15], [23], [24], [26], [35], [41]), estimates of the relative projection constants ([5], [17], [21], [25], [33], [38], [42]), construction of spaces with large relative projection constants ([4], [5], [20] [22]). For basic information concerning this topic the reader is referred to [32]

While a minimal projection will, in general, provide good approximations, it may fail to preserve particular properties of elements, as illustrated below. As such, we are motivated to look for projections which leave invariant (or preserve) a particular functional characteristic (or 'shape'). These characteristics are often described using *cones*.

More precisely, a *cone* in X is convex set closed under nonnegative scalar multiplication. Assuming $\mathcal{P} \neq \emptyset$, we may fix cone $S \subset X$ and ask if any element from $\mathcal{P}$ leaves S invariant; i.e, let

$$\mathcal{P}_S = \mathcal{P}_S(X,V) = \{P \in \mathcal{P} \mid PS \subset S\}$$

and determine if $\mathcal{P}_S \neq \emptyset$. When $P \in \mathcal{P}_S$ we say P is *shape-preserving* (in the sense of S). Some basic results on the existence of shape-preserving projections can be found in [10], [31], [12] and [34]. Not surprisingly, for given X, V and S, the problem of determining if $\mathcal{P}_S \neq \emptyset$ is nontrivial in general.

In this paper we first characterize, for a large collection of X, V and S, when $\mathcal{P}_S \neq \emptyset$; then, for each setting in which $\mathcal{P}_S \neq \emptyset$, we calculate $\inf_{P \in \mathcal{P}_S} \|P\|$. Moreover we construct a *minimal shape-preserving projection*.

Specifically, for positive integer L let X denote the L-th continuously differentiable functions on $[a,b]$, $C^L[a,b]$, normed by

$$\|f\|_L = \max_{i=0..L}\{\|f^{(i)}\|_\infty\}.$$

In this case we simply write $X = (C^L[a,b], \|\cdot\|_L)$. We denote by X^* the dual space of X. In this setting, note that δ_t^k, k-th derivative evaluation at

t, belongs to sphere of X^* for $k = 0 \ldots L$ and $t \in [a, b]$. For fixed k, consider the cone $S \subset X$ of all $f \in X$ with nonnegative k-th derivative on $[a, b]$. We refer to this set as the *cone of k-convex functions*. With $V = \Pi_n$, the n-th degree algebraic polynomials, it was shown in [11] that

$$\mathcal{P}_S \neq \emptyset \iff k \geq n - 1. \tag{1}$$

For example, with $X = (C^1[0, 1], \| \cdot \|_1)$ and $k = 1$ we see that there is no monotonicity-preserving (1-convex preserving) projection from X onto $V = \Pi_3$. There is however a projection preserving convexity (or 2-convexity) onto V. Moreover, Theorem 4.2 in [11] constructs a minimal norm element of $\mathcal{P}_S$ for $k = n - 1$ (with norm $3/2$ for every n) using techniques from minimal projection theory found in [7].

As we will see in the Section 3, the existence of a projection preserving k-convexity onto Π_n can be determined via a geometric consideration; in the case $k = n$ or $k = n - 1$, this geometric approach reduces (respectively) to a 1-dimensional or 2-dimensional problem and, as such, is relatively easy to solve. That is, the geometric approach quickly reveals the result in (1).

We now look to generalize k-convexity. Using notation similar to that of [30], for fixed positive integer n let $\boldsymbol{\sigma} = (\sigma_0, \sigma_1, \ldots, \sigma_n)$ be an $(n + 1)$-tuple with $\sigma_i \in \{0, 1\}$; let $M = \max_{\sigma_i = 1} i$. With $X = C^L[a, b]$, $(L \geq M)$ define

$$S_{\boldsymbol{\sigma}} := \{f \in X \mid \sigma_i f^{(i)} \geq 0, \ i = 0, \ldots, n\}.$$

We say $f \in X$ is *multi-convex* if f belongs to the cone $S_{\boldsymbol{\sigma}}$. In this paper we fix $V = \Pi_n$ and consider projections from X onto V leaving invariant a cone of multi-convex functions - so-called *multi-convex projections*. We denote this set of projections by $\mathcal{P}_{S_{\boldsymbol{\sigma}}}$ and look to construct minimal norm elements from this set.

This paper is organized into five sections. Following these introductory remarks, the main content of this paper is described in Section 2. Here we characterize those $\boldsymbol{\sigma}$ for which $\mathcal{P}_{S_{\boldsymbol{\sigma}}} \neq \emptyset$, where $\mathcal{P}_{S_{\boldsymbol{\sigma}}} \subset X = (C^L[a, b], \| \cdot \|_L)$. Furthermore, we develop an iterative, norm-preserving construction of multi-convex projections from X onto $(n+1)$-dimensional subspaces V, where the iteration is with respect to n. This construction yields minimal norm multi-convex projections in the case $V = \Pi_n$. Sections 3 and 5 provide proofs of the results of Section 2. The proofs in Section 5 require basic, non-trivial facts about the unit ball of $X = (C^L[a, b], \| \cdot \|_L)$. For sake of completeness, we prove the needed results in Section 4 (indeed we found no single source

which described the geometry of this ball and thus hope that Section 4 may be of independent utility to others).

As a summary of notation used in following, the dual space of Banach space X is denoted by X^*; we denote by $B(X)$ and $S(X)$, respectively, the unit ball of X and the unit sphere of X. For convex set $K \subset X$, we denote the set of extreme points of K by $\operatorname{ext}(K)$. The convex hul of subset $A \subset X$ is denoted by $\operatorname{co} A$ while the conex cone generated by A is denoted and defined as $\operatorname{cone}(A) = \{\rho a \mid \rho \in [0, \infty) \text{ and } a \in A\}$.

2 Main Results

Let L and n denote positive integers such that $L \geq n - 1$ (the reason for this inequality will be made clear). Let $\boldsymbol{\sigma} = (\sigma_0, \sigma_1, \ldots, \sigma_n)$ with $\sigma_i \in \{0, 1\}$; let $M = \max_{\sigma_i = 1} i$ and $m = \min_{\sigma_i = 1} i$. We say that $\boldsymbol{\sigma}$ is *1-connected* if whenever $\sigma_i = \sigma_j = 1$ for $i < j$, we have $\sigma_k = 1$ for all $k = i, i + 1, \ldots, j$.

THEOREM 2.1 *Let* $X = (C^L[0,1], \|\cdot\|_L)$. $\mathcal{P}_{S_{\boldsymbol{\sigma}}}(X, \Pi_n) \neq \emptyset$ *iff* $M \geq n - 1$ *and* $\boldsymbol{\sigma}$ *is 1-connected.*

The next theorems describe minimal norm multi-convex projections. But first a few comments are in order. By defintion we always have $m \leq M$. Whenever $M = n$, we automatically assume $L \geq n$. Theorem 2.1 indicates that there two possible situations (of interest to us) in which $m = M$: they are $m = M = n - 1$ and $m = M = n$. These cases are actually 'k-convex' shapes (regarded as specific multi-convex shapes); moreover these situations constitute somewhat extreme cases in the multi-convex realm. The case $m = M = n - 1$ has been handled in [11]. For sake of completeness, we give the result on minimality for this cases in Theorem 2.2.

In the case $m = M = n$, the minimal shape-preserving projection problem is completely unsolved for n$\geq$ 2 (the projection given in [8] partially solves the problem in the $n = 2$ case). Indeed, it is conjectured in [11] that a minimal norm projection from $X = C^L[a, b]$ onto $V = \Pi_n$ preserves n-convexity for every $L = 0, 1, \ldots$. That is, in the case of n-convexity, the minimal shape-preserving projection problem is perhaps equivalent to the minimal projection problem. As such, this paper does not address this case.

There is one other exceptional case: $m = n - 1$ and $M = n$. It turns out that the results concerning minimal shape-preserving projections in this case are similar to those in the case where $m = M = n - 1$ but the method of proof

differs substantially from the approach in [11] as well as here. Consequently, the $m = n - 1$, $M = n$ case is handled in [28]. Howerver, Theorem 2.2 below states the result for this case.

Through the remainder of this paper we will assume $n \geq 2$. In the $n = 1$ case, there is a projection of norm one in $\mathcal{P}_{S_{\boldsymbol{\sigma}}}(X, \Pi_1)$.

THEOREM 2.2 (see [11] and [28]) *Let $X = (C^L[a, b], \| \cdot \|_L)$. For fixed n let $m = n - 1 \leq M$. Then there exists $P_m \in \mathcal{P}_{S_{\boldsymbol{\sigma}}}(X, \Pi_n)$ such that $\|P_m\| = 3/2$ and $\|P_m\| \leq \|P\|$ for every $P \in \mathcal{P}_{S_{\boldsymbol{\sigma}}}(X, \Pi_n)$.*

THEOREM 2.3 *Let $X = (C^L[a, b], \| \cdot \|_L)$. For fixed n, assume $M \geq n - 1$ and $\boldsymbol{\sigma}$ is 1-connected. Suppose $m = 0$ and define $P_{0,n} = \sum_{i=0}^{n} u_i \otimes v_i$ where*

$$u_i = \delta_0^i \text{ for } i \neq n,$$

$$u_n = \delta_1^{n-1},$$

$$v_i = \frac{x^i}{i!} \text{ for } i \neq n - 1$$

and

$$v_{n-1} = \frac{x^{n-1}}{(n-1)!} - \frac{x^n}{n!};$$

i.e.,

$$P_{0,n} = \delta_0 \otimes 1 + \delta_0^1 \otimes \frac{x}{1!} + \cdots + \delta_0^{n-1} \otimes \left(\frac{x^{n-1}}{(n-1)!} - \frac{x^n}{n!} \right) + \delta_1^{n-1} \otimes \frac{x^n}{n!}. \quad (2)$$

Then $P_{0,n}$ has minimal norm in $\mathcal{P}_{S_{\boldsymbol{\sigma}}}(X, \Pi_n)$ and

$$\|P_{0,n}\| = \sum_{k=0}^{n-1} \frac{1}{k!}. \quad (3)$$

Moreover, in the case that $M = n - 1$, we have $\{P_{0,n}\} = \mathcal{P}_{S_{\boldsymbol{\sigma}}}(X, \Pi_n)$.

THEOREM 2.4 *Let $X = (C^L[a, b], \| \cdot \|_L)$ and Let $X_1 = (C^{L+1}[a, b], \| \cdot \|_{L+1})$. For fixed integer n, assume $0 < m < n - 1 \leq M$ and $\boldsymbol{\sigma}$ is 1-connected. Let $Y \subset X$ denote an $(n + 1)$-dimensional subspace, spanned by $\{w_0, w_1, \ldots, w_n\}$; i.e., $Y = [w_0, \ldots, w_n]$. Let $P_{m,n} = \sum_{i=0}^{n} q_i \otimes w_i$ denote a projection from X onto Y such that $P_{m,n}$ preserves $S_{\boldsymbol{\sigma}}$; i.e.,*

$$P_{m,n} \in \mathcal{P}_{S_{\boldsymbol{\sigma}}}(X, Y).$$

Define the operator $P_{m+1,n+1}$ on X_1 by

$$(P_{m+1,n+1}f)(x) = \frac{f(0) + f(1)}{2} + \int_0^x (P_{m,n}f')(t) \ dt$$
$$- \frac{1}{2} \int_0^1 (P_{m,n}f')(t) \ dt \tag{4}$$

where f' denotes the derivative of f. Then

$$P_{m+1,n+1} \in \mathcal{P}_{S_{\widehat{\boldsymbol{\sigma}}}}(X_1, [1, W_0, W_1, \ldots, W_n])$$

where

$$W_i(t) = \int_0^t w_i(s) \ ds$$

and $\widehat{\boldsymbol{\sigma}}$ is the 1-connected $(n+2)$-tuple such that $\max_{\sigma_i=1} i = M + 1$ and $\min_{\sigma_i=1} i = m + 1$. Moreover, if $\|P_{m,n}\| \geq 2$ then

$$\|P_{m+1,n+1}\| = \|P_{m,n}\| \tag{5}$$

THEOREM 2.5 *Let k be a nonnegative integer. Let $X = (C^{L+k}[a,b], \|\cdot\|)$ where $\|\cdot\|$ is any norm such that*

$$\|\cdot\|_{2,L+k} \leq \|\cdot\| \leq \|\cdot\|_{L+k}$$

where

$$\|f\|_{2,L+k} = \max \left\{ \max_{j=0,\ldots,L+k-1} \{|f^{(j)}(0)|, |f^{(j)}(1)|\}, \|f^{L+k}\|_\infty \right\}$$

and

$$\|f\|_{L+k} = \max_{i=0..L+k} \{\|f^{(i)}\|_\infty\}.$$

Let $P_{k,n+k}$ denote the specific operator obtained by k applications of (4) beginning with $P_{0,n}$ given in Theorem 2.3. Then $P_{k,n+k}$ is a minimal norm element of $\mathcal{P}_{S_{\boldsymbol{\sigma}}}(X, \Pi_{n+k})$ where $\boldsymbol{\sigma}$ is the 1-connected $(n+k+1)$-tuple such that $\max_{\sigma_i=1} i \geq n + k - 1$ and $\min_{\sigma_i=1} i = k$.

In general, given two norms that are equivalent (but not proportional), we should not expect a projection that has minimal operator norm with respect to first norm to be minimal in the operator norm determined by the second. From this viewpoint, we note that Theorem 2.5 is quite surprising.

The proofs of Theorems 2.1, 2.3 and 2.4 and Theorem 2.5 are contained in the sections that follow. We first verify existence in Section 3. Then, in Section 4, we show how to calculuate the norms of functionals from a particular family. This calculation will play a crucial role in Section 5, where we verify shape-preserving properties and norm minimality of constructed projections.

3   Proof of Existence

We employ results from [31] to establish existence. The relevant material from this paper is included below.

A *cone* K in a Banach space is defined to be a convex set which is closed under nonnegative scalar multiplication. K is said to be pointed if K contains no lines through 0.

For $\phi \in K$, let $[\phi]^+ := \{\alpha\phi \mid \alpha \geq 0\}$. We say $[\phi]^+$ is an *extreme ray* of K if $\phi = \phi_1 + \phi_2$ implies $\phi_1, \phi_2 \in [\phi]^+$ whenever $\phi_1, \phi_2 \in K$. We let $E(K)$ denote the union of all extreme rays of K. When K is a closed, pointed cone of finite dimension we always have $K = \mathrm{co}(E(K))$.

We say a finite (possibly) signed measure μ with support $E \subset X^*$ is a *generalized representing measure* for $\phi \in X^*$ if $\langle x, \phi \rangle = \int_E \langle s, x \rangle \, du(s)$ for all $x \in X$. A nonnegative measure μ satisfying this equality is simply a *representing measure*.

DEFINITION 3.1 Let X be a Hausdorff topological vector space over $\mathbb{R}$ and let X^* be the topological dual of X. We say that a pointed closed cone $K \subset X^*$ is *simplicial* if K can be recovered from its extreme rays, (i.e., $K = \overline{co}(E(K))$) and the set of extreme rays of K form an independent set (independent in the sense that any generalized representing measure for $x \in K$ supported on $E(K)$ must be a representing measure.)

PROPOSITION 3.1 *A pointed closed cone $K \subset X^*$ of finite dimension d is simplicial iff K has exactly d extreme rays.*

For given $S_{\boldsymbol{\sigma}}$, we define its dual cone as

$$S^* = \{u \in X^* \mid u(f) \geq 0 \ \forall f \in S_{\boldsymbol{\sigma}}\}.$$

Note that for each $S_{\boldsymbol{\sigma}}$, the cone dual S^* is simplicial with $[\phi]^+ \in E(S^*)$ iff $\phi = \rho\delta_t^k$ where $\rho > 0$, $t \in [0, 1]$ and integer $k \in [0, 1, \ldots, L]$. The result we will need is the following.

THEOREM 3.1 *Let $X = (C^L[0,1], \| \cdot \|_L)$. $\mathcal{P}_{S_{\boldsymbol{\sigma}}}(X, \Pi_n) \neq \emptyset$ if and only if the cone $S^*_{|V}$ is simplicial.*

(*Proof of Theorem 2.1*) Throughout this proof we denote the dual cone of $S_{\boldsymbol{\sigma}}$ by S^*.

($\Leftarrow$) We verify that $S^*_{|V}$ is simplicial. Note that for each $t \in [0,1]$ and each integer $j \in [m, M]$ the functional δ_t^j belongs to an extreme ray of S^*. As such, we need only demostrate that a simplicial sub-cone of $S^*_{|V}$ captures all restrictions $(\delta_t^j)_{|V}$; this will then imply that $S^*_{|V}$ is itself simplicial. To this end, we fix for V the basis $\{v_i\}_{i=0}^n$ where $v_i = x^i/i!$ and embed $S^*_{|V}$ into (the positive orthant of) $\mathbb{R}^{n+1}$ via the identification

$$\phi_{|V} \equiv \langle \boldsymbol{v}, \phi \rangle = \begin{pmatrix} \langle v_0, \phi \rangle \\ \langle v_1, \phi \rangle \\ \vdots \\ \langle v_n, \phi \rangle \end{pmatrix};$$

with this understanding we will regard $S^*_{|V} \subset \mathbb{R}^{n+1}$. Let us now consider the case $M = n-1$ (we will see that the $M = n$ case follows in an identical way). Notice that, for integer $j \in [m.n-1]$ and $t \in [0,1]$, we have

$$(\delta_t^j)_{|V} = \begin{pmatrix} 0_1 \\ \vdots \\ 0_j \\ 1 \\ \frac{t}{1!} \\ \vdots \\ \frac{t^{k-j}}{(k-j)!} \\ \vdots \\ \frac{t^{n-j}}{(n-j)!} \end{pmatrix}. \tag{6}$$

Denote by $\boldsymbol{e}_i$ the vector $(0_1, \ldots, 0_{i-1}, 1, 0_{i+1}, \ldots, 0_{n+1})^T \in R^{n+1}$; from (6) it is clear that, for every integer $j \in [m, n-1]$, $\boldsymbol{e}_{j+1} \in S^*_{|V}$ (given by $(\delta_0^j)_{|V}$) as well as $\boldsymbol{e}_n + \boldsymbol{e}_{n+1} \in S^*_{|V}$ (given by $(\delta_1^{n-1})_{|V}$). Moreover, for integer $j \in [m, n-1]$ we have

$$(\delta_t^j)_{|V} = \sum_{k=j}^{n-2} \frac{t^{k-j}}{(k-j)!} \boldsymbol{e}_{k+1} + \frac{t^{n-j-1}}{(n-j-1)!} \left(1 - \frac{t}{n-j}\right) \boldsymbol{e}_n + \frac{t^{n-j}}{(n-j)!} (\boldsymbol{e}_n + \boldsymbol{e}_{n+1}).$$

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Since the coefficient functions of $\boldsymbol{e}_n + \boldsymbol{e}_{n+1}$ and each $\boldsymbol{e}_k$, $k = j, \ldots, n$ are nonnegative, we have that $S^*_{|V}$ is simplicial. In the case that $M = n$, we again note from (6) that $\boldsymbol{e}_j \in S^*_{|V}$ for every integer $j = m+1, \ldots, n+1$ and thus $S^*_{|V}$ is simplicial.

($\Rightarrow$) Assume $\mathcal{P}_{S_{\boldsymbol{\sigma}}} \neq \emptyset$. By Theorem 3.1, we know $S^*_{|V}$ is simplicial. Let $E = \{[x_1]^+, \ldots, [x_d]^+\}$ be the set of extreme rays of $S^*_{|V}$. We first show that M must be at least $n-1$. Suppose, to contrary, that $M \leq n-2$. For convenience, fix for V the basis $\{v_i\}_{i=0}^n$ where

$$v_i = \begin{cases} \frac{x^i}{i!} & i < M \\ \frac{(i-M)!}{i!} x^i & i \geq M \end{cases}. \tag{7}$$

For $t \in [0,1]$, let $\Delta_t := (\delta_t^M)_{|V}$ and, via the embedding into $\mathbb{R}^{n+1}$ described above, notice that $\Delta_t = (0_1, \ldots, 0_M, 1, t, t^2, \ldots, t^{n-M})$. Consider the subcone K of $S^*_{|V} \subset R^{n+1}$ generated by rays $[\Delta_t]^+$; i.e., let

$$K = \overline{\mathrm{co}}(\{[\Delta_t]^+ \mid t \in [0,1]\}). \tag{8}$$

K has infinitely many extreme rays since each Δ_t is an extreme point of $C = \overline{\mathrm{co}}(\{\Delta_t \mid t \in [0,1]\})$ (C is a translate of the convex hull of the moment curve $(t, t^2, \ldots, t^{n-M})$). If $m = M$ then we have an immediate contraction; assume then that $0 \leq m < M$. Consequently, E cannot belong entirely to K; without loss assume $\{[x_1]^+, \ldots, [x_k]^+\} = E - (E \cap K)$. Since each such ray is extreme, we must have for each $i = 1, \ldots, k$, $[x_i]^+ = [(\delta_t^j)_{|V}]^+$ for some $t \in [0,1]$ and some integer $j \in [m, M-1]$. But because $j < M$, we see that every (non-zero) element $(a_1, a_2, \ldots, a_{n+1})^T$ from $\mathrm{co}(\{[x_1]^+, \ldots, [x_k]^+\})$ contains at least one non-zero entry in the first M coordinates; i.e., there exists integer $s \in [0, M]$ such that $a_s \neq 0$. This implies $\mathrm{co}(\{[x_1]^+, \ldots, [x_k]^+\}) \cap K = \emptyset$. Thus $K \subset S^*_{|V}$ cannot be captured in a simplicial subcone of $S^*_{|V}$, and this contradicts the fact that $S^*_{|V}$ simplicial. Therefore $M \geq n-1$. We now show $\boldsymbol{\sigma}$ is 1-connected. Suppose it is not; let $Z := \max\{i \mid \sigma_{i+1} = 0 \text{ and } i < M\}$ (Z marks the location of the last 1 in $\boldsymbol{\sigma}$ before the last break of the sequence of 1's). For convenience fix for V the basis in (7) using Z rather than M. Similar to the above, define $\Delta_t := (\delta_t^Z)_{|V}$ and K as in (8). K has infinitely many extreme rays and thus, as before, the set of extreme rays of $S^*_{|V}$, $E = \{[x_1]^+, \ldots, [x_d]^+\}$, cannot belong entirely to K. Every x_i must be of the form $(\delta_t^j)_{|V}$ for some $t \in [0,1]$ and some integer j as prescribed by $\boldsymbol{\sigma}$. The convex hull of the set $\{[x_i]^+ \mid x_i = (\delta_t^j)_{|V} \text{ for } j \geq Z+2\}$

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misses every (non-zero) element of K since every element $(a_1, \ldots a_{n+1})$ of this convex hull is such that $a_{Z+1} = 0$. Similarly, the convex hull of the set $\{[x_i]^+ \mid x_i = (\delta_t^j)_{|V} \text{ for } j \leq Z - 1\}$ misses every (non-zero) element of K since every element $(a_1, \ldots a_{n+1})$ of this convex hull is such that $a_s \neq 0$ for some integer $s \in [m, Z - 1]$. If $\boldsymbol{\sigma}$ is not 1-connected then we have exhausted all possible choices for x_i (in particular, there is not x_i of the form $(\delta_t^{Z+1})_{|V}$) and we find that $S^*_{|V}$ cannot be simplicial. This contradiction forces us to conclude that $\boldsymbol{\sigma}$ is 1-connected. $\blacksquare$

4   Results on the Geometry of $B(C^L[a,b], \|\cdot\|_L)$

We start with two well-known lemmas, which straightforward proofs will be omitted.

LEMMA 4.1 *Let $(X, \|\cdot\|)$ be a normed space. Suppose that $(\|\cdot\|_k)$ is a sequence of equivalent norms on X such that*

$$\|x\|_k(1 - a_k) \leq \|x\| \leq (1 + a_k)\|x\|_k \tag{9}$$

for any $x \in X$. Assume $a_k \to 0$. Let $\mathcal{L}_k(X)$ denote the space of linear, continuous with respect to $\|\cdot\|$ operators defined on X with the norm induced by $\|\cdot\|_k$. Then for any $T \in \mathcal{L}(X)$,

$$\|T\|_k \to \|T\|,$$

where $\|T\|$ denotes the operator norm of T induced by $\|\cdot\|$.

LEMMA 4.2 *Let $(X, \|\cdot\|)$ be a normed space and let $\|\cdot\|_k$ be a sequence of norms on X satisfying (9) such that $a_k \to 0$. Let X^*_k denote the dual space X^* equipped with the norm induced by $\|\cdot\|_k$. Then for any $f \in X^*$,*

$$\|f\|_k \to \|f\|,$$

where $\|f\|$ denotes the norm of f in X^*.

DEFINITION 4.1 *Let $\{t_i\}$ be a countable, dense subset of $[0, 1]$ such that $t_0 = 0$ and $t_1 = 1$. Let us define for $k \in \mathbb{N}$ a norm $\|\cdot\|_{k,L}$ on $C^L[0, 1]$ by:*

$$\|f\|_{k,L} = max_{i=0,\ldots,L} A_{ik}(f),$$

where for $i = 0, ..., L - 1$

$$A_{ik} = max_{j=1,...,k} |f^{(i)}(t_j)|$$

and

$$A_{L,k} = \|f^{(L)}\|_\infty.$$

LEMMA 4.3 *Let $\| \cdot \|_{k,L}$ be as in Definition 4.1. Then for any $\epsilon > 0$ there exists $k_o \in \mathbb{N}$ such that for any $f \in C^L[0,1]$ and $k \geq k_o$*

$$\|f\|_{k,L} \leq \|f\|_L \leq (1 + \epsilon)\|f\|_{k,L}, \tag{10}$$

where

$$\|f\|_L = \max_{i=0..L}\{\|f^{(i)}\|_\infty\}.$$

Proof. Fix $k \in \mathbb{N}$, $k \geq 3$. Without loss of generality, we can assume that

$$1 = t_o < t_2 < ... < t_k < t_1 = 1.$$

Set

$$\Delta_k = \max\{t_2 - t_o, t_1 - t_k, t_j - t_{j-1}, j = 3, ..., k\}.$$

By the density of $\{t_j\}$, $\Delta_k \to 0$. Fix $k_o \in \mathbb{N}$ such that

$$(1 + \Delta_k)^{L+1} < 1 + \epsilon$$

for $k \geq k_o$. Take any $k \geq k_o$. Fix $f \in X$. First we show that

$$\|f^{(L-1)}\|_\infty \leq (1 + \Delta_k)\|f\|_{k,L}. \tag{11}$$

Let $t \in [0,1]$ be so chosen that $\|f^{(L-1)}\|_\infty = |f^{(L-1)}(t)|$. Then $t \in [t_o, t_2]$ or $t \in [t_k, t_1]$ or $t \in [t_i, t_{i+1}]$ for some $i = 2, ..., k$. Hence by the definition of Δ_k and the mean value theorem, for a properly chosen t_i i=0,...,k,

$$|f^{(L-1)}(t) - f^{(L-1)}(t_i)| \leq \|f^{(L)}\|_\infty |t - t_i|.$$

Hence

$$\|f^{(L-1)}\|_\infty = |f^{(L-1)}(t)| \leq |f^{(L-1)}(t_i)| + \|f^{(L)}\|_\infty \Delta_k$$
$$\leq \|f\|_{k,L}(1 + \Delta_k),$$

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which proves (11). Analogously, by the mean value theorem,

$$\|f^{(L-j)}\|_\infty \leq (1 + \Delta_k)^j \|f\|_{k,L}$$

for $j = 0, ..., L$. Hence

$$max_{i=0,...,L}\{\|f^{(i)}\|_\infty\} \leq (1 + \Delta_k)^{L+1} \|f\|_{k,L}$$

which gives

$$\|f\| \leq (1 + \epsilon)\|f\|_{k,L}$$

for $k \geq k_o$, as required. ∎

THEOREM 4.1 *Let* $X = (C^L[0,1], \| \cdot \|_L)$ *and* $X_k = (C^L[0,1], \| \cdot \|_{k,L})$. *Then*

$$\text{ext}\,(B(X^*)) \subset \{\pm\delta_t^i \mid i = 0, .., L, t \in [0,1]\}$$

and

$$\text{ext}\,(B(X_k^*)) \subset \{\pm\delta_{t_j}^i \mid i = 0, .., L-1, j = 0, ..., k\} \cup \{\delta_t^L \mid t \in [0,1]\}.$$

Proof. Note that X can be isometrically embeded in $Z = (C[0,1])^{L+1}$ with a norm

$$\|(f_1, ..., f_{L+1})\| = max_{i=1,...,L+1}\|f_i\|_\infty.$$

The embedding is given by a formula

$$T(f) = (f, f^{(1)}, ..., f^{(L)}).$$

Note that

$$\text{ext}\,(B(Z^*)) = \{(0, ..., \pm\delta_t, 0, ..., 0) : t \in [0,1]\}.$$

To show our claim we prove that X is a weakly separating subspace of Z. Recall that a linear subspace V of a Banach space W is called *weakly separating* if any point from $\text{ext}\,(B(V^*))$ has only one Hahn-Banach extension in $B(W^*)$.
So assume $x^* \in \text{ext}\,B(X^*)$. Set

$$K = \{f \in B(Z^*) : f|_X = x^*\}.$$

We show that K consists of exaclty one element from $\text{ext}\,(B(Z^*))$. It is easy to see that K is a convex, weak* closed subset of $B(Z^*)$. By the Banach-Alaoglu

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theorem K is weakly* compact. By the Krein-Milman theorem $\mathrm{ext}\,(K) \neq \emptyset$. First we show that $\mathrm{ext}\,(K) \subset \mathrm{ext}\,(B(Z^*))$. Take any $g \in \mathrm{ext}\,(K)$ and assume $g = (g_1 + g_2)/2$, where $g_1, g_2 \in B(Z^*)$. Then

$$f = g_X = \frac{(g_1)|_X + (g_2)|_X}{2}.$$

Since $f \in \mathrm{ext}\,(B(X^*))$, $((g_1)|_X = f$ and $(g_2)|_X = f$. Hence $g_1, g_2 \in K$. Since $g \in \mathrm{ext}\,(K)$, $g_1 = g_2$, as required.

Now assume on the contrary that K consinsts of more than one element. Then we can find at least two different points from $ext(B(Z^*))$

$$z_1 = \pm(0, , , .(\delta_s)_i, 0, ...0)$$

$$z_2 = \pm(0, , , .(\delta_t)_j, 0, ...0)$$

belonging to $\mathrm{ext}\,(K)$. Hence $(z_1)|_X = (z_2)_X$. But if $i < j$ taking $f_i(t) = t^i$ we get (via the isometric embedding) $z_1(f_i) = 1$ and $z_2(f_i) = 0$. Hence $i = j$. If $s \neq t$ taking $f_i(t) = t^{i+1}$, we also get $z_1(f_i) \neq z_2(f_i)$. Finally if

$$z_1 = (0, , , .(\delta_t)_i, 0, ...0)$$

$$z_2 = -(0, , , .(\delta_t)_i, 0, ...0)$$

then also $z_1(f_i) \neq z_2(f_i)$, where $f_i(t) = t^i$. Consequently, $z_1 = z_2$; a contradiction. Hence K consists of exactly one element z. By the previous reasoning $z \in ext(B(Z^*)$. Consequently, (via isometric embedding) $x^* = \pm\delta_t^i$, for some $t \in [0, 1]$ and $i = 0, ..., L$, as required.

Now we consider the case of X_k. Note that X_k can be isometrically embeded into

$$C = \mathbb{R}^{(k+1)L} \times C[0, 1]$$

equipped with a norm

$$\|(r_1, ..., r_{(k+1)L}, f)\| = max\{|r_i|, \ i = 1, ..., (k+1)L, \|f\|_\infty\}.$$

The embedding is given by

$$T(f) = (f(t_j)^{(i)}, i = 0, ..., L - 1, j = 0, ..., k, f^{(L)}).$$

Reasoning in the same way as in the case of X we get our result. ∎

THEOREM 4.2 *Let $X_k = (C^L[0,1], \|\cdot\|_{k,L})$, $L \geq 2$. Set*

$$g = \frac{\sum_{i=0}^{L-1} \delta_0^i}{L} \in (X_k)^*$$

Then $\|g\| = 1$.

Proof. Without loss of generality, we can assume that

$$0 = t_o < t_1 < ... < t_k = 1.$$

It is clear that $\|g\| \leq 1$. Assume that $\|g\| < 1$. Then $\|bg\| = 1$ for some $b > 1$, since $g \neq 0$. By Theorem 4.1 and the Krein -Milman Theorem

$$bg = \sum_{i=0}^{L-1} \sum_{j=0}^{k} a_{ij} \delta_{t_j}^i + u. \tag{12}$$

Here u is a Radon measure on [0,1] acting as a functional on X_k as

$$\hat{u}(f) = \int_{[0,1]} f^{(L)}(t) du(t)$$

and

$$\sum_{i=0}^{L-1} \sum_{j=0}^{k} |a_{ij}| + \|u\| = 1, \tag{13}$$

where $\|u\|$ denotes the total variation of u. First we show that $u = 0$. Assume on the contrary that $u \neq 0$. Then

$$u = bg - \sum_{i=0}^{L-1} \sum_{j=0}^{k} a_{ij} \delta_{t_j}^i.$$

Note that $u = \sum_{l=0}^{k-1} u_l$, where u_l is a Radon measure defined by $u_l(A) = u(A \cap (E_l = [t_l, t_{l+1}))$ for $l = 0, ..., k-2$ and $u_{k-1}(A) = u(A \cap (E_{k-1} = [t_{k-1}, 1])$. Let us first assume that for every $l = 0, ..., k-1$ $u_l = c_l m_l$, where m_l is the Lebesgue measure on E_l and $c_l \in \mathbb{R}$. By the Fundamental Theorem of Calculus

$$m_l(f) = c_l \int_{E_l} f^{(L)}(s) ds = c_l \left(f^{(L-1)}(t_{l+1}) - f^{(L-1)}(t_l) \right)$$

for $l = 0, ..., k - 1$. By the Hermite interpolation theorem there exists a polynomial p such that

$$p^{(i)}(0) = 0 \text{ for } i = 0, ..., L - 1,$$

$$p^{(i)}(t_j) = sgn(a_{ij}) \text{ for } i = 0, ..., L - 2, j = 1, ..., k,$$

and

$$p^{(L-1)}(t_j) = sgn(a_{L-1,j} - c_{j-1} + c_j)$$

for $j = 1, ..., k$. Observe that

$$0 = (bg)(p) = \sum_{i=0}^{L-2} \sum_{j=1}^{k} |a_{ij}| + \sum_{j=1}^{k} |a_{L-1,j} + c_{j-1} - c_j|.$$

Hence all coefficients in the above sum are equal to 0. But this implies that

$$bg = \sum_{i=0}^{L-2} a_{i,0} \delta_0^i + (a_{L-1,0} - c_0) \delta_0^{(L-1)}.$$

Since $b > 1$ and the set $\{\delta_0^i : i = 0, ..., L-1\}$ is linearly independent this leads to a contradiction with (13). So assume that there exists $l \in \{0, ..., k - 1\}$ such that $u_l \neq 0$ and u_l is not a constant multiple of m_l. By (12)

$$u(f) = \int_{[0,1]} f^{(L)}(t) du(t) = 0 \tag{14}$$

for any $f \in X_k$ satisfying

$$(f)^{(i)}(t_j) = 0 \text{ for } i = 0, ..., L - 1, j = 0, ..., k. \tag{15}$$

Fix any $f \in X_k$ satisfying (14). Suppose there exists $D_1 \subset E_l$ and $D_2 \subset E_l$ such that

$$u(D_1) \cdot u(D_2) < 0. \tag{16}$$

Assume $u(D_1) > 0$. Modifying D_1 and D_2, if necessary, we can assume that $D_1 \cap D_2 = \emptyset$. By the properties of Radon measures there exists two disjoint subintervals (let us also denote them by D_1, D_2) of E_l of the same lengths $c > 0$ satisfying (16). Set

$$h_L(t) = \chi_{D_1} - \chi_{D_2}. \tag{17}$$

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We now modify h_L to a continuous function h_L^l on [0,1] with support containing in $D_1 \cup D_2$. To do this fix $l \in \mathbb{N}$ such that $c - 2/l > 0$. S Assume that $D_1 = (s_1, s_2)$, $D_2 = (w_1, w_2)$ and $s_2 < w_1$. Set $h_L^l(t) = h_L(t)$ if $t \notin D_1 \cup D_2$, $h_L^l(t) = 1$ if $t \in (s_1 + 1/l, s_2 - 1/l)$, $h_L^l(t) = -1$ if $t \in (w_1 + 1/l, w_2 - 1/l)$ and define it in the linear way for other t. Note that, for Lebesgue measure m,

$$\int_{[0,1]} h_L^l(t)dm(t) = 0, \tag{18}$$

and l can be increased so large that

$$\int_{[0,1]} h_L^l(t)du(t) > 0. \tag{19}$$

Set $H_L(t) = h_L^l(t)$, where l is so chosen that (16) and (19) are satisfied. Set

$$H_{L-1}(t) = \int_{[0,t]} H_L(s)dm(s)$$

and

$$H_{j-1}(t) = \int_{[0,t]} H_j(s)dm(s)$$

for $j = L - 1, ..., 1$. Define

$$G = f + H_0.$$

By the construction of H_L

$$\int_{E_l} H_j(s)dm(s) = 0,$$

for $j = 1, ..., L$. Consequently,

$$(G)^{(i)}(t_j) = 0 \text{ for } i = 0, ..., L - 1, j = 0, ..., k$$

and thus G satisfies (15). As such G should satisfy (14) - but this is in contradiction with (19). To end the proof that $u = 0$, assume that u_l does not satisfy (16). Hence u_l or $-u_l$ is a nonzero measure on E_l which is not a constant multiple of the Lebesgue measure of E_l. Without loss of generality we can assume that u_l is a measure. By the above conditon u_l is not a Haar measure of E_l. Hence exists an open interval $D_1 \subset E_l$ and $t > 0$ such that

$$u_l(D_1) \neq u_l(D_1 + t).$$

Hence there exists two open disjoint intervals $D_1, D_2 \subset E_l$ of the same lengths $c > 0$ satisfying

$$u_l(D_1) \neq u_l(D_2). \tag{20}$$

Since obviously $m(D_1) = m(D_2) = c$, reasoning as above and replacing (16) by (20) we get a contradiction with (19). This finally shows that $u = 0$. Hence (12) reduces to

$$bg = \sum_{i=0}^{L-1} \sum_{j=0}^{k} a_{ij} \delta_{t_j}^i. \tag{21}$$

By the Hermite interpolation theorem there exists a polynomial p such that

$$p^{(i)}(0) = 0 \text{ for } i = 0, ..., L - 1,$$

$$p^{(i)}(t_j) = sgn(a_{ij}) \text{ for } i = 0, ..., L - 1, j = 1, ..., k.$$

Observe that

$$0 = (bg)(p) = \sum_{i=0}^{L-1} \sum_{j=1}^{k} |a_{ij}|.$$

Hence

$$a_{ij} = 0$$

for $j = 1, ..., k$ and $i = 0, ..., L - 1$. By (21),

$$\sum_{i=0}^{L-1} (b/n - a_{i,0}) \delta_0^i = 0.$$

By the linear independence of functionals δ_0^i, $i = 0, ..., L-1$, we get $a_{i,0} = b/n$ for $i = 0, .., L - 1$. Hence

$$\sum_{i=0}^{L-1} a_{i,0} = b > 1;$$

a contradiction with (13). The proof is complete. ∎

COROLLARY 4.1 *Let* $X = (C^L[0,1], \|\cdot\|_L)$. *Set*

$$g = \frac{\sum_{i=0}^{L} \delta_0^i}{(L+1)} \in X^*.$$

Then $\|g\| = 1$.

Proof. Applying Theorem 4.2 to $Y_k = (C^{L+1}[0,1], \|\cdot\|_{k,L+1})$ and g we get $\|g\|_{Y_k^*} = 1$, where $\|\cdot\|_{Y_k^*}$ denotes the norm on $(Y_k)^*$. By Lemma 4.2, Lemma 4.3 and Theorem 4.2,

$$\|g\|_{Y^*} = 1,$$

where $\|\cdot\|_{Y^*}$ denotes the norm on Y^* where $Y = (C^{L+1}[0,1], \|\cdot\|_{L+1})$. Hence, using the weak* density of Y in Y^{**}, there exists a sequence $\{f_k\} \subset Y$ with

$$\|f_k\|_{L+1} = \max_{i=0,\ldots,L+1} \|f_k^{(i)}\|_\infty = 1,$$

such that $g(f_k) \to 1$. Note that $f_k \in X$ and

$$\|f_k\|_L = \max_{i=0,\ldots,L} \|f^{(i)}\|_\infty \le \|f_k\|_{L+1} = 1.$$

Hence $\|g\| = 1$, as required. ∎

LEMMA 4.4 *Let $X = (C^L[0,1], \|\cdot\|_L)$. Fix integer $k \in [0,L]$ and set*

$$g = \sum_{i=0}^{k} \delta_0^{(i)} + \delta_1^{(k)} \in X^*.$$

Set
$$W = \{F \in X^{**} \mid F(g) = k+2 \text{ and } \|F\| = 1\}.$$

Then $W \ne \emptyset$.

Proof. By Corollary 4.1, there exists $F \in (X^L)^{**}$, $\|F\| = 1$ such that $F(\delta_0^{(i)}) = 1$ for $i = 0,\ldots,k$. By the weak* density of X in X^{**}, there exists a sequence $\{f_j\} \subset X$, $\|f_j\|_L \le 1$ such that $f_j^{(i)}(0) \to 1$ for $i = 0,\ldots,k$. Let us define a sequence of continuous functions $\{g_j^{(k)}\}$ by $g_j^{(k)}(t) = f_j^{(k)}(t)$ if $t \in [0, 1-1/j]$ $g_j^{(k)}(1) = 1$, and in the linear way on the interval $[1-1/j, 1]$. Note that for any $j \in \mathbb{N}$,
$$g_j^{(k)} = f_j^{(k)} + h_j^{(k)}$$
where $h_j^{(k)}(t) = 0$ for $t \in [0, 1-1/j]$ and $\|h_j^{(k)}\|_\infty \le 2$. Let

$$h_j^{(k-1)}(t) = \int_0^t h_j^{(k)}(s)ds$$

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and

$$h_j^{(i-1)}(t) = \int_0^t h_j^{(i)}(s)ds$$

for $i = k - 2, ..., 1$. Set

$$g_j = f_j + h_j^{(0)}.$$

By the Mean Value Theorem

$$\|g_j^{(i)}\|_\infty \leq 1 + 2/j$$

for $i = 0, ..., k - 1$ and

$$\|g_j^{(k)}\|_\infty \leq 1$$

by definition. Hence $\|g_j\| \leq 1 + 2/j$ for $j \in \mathbb{N}$. By the Banach Alaoglu Theorem $\{g_j\}$ has a cluster point $F_1 \in X^{**}$, $\|F_1\| = 1$. It is clear by definition of g_j that $F_1 \in W$. ∎

THEOREM 4.3 *Let $X = C^L[0,1], \|\cdot\|_L)$ and fix integer $k \in [0, L-1]$. Set*

$$W = \{F \in X^{**} : F(\delta_0^{(k)}) = 1, \|F\| = 1\}.$$

Assume u is a Borel measure on [0,1]. Define $u^k \in X^*$ by

$$u^k(f) = \int_{[0,1]} f^{(k)}(t)du(t). \tag{22}$$

Then for any $F \in W$ and for any Borel measure u on [0,1],

$$F(u^k) \geq 0.$$

Proof. Fix $F \in W$ and a Borel measure u. By the Goldstine Theorem there exists a sequence $\{f_l\} \subset X$, $\|f_l\|_L \leq 1$, such that

$$f_l(\delta_o^{(k)}) = f_l^{(k)}(0) \to F(\delta_o^{(k)}) = 1 \tag{23}$$

and

$$f_l(u^{(k)}) = \int_{[0,1]} f_l^{(k)}(t)du(t) \to F(u^{(k)}). \tag{24}$$

In particular, $\|f_l^{(k+1)}\|_\infty \leq 1$. Hence by the Mean Value Theorem for any $s, t \in [0,1]$ $l \in \mathbb{N}$,

$$|f_l^{(k)}(s) - f_l^{(k)}(t)| \leq |t - s|.$$

Also $\|f_j^{(k)}\|_\infty \le 1$. By the Ascoli-Arzela Theorem, passing to a subsequence, if necessary, we can assume that there exists $f \in C[0,1]$ such that

$$\|f_l^{(k)} - f\|_\infty \to 0. \tag{25}$$

Now we show that $f(t) \ge 0$ for any $t \in [0,1]$. By (23), $f(0) = 1$. Assume on the contrary, that there exists $t_0 \in (0,1]$ such that $f(t_o) < 0$. By (23) there exists $\delta > 0$ such that

$$|f_l^{(k)}(0) - f_l^{(k)}(t_0)| > 1 + \delta.$$

for $l \ge l_o$. By The Mean Value Theorem,

$$1 + \delta < |f_l^{(k)}(0) - f_l^{(k)}(t_o)| \le \|f_l^{(k+1)}\|_\infty t_0 \le 1;$$

a contradiction. Hence $f(t) \ge 0$ for any $t \ge 0$. By (25)

$$f_l(u^{(k)}) = \int_{[0,1]} f_l^{(k)}(t)du(t) \to \int_{[0,1]} f(t)du(t) \ge 0,$$

since f is nonegative and u is a measure. By (24),

$$F(u^{(k)}) = \int_{[0,1]} f(t)du(t) \ge 0,$$

which completes the proof. ∎

THEOREM 4.4 *Let $X = (C^L[0,1], \|\cdot\|_L)$ and fix integer $k \in [0,L]$. Let $g = \sum_{i=0}^k \delta_o^i$ and*

$$W_1 = \{F \in X^{**} : F(g) = k + 1, \|F\| = 1\}.$$

Assume u is a Borel measure on [0,1]. Define $u^k \in X^*$ by

$$u^k(f) = \int_{[0,1]} f^{(k)}(t)du(t).$$

Then there exists $F \in W_1$ such that

$$F(u^k) \ge 0$$

for any Borel measure u on [0,1]. Moreover $F(m_t^k) = 0$, for any $t \in [0,1]$, where

$$m_t^k(f) = \int_{[0,t]} f^{(k)}(t)dm(t)$$

and m is the Lebesgue measure on $[0,1]$.

Proof. Let $Z = (C^{L+1}[0,1], \|\cdot\|)$. Set $g = \sum_{i=0}^{k+1} \delta_0^i$ and

$$W_2 = \{F \in Z^{**} : F(g) = k+2, \|F\| = 1\}.$$

By Corollary 4.1, $W_2 \neq \emptyset$. Take any $G \in W_2$. Since $W_2 \subset W$, by Theorem 4.7, for any Borel measure u, $G(u^k) \geq 0$, where u^k is a functional defined on Z by (22). By the Goldstine Theorem applied to $B(Z^{**})$, there exists a net $\{f_\beta\} \subset Z$, $\|f_\beta\| \leq 1$ for any β such that $f_\beta \to G$ weak-$*$ in Z^{**}. Since $Z \subset X$ (as sets) $\{f_\beta\} \subset X$. Moreover, each f_β has norm one in X, since its norm in Z is at most one. By the Banach-Alaoglu Theorem applied to $B(X^{**})$ the set $\{f_\beta\}$ has an accumulation point $F \in B(X^{**})$. Since $G \in W_2$,

$$1 = G(\delta_0^i) = \lim_\beta f_\beta^{(i)}$$

for $i = 0, ..., k+1$. Hence obviously $F(\delta_o^{(i)}) = 1$, for $i = 0, ..., k$. Since $\|F\| \geq 1$, $F \in W_1$. Moreover by Theorem 4.7, for any Borel measure u on $[0,1]$

$$F(u^k) = \lim_\beta \left(\int_{[0,1]} f_\beta^{(k)}(t) du(t) \right) = G(u^k) \geq 0,$$

which proves our claim. In particular, for any $t \in [0,1]$ $F(m_t^k) \geq 0$. By the Fundamental Theorem of Calculus, for any $f \in X$,

$$m_t^k(f) = \int_{[0,t]} f^{(k)}(t) dm(t) = f^{(k-1)}(t) - f^{(k-1)}(0).$$

Since $F \in W_1$,

$$0 \leq F(m_t^k) = F(\delta_t^{(k-1)} - \delta_0^{(k-1)}) = F(\delta_t^{(k-1)}) - 1.$$

Hence $F(\delta_t^{(k-1)}) \geq 1$. Since $\|F\| = 1$,

$$F(\delta_t^{(k-1)}) = 1.$$

Consequently, $F(m_t^k) = 0$, which completes the proof. ∎

5 Proofs of Minimality

Proof of Theorem 2.3

The fact that $P_{0,n} = \sum_{i=0}^{n} u_i \otimes v_i$ is a projection follows from the definition of each u_i and v_i; it is easy to check that $\langle v_i, u_j \rangle = 0$ unless $i = j$, in which case the result is 1.

The verification that $P_{0,n}$ preserves the multi-convex shape described by $\boldsymbol{\sigma}$ (i.e., $P_{0,n} S_{\boldsymbol{\sigma}} \subset S_{\boldsymbol{\sigma}}$) consists of a direct calculation; let $f \in S_{\boldsymbol{\sigma}}$, $t \in [0,1]$, integer $j \leq M$ and consider

$$
\begin{aligned}
\langle P_{0,n} f, \delta_t^j \rangle &= \left\langle \sum_{k=0}^{n-1} f^{(k)}(0) v_k + f^{(L-1)}(1) v_n, \delta_t^j \right\rangle \\
&= \sum_{k=j}^{n-1} f^{(k)}(0) v_k^{(j)}(t) + f^{(L-1)}(1) v_n^{(j)}(t) \\
&\geq 0
\end{aligned}
$$

since every term in the sum is nonnegtive. Thus $P_{0,n} \in \mathcal{P}_{S_{\boldsymbol{\sigma}}}(X, \Pi_n)$

To verify (3) of Theorem 2.3, note that, from the form of $P_{0,n}$ in (2), we have $\|P_{0,n}\| \leq \sum_{k=0}^{n-1} \frac{1}{k!}$. If $L = n - 1$ then by Lemma 4.4 then $\|P_{0,n}\| = \sum_{k=0}^{n-1} \frac{1}{k!}$. If $L \geq n$ Theorem 4.4 guarantees the existence of $F \in B(X^{**})$ such that $F(\delta_0^i) = 1$ for $i = 0, \ldots, n-1$ and $F(m) = 0$, where m denotes the Lebesgue measure. But we see from the proof of Theorem 4.4 that F vanishing on m implies $F(\delta_1^{n-1}) = 1$, which implies $\|P_{0,n}\| \geq \sum_{k=0}^{n-1} \frac{1}{k!}$ and thus (3) follows.

To show $P_{0,n}$ has minimal norm in $\mathcal{P}_{S_{\boldsymbol{\sigma}}}(X, \Pi_n)$, we consider two cases: $M = n - 1$ and $M = n$. We handle the $M = n - 1$ case first, using the following uniqueness argument.

We begin with a corollary given in [9]; it describes how the functionals that define a projection must be chosen in order for the projection to preserve shape.

COROLLARY 5.1 (see [9]) *Suppose $P \in \mathcal{P}_S$. If $S_{|_V}^*$ is k-dimensional then there exists a basis $\boldsymbol{v} = (v_1, \ldots, v_n)^T$ for V such that whenever $P = \boldsymbol{u} \otimes \boldsymbol{v} \in \mathcal{P}_S$, where $\boldsymbol{u} = (u_1, \ldots, u_n) \in (X^*)^n$, we have, for $i = n-k+1, \ldots, n$, $u_i \in S^*$. Moreover, each such u_i restricts to a distinct extreme ray of $S_{|_V}^*$.*

To utilize this result, we note that the proof of Theorem 2.1 demonstrates

that the simplicial cone $S^*_{|V}$ has easily-described extreme rays; they are generated (via nonnegative scalar multiplication) by

$$\{(\delta_0)_{|V}, (\delta_0^1)_{|V}, \ldots, (\delta_0^{n-1})_{|V}, (\delta_1^{n-1})_{|V}\}. \tag{26}$$

Thus, by Corollary 5.1, every projection $P = \sum_{i=0}^n u_i \otimes \widehat{v}_i \in \mathcal{P}_{S_{\boldsymbol{\sigma}}}(X, \Pi_n)$ must be such that $u_i \in S^*$ and $(u_i)_{|V} = (\delta_0^j)_{|V}$ for some j or $(u_i)_{|V} = (\delta_1^{n-1})_{|V}$. However, it is easy to check that for every j there exists a unique element of S^* whose restriction to V is $(\delta_0^j)_{|V}$ - namely δ_0^j. Similarly, $\delta_1^{(n-1)}$ is the unique element of S^* with restriction to V given by $(\delta_1^{n-1})_{|V}$. Consequently, the form of $P_{0,n}$ given in (2) implies that $P_{0,n}$ is the unique element of $\mathcal{P}_{S_{\boldsymbol{\sigma}}}(X, \Pi_n)$ and therefore of minimal norm.

We consider now the case $M = n$. Unlike the previous case, the projection $P_{0,n}$ is not unique in $\mathcal{P}_{S_{\boldsymbol{\sigma}}}(X, \Pi_n)$; indeed consider $P_{0,n}$ written in the following way:

$$P_{0,n} = \delta_0 \otimes 1 + \delta_0^1 \otimes \frac{x}{1!} + \cdots + \delta_0^{n-1} \otimes \frac{x^{n-1}}{(n-1)!} + (\delta_1^{n-1} - \delta_0^{n-1}) \otimes \frac{x^n}{n!}.$$

Replacing the functional $(\delta_1^{n-1} - \delta_0^{n-1})$ with (a positive scalar multiple of) any nonzero element from the weak* closure of $\mathrm{cone}\{\delta_t^n\}_{t \in [0,1]}$ will result in a element of $\mathcal{P}_{S_{\boldsymbol{\sigma}}}(X, \Pi_n)$. In fact, by Corollary 5.1 every element of $\mathcal{P}_{S_{\boldsymbol{\sigma}}}(X, \Pi_n)$ can be constructed in this way. And it is because of this that we are unable to appeal to standard theory of minimal projections, (described for example in [9]) which relies on best approximations from a linear space (and not from a cone). Thus we proceed in the following way: we show that replacing $(\delta_1^{n-1} - \delta_0^{n-1})$ in $P_{0,n}$ with any other allowable functional from S^* results in an element of $\mathcal{P}_{S_{\boldsymbol{\sigma}}}(X, \Pi_n)$ with norm at least as large as $\|P_{0,n}\|$. The following summarizes the form of an element from $\mathcal{P}_{S_{\boldsymbol{\sigma}}}(X, \Pi_n)$ in the $M = n$ case.

LEMMA 5.1 *Let $Q \in \mathcal{P}_{S_{\boldsymbol{\sigma}}}(X, \Pi_n)$. Then there exists $u \in X^*$ such that*

$$Q = \delta_0 \otimes 1 + \delta_0^1 \otimes \frac{x}{1!} + \cdots + \delta_0^{n-1} \otimes \frac{x^{n-1}}{(n-1)!} + u \otimes \frac{x^n}{n!}.$$

Moreover, there exists a probabilistic Borel measure μ such that for every $f \in X$ we have

$$u(f) = \int_0^1 f^{(n)}(t) \, d\mu(t) \tag{27}$$

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Proof. Fix the basis $\{1, x, x/2!, \ldots, x^n/n!\}$; then Corollary 5.1 guarantees this representation of Q and implies that $u \in S^* \subset X^*$. Furthermore $E(S^*)$, the set of extreme rays of S^*, is (strictly) contained in the set of rays

$$\{[\delta_t^j]^+ \mid t \in [0,1], \ j = 0, \ldots, n\}.$$

Let

$$C = \overline{\text{co}} \left(E(S^*) \cap S(X^*) \right)$$

where the closure is taken with respect to the weak* topology. Note that $\text{ext}(C) \subset \{\delta_t^j \mid t \in [0,1], \ j = 0, \ldots, n\}$. Then by Proposition 2 from [31], for every non-zero $\phi \in S^*$, there exists a positive scalar $c \in R$ and a probabilistic Borel measure μ supported on $\text{ext}(C)$ such that μ represents $c\phi$ (in the sense of Choquet); i.e.,

$$\phi(f) = \int_{\text{ext}(C)} f(\delta) \ d\mu(\delta)$$

for every $f \in X$. Consider now our u above; the fact that $\langle x^i, u \rangle = 0$ for every $i = 0, \ldots, n-1$ implies that a representing measure μ for u cannot have any positive support on the set of extreme points of C of the form $\{\delta_t^j \mid t \in [0,1], \ j = 0, \ldots, n-1\}$. And thus the representation in (27) is the only choice. $\blacksquare$

Now from Theorem 4.4, we have the existence of an $F \in B(X^{**})$ such that $F(\delta_0^i) = 1$ for $i = 0, \ldots, n-1$ and $F(u) \geq 0$. Therefore $\|Q\| \geq \sum_{k=0}^{n-1} \frac{1}{k!}$ which implies $P_{0,n}$ is of minimal norm in the $M = n$ case. This completes the proof of Theorem 2.3. $\blacksquare$

REMARK 5.1 Let $X = (C^L[0,1], \|\cdot\|_{2,L})$, where

$$\|f\|_{2,L} = \max \left\{ \max_{j=0,\ldots,L-1} \{|f^{(j)}(0)|, |f^{(j)}(1)|\}, \|f^{(L)}\|_\infty \right\}.$$

Note that $\|\cdot\|_L$ and $\|\cdot\|_{2,L}$ are equivalent since

$$\left(\frac{2}{3}\right)^L \|\cdot\|_L \leq \|\cdot\|_{2,L} \leq \|\cdot\|_L.$$

And so $P_{0,n} \in \mathcal{P}_{S_{\boldsymbol{\sigma}}}(X, \Pi_n)$. Moreover, an arguement identical to the above shows $P_{0,n}$ is minimal in $\mathcal{P}_{S_{\boldsymbol{\sigma}}}(X, \Pi_n)$ (for either $M = n-1$ or $M = n$) and and (by Lemma 4.4 and Theorem 4.4) $\|P_{0,n}\| = \sum_{k=0}^{n-1} \frac{1}{k!}$.

Proof of Theorem 2.4

To simplify notation, let P denote the operator $P_{m+1,n+1}$ defined in (4). Also, for positive integer k, we will denote the Banach space $C^k[0, 1]$ as simply C^k.

We begin by verifying that P is a projection onto $[1, W_0, W_1, \ldots, W_n]$. Let $k(x)$ denote a (non-zero) constant function and note that

$$(Pk)(x) = (k(0) + k(1))/2 = k(x)$$

since $k' \equiv 0$; thus $(P1)(x) = 1$. Moreover, using the fact that $P_{m,n}$ is a projection onto Y, we have for each integer $j \in [0, n]$

$$
\begin{aligned}
(PW_j)(x) &= (W_j(0) + W_j(1))/2 + \int_0^x w_j(t)\, dt - W_j(1)/2 \\
&= W_j(x)
\end{aligned}
$$

since $W_j(0) = 0$. Thus P is a projection onto $[1, W_0, W_1, \ldots, W_n]$. Note the following (derivative) relationships between projections P and $P_{m,n}$: for any $f \in C^{L+1}$, integer $j \in [1, L+1]$ and $x \in [0, 1]$ we have

$$(Pf)^j(x) = (P_{m,n} f')^{(j-1)}(x). \tag{28}$$

To see that P preserves shape, let $f \in S_{\widehat{\boldsymbol{\sigma}}} \subset C^{L+1}$ and fix integer $j \in [m+1, M+1]$. Then (28) implies $(Pf)^j(x) \geq 0$ since $f' \in S_{\boldsymbol{\sigma}} \subset C^L$ and $P_{m,n} \in \mathcal{P}_{S_{\boldsymbol{\sigma}}}(C^L, W)$. Hence

$$P \in \mathcal{P}_{S_{\widehat{\boldsymbol{\sigma}}}}(C^{L+1}, [1, W_0, W_1, \ldots, W_n].$$

We now verify (5). Let $A = \{f' \mid f \in B(C^{L+1})\}$. We claim

$$A = B(C^L); \tag{29}$$

clearly $A \subset B(C^L)$. Let $g \in B(C^L)$ and define $f(x) = \int_0^x g(t)\, dt$. Note that $f \in C^{L+1}$, since for each integer $j \in [1, L+1]$ we have

$$f^{(j)}(x) = g^{(j-1)}(x). \tag{30}$$

Furthermore, from (30) if follows that $\|f^{(j)}\|_\infty \leq 1$ for each integer $j \in [1, L+1]$. And finally using the definition of f we have $\|f\|_\infty \leq \|g\|_\infty \leq 1$

and thus $f \in B(C^{L+1})$. This establishes our claim in (29) since $g = f'$. We are now ready to compare $\|P\|$ and $\|P_{m,n}\|$. Recall that

$$\|P\| = \sup_{f \in B(C^{L+1})} \|Pf\| = \sup_{f \in B(C^{L+1})} \max_{j=0,\dots,L+1} \|(Pf)^{(j)}\|_\infty$$
$$= \max_{j=0,\dots,L+1} \sup_{f \in B(C^{L+1})} \|(Pf)^{(j)}\|_\infty.$$

Consider first the case in which $j \geq 1$; for each such j we have

$$\sup_{f \in B(C^{L+1})} \|(Pf)^{(j)}\|_\infty = \sup_{f \in B(C^{L+1})} \sup_{x \in [0,1]} |(Pf)^{(j)}(x)|$$
$$= \sup_{f \in B(C^{L+1})} \sup_{x \in [0,1]} |(P_{m,n}f')^{(j-1)}(x)| \qquad \text{by (30)}$$
$$= \sup_{f \in B(C^{L+1})} \|(P_{m,n}f')^{(j-1)}\|_\infty$$
$$= \sup_{f \in A} \|(P_{m,n}f)^{(j-1)}\|_\infty$$
$$= \sup_{f \in B(C^L)} \|(P_{m,n}f)^{(j-1)}\|_\infty \qquad \text{by (29)}.$$

Consequently we have

$$\max_{j=1,\dots,M+1} \sup_{f \in B(C^{L+1})} \|(Pf)^{(j)}\|_\infty = \max_{k=0,\dots,M} \sup_{f \in B(C^L)} \|(P_{m,n}f)^{(k)}\|_\infty \tag{31}$$
$$= \|P_{m,n}\|.$$

To finish the comparison, we must check the $j = 0$ case. Recalling the form of P (or equivalenly $P_{m+1,n+1}$) given in (4), we find

$$\sup_{f \in B(C^{L+1})} \|(Pf)\|_\infty \leq 1 + \sup_{f \in B(C^{L+1})} \left\| \int_0^x (P_{m,n}f')(t) \, dt - \frac{1}{2} \int_0^1 (P_{m,n}f')(t) \, dt \right\|. \tag{32}$$

However, the right-hand side of (32) becomes

$$1 + \sup_{f \in B(C^{L+1})} \left\| \frac{1}{2} \int_0^x (P_{m,n}f')(t) \, dt - \frac{1}{2} \left(\int_0^1 (P_{m,n}f')(t) \, dt - \int_0^x (P_{m,n}f')(t) \, dt \right) \right\|_\infty$$

$$= 1 + \frac{1}{2} \sup_{f \in B(C^{L+1})} \left\| \int_0^x (P_{m,n} f')(t)\, dt - \int_x^1 (P_{m,n} f')(t)\, dt \right\|_\infty$$

$$= 1 + \frac{1}{2} \sup_{f \in B(C^{L+1})} \sup_{x \in [0,1]} \left| \int_0^x (P_{m,n} f')(t)\, dt - \int_x^1 (P_{m,n} f')(t)\, dt \right|$$

$$\leq 1 + \frac{1}{2} \sup_{f \in B(C^{L+1})} \int_0^1 |P_{m,n} f')(t)|\, dt$$

$$= 1 + \frac{1}{2} \sup_{f \in B(C^L)} \int_0^1 |P_{m,n} f)(t)|\, dt \qquad \text{by (29)}$$

$$\leq 1 + \frac{1}{2} \|P_{m,n}\|.$$

Thus

$$\sup_{f \in B(C^{L+1})} \|(Pf)\|_\infty \leq 1 + \frac{1}{2}\|P_{m,n}\| \leq \|P_{m,n}\|$$

since, by assumption, $\|P_{m,n}\| \geq 2$. This result, in combination with (31), establishes (5) and completes the proof of Theorem 2.4. ∎

REMARK 5.2 This proof demonstrates that when $C^L[0,1]$ is normed by $\|\cdot\|_L$, the construction given in (4) is (operator) norm-preserving. It is a staightforward verification that this proof can be repeated when $\|\cdot\|_L$ is replaced by $\|\cdot\|_{2,L}$ and thus we have norm preservation in this case as well.

Proof of Theorem 2.5
We begin by verifying this theorem in the $(L+k)$-norm case. For notation sake, let $X_{L+k} = (C^{L+k}[0,1], \|\cdot\|_{L+k})$. From our assumption on the construction of $P_{k,n+k}$ and Theorem 2.4 we have that $P_{k,n+k} \in \mathcal{P}_{S_{\boldsymbol{\sigma}}}(X_{L+k}, \Pi_{n+k})$ and

$$\|P_{k,n+k}\| = \|P_{0,n}\|. \qquad (33)$$

In fact, we can say more; a straightforward generalization of (29) gives

$$\left\{ f^{(k)} \mid f \in B(X_{L+k}) \right\} = B(X_L)$$

and so

$$\|P_{0,n}\| = \sup_{f \in B(X_L)} \|P_{0,n} f\|_\infty$$

$$= \sup_{f \in B(X_{L+k})} \|P_{0,n}(f^{(k)})\|_\infty$$

$$= \sup_{f \in B(X_{L+k})} \|(P_{k,n+k} f)^{(k)})\|_\infty.$$

This result, together with (33), yields

$$\|P_{k,n+k}\| = \sup_{f \in B(X_{L+k})} \|(P_{k,n+k}f)^{(k)})\|_\infty. \tag{34}$$

Let $Q \in \mathcal{P}_{S_{\boldsymbol{\sigma}}}(X_{L+k}, \Pi_{n+k})$. Recall $M = \max_{\sigma_i = 1} i$. Then by Corollary 5.1 there exists $\{\widehat{v}_0, \widehat{v}_1, \ldots, \widehat{v}_{n+k}\}$ for Π_{n+k} such that Q may be represented as

$$Q = \sum_{i=0}^{k-1} \phi_i \otimes \widehat{v}_i + \sum_{i=k}^{n+k-1} \delta_0^i \otimes \widehat{v}_i + \Delta \otimes \widehat{v}_{n+k} \tag{35}$$

where, in the case $M = n + k - 1$, $\Delta = \delta_1^{n+k-1}$ and otherwise $(M = n + k)$ Δ is any non-zero element of the weak* closure of the cone generated by the set $\{\delta_t^{n+k}\}_{t \in [0,1]}$. We claim that for $i = 0, \ldots, k-1$, the degree of $\hat{v}_i$ is strictly less than k. To the contrary, suppose for some i we have $k \le \deg(\widehat{v}_i) \le n+k$. If $\deg(\widehat{v}_i) < n + k$ then $\langle \widehat{v}_i, \delta_0^{deg(\hat{v}_i)} \rangle \ne 0$ which is a contradiction; a similar conclusion is obtained if $\deg(\widehat{v}_i) = n + k$ since $\langle \widehat{v}_i, \Delta \rangle \ne 0$. Thus $\deg(\widehat{v}_i) < k$ for $i = 0, \ldots, k-1$. For $i = k, \ldots, n+k$, write $\widehat{v}_i = a_i v_i + p_i$ where $a_i \in \mathbb{R}$, v_i is as in Theorem 2.3, and $p_i \in \Pi_{n+k}$. Using an orthogonality argument identical to that above (e.g., $\langle \widehat{v}_i, \delta_0^j \rangle = 0$ whenever $i \ne j$), we conclude that $a_i = 1$ and $\deg(p_i) < k$. Thus

$$(Qf)^{(k)} = (P_{k,n+k}f)^{(k)} \tag{36}$$

and so by (34) we find

$$\|Q\| \ge \sup_{f \in B(X_{L+k})} \|(Qf)^{(k)}\|_\infty = \sup_{f \in B(X_{L+k})} \|(P_{k,n+k}f)^{(k)}\|_\infty = \|P_{k,n+k}\|.$$

Therefore $P_{k,n+k}$ has minimal norm in $\mathcal{P}_{S_{\boldsymbol{\sigma}}}(X_{L+k}, \Pi_{n+k})$.

Consider now the case $X_{2,L+k} = (C^{L+k}[0,1], \| \cdot \|_{2,L+k})$. From our assumption on the construction of $P_{k,n+k}$ and Remarks 5.1 and 5.2 we have that $P_{k,n+k} \in \mathcal{P}_{S_{\boldsymbol{\sigma}}}(X_{2,L+k}, \Pi_{n+k})$ and

$$\|P_{k,n+k}\|_{2,L+k} = \|P_{0,n}\|_{2,L}, \tag{37}$$

where $\|Q\|_{2,L+k}$ denotes the operator norm of Q defined on $X_{2,L+k}$. From Remark 5.1 we have that $\|P_{0,n}\|_{2,L}$ has minimal norm (in the context of Theorem 2.3 and Remark 5.1) and therefore, from an arguement identical to that above in the L-norm case, we can conclude that $P_{k,n+k}$ is a minimal norm element from $\mathcal{P}_{S_{\boldsymbol{\sigma}}}(X_{2,L+k}, \Pi_{n+k})$.

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We now make the following observation; note that

$$P_{k,n+k} : X_{L+k} = (C^{L+k}[0,1], \|\cdot\|_{L+k}) \to Y = (\Pi_{n+k}, \|\cdot\|_{2,L+k})$$

is an operator which preserves the multi-convex shape $S_{\boldsymbol{\sigma}}$. From the form of $P_{k,n+k}$, it follows that the operator norm of $P_{k,n+k} : X_{L+k} \to Y$ is equal to the operator norm of $P_{k,n+k} \in \mathcal{P}_{S_{\boldsymbol{\sigma}}}(X_{2,L+k}, \Pi_{n+k})$. Let $\|P_{k,n+k}\|$ denote this common value. Again using the form of $P_{k,n+k}$ we find

$$\|P_{k,n+k}\| = \sup_{f \in B(X_{2,L+k})} \sup_{t \in \{0,1\}} |(P_{k,n+k}f)^{(k)}(t)|.$$

We claim that $P_{k,n+k}$ has minimal (operator) norm among all operators between spaces X_{L+k} and Y preserving $S_{\boldsymbol{\sigma}}$. Indeed, let Q be any such operator. From Remark 5.1 we recall that norms $\|\cdot\|_{L+k}$ and $\|\cdot\|_{2,L+k}$ are equivalent and therefore we may consider Q as an element of $\mathcal{P}_{S_{\boldsymbol{\sigma}}}(X_{L+k}, \Pi_{n+k})$ - i.e., a projection from X_{L+k} onto subspace Π_{n+k} such that $QS_{\boldsymbol{\sigma}} \subset S_{\boldsymbol{\sigma}}$. This implies that $Q : X_{L+k} \to Y$ has the form described in (35) and therefore we have the relation in (36). The minimal of $P_{k,n+k} : X_{L+k} \to Y$ follows since

$$\begin{aligned}
\|Q\| &= \sup_{f \in B(X_{L+k})} \|Qf\|_{2,L+k} \\
&\geq \sup_{f \in B(X_{L+k})} \sup_{t \in \{0,1\}} |(Qf)^{(k)}(t)| \\
&= \sup_{f \in B(X_{L+k})} \sup_{t \in \{0,1\}} |(P_{k,n+k}f)^{(k)}(t)| \\
&= \sup_{f \in B(X_{2,L+k})} \sup_{t \in \{0,1\}} |(P_{k,n+k}f)^{(k)}(t)| \\
&= \|P_{k,n+k}\|.
\end{aligned}$$

Finally, let $X = (C^{L+k}[0,1], \|\cdot\|)$ such that

$$\|\cdot\|_{2,L+k} \leq \|\cdot\| \leq \|\cdot\|_{L+k}.$$

From the definitions of the L- and $(2,L)$-norms, we have

$$\|P_{k,n+k}\|_{2,L+k} \leq \|P_{k,n+k}\|_X \leq \|P_{k,n+k}\|_{L+k}$$

where $\|P_{k,n+k}\|_X$ is the operator norm of $P_{k,n+k}$ defined on X. But from (33), (37) and Remark 5.1 we find

$$\|P_{k,n+k}\|_X = \|P_{k,n+k}\|_{2,L+k} = \|P_{k,n+k}\|_{L+k}.$$

As done above, let $\|P_{k,n+k}\|$ denote this common value. To show $P_{k,n+k}$ has minimal norm in $\mathcal{P}_{S_{\boldsymbol{\sigma}}}(X, \Pi_{n+k})$ let $Q \in \mathcal{P}_{S_{\boldsymbol{\sigma}}}(X, \Pi_{n+k})$. Note that we have $Q : X_{L+k} \to Y$ such that $QS_{\boldsymbol{\sigma}} \subset S_{\boldsymbol{\sigma}}$ and so

$$\|Q\|_X = \sup_{f \in B(X)} \|Qf\| \geq \sup_{f \in B(X_{L+k})} \|Qf\|_{2,L+k} \geq \|P_{k,n+k}\|$$

∎

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